Abstract. The goal of this work is to construct, for a smooth variety $X$ over a perfect field $k$ of finite characteristic, an overconvergent de Rham-Witt complex $W^1\Omega^*_X/k$ as a suitable subcomplex of the de Rham-Witt complex of Deligne-Illusie. This complex, which is functorial in $X$, is a complex of étale sheaves and a differential graded algebra over the ring $W^1(O_X)$ of overconvergent Witt-vectors. If $X$ is affine one proves that there is a canonical isomorphism between Monsky-Washnitzer cohomology and (rational) overconvergent de Rham-Witt cohomology. In general, we compare this cohomology with the rigid cohomology of $X$.

Résumé. Le but de ce travail est de construire, pour $X$ une variété lisse sur un corps parfait $k$ de caractéristique finie, un complexe de de Rham-Witt surconvergent $W^1\Omega^*_X/k$ comme un sous-complexe convenable du complexe de de Rham-Witt de Deligne-Illusie. Ce complexe qui est fonctoriel en $X$ est un complexe des faisceaux étalés et une algèbre différentielle graduée sur l’anneau $W^1(O_X)$ des vecteurs de Witt surconvergents. Lorsque $X$ est affine, on démontre qu’il existe un isomorphisme canonique entre la cohomologie de Monsky-Washnitzer et la cohomologie (rationnelle) de de Rham-Witt surconvergente. En général, on compare cette cohomologie avec la cohomologie rigide de $X$.

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Let $X$ be a smooth variety over a perfect field $k$ of finite characteristic. The purpose of this work is to define an overconvergent de Rham-Witt complex $W\hat{}Ω_{X/k}$ suitable enough to compute the Monsky-Washnitzer cohomology in the case that $X$ is affine and more generally the rigid cohomology of $X$. The complex $W\hat{}Ω_{X/k}$ is a differential graded algebra over the sheaf of rings $W\hat{}(𝒪_X)$ of overconvergent Witt vectors and is functorial in $X$. We prove that the hypercohomology groups $H^•(X,W\hat{}Ω_{X/k})\otimes\mathbb{Q}$ are canonically isomorphic to rigid cohomology groups of $X$ defined by Berthelot [3]. It is known that the these groups are finite dimensional vector spaces over $W(k)\otimes\mathbb{Q}$. We conjecture that the image of $H^•(X,W\hat{}Ω_{X/k})$ is a $W(k)$-lattice in the rigid cohomology group. This would yield an integral structure on rigid cohomology. We note that Lubkin [14] used another growth condition on Witt vectors. His bounded Witt vectors are different from our overconvergent Witt vectors.

The overconvergent de Rham-Witt complex is defined as a subcomplex of the classical de Rham-Witt complex of Deligne-Illusie [10] as follows. One first considers the case of a polynomial algebra $A = k[T_1,\ldots,T_d]$. For each real $\epsilon > 0$ a certain norm that we call the Gauss norm is defined on the de Rham-Witt complex $WΩ^•_A/k$. A Witt differential from $WΩ^•_A/k$ is called overconvergent if its Gauss norm is finite for some $\epsilon > 0$. We denote the subcomplex of all overconvergent Witt differentials by $W\hat{}Ω^•_A/k$. Following the description in [12], $WΩ^•_A/k$ decomposes canonically into an integral part and an acyclic fractional part and this decomposition continues to hold for the complex of overconvergent Witt differentials. The integral part is easily identified with the de Rham complex associated to the weak completion of the polynomial algebra $W(k)[T_1,\ldots,T_d]$ in the sense of Monsky and Washnitzer. This explains the terminology “overconvergent” for Witt differentials. For an arbitrary smooth $k$-algebra $B$ we choose a presentation $A \to B$. We define the complex of overconvergent Witt differentials $W\hat{}Ω^•_{B/k}$ as the image of $W\hat{}Ω^•_{A/k}$. This is independent of the presentation. It is a central result that the functor which associates to a smooth affine scheme $Spec B$ the group $W\hat{}Ω^m_{B/k}$ is a sheaf for the étale topology, and that $H^i_\text{Zar}(Spec B,W\hat{}Ω^m_{B/k}) = 0$ for $i \geq 1$. For this we generalize ideas of Meredith [15]. One also uses that the ring of overconvergent Witt vectors is weakly complete in the sense of Monsky-Washnitzer [6] and the complex of overconvergent Witt differentials satisfies a similar property of weak completeness. The étale sheaf property depends on an explicit description - for a finite étale extension $C/B$ - of $W\hat{}Ω^•_{C/k}$ in terms of $W\hat{}Ω^•_{B/k}$. The result is as nice as one can hope for. By a result of Kedlaya [11] any smooth variety can be covered by affines which are finite étale over a localized polynomial algebra. It then remains to show a localization property of overconvergence; namely a Witt differential of a localized polynomial algebra which becomes overconvergent after further localization is already overconvergent. This requires a detailed study of suitable Gauss norms (that are all equivalent) on the truncated de Rham-Witt complex of a localized polynomial algebra.
For the comparison with the Monsky-Washnitzer cohomology we consider
the notion of an overconvergent Witt lift of a smooth $k$-algebra $C$. This is
a homomorphism from a Monsky-Washnitzer lift $\tilde{C}^\dagger$ to $W^\dagger(C)$. We prove
that any smooth algebra admits an overconvergent Witt lift. This yields
an integral comparison map between the de Rham complex of $\tilde{C}^\dagger$ and the
overconvergent de Rham-Witt complex of $C$. For a finite étale extension of
a localized polynomial algebra one then shows explicitly that the integral
comparison map is a quasi-isomorphism. Here a modified but equivalent
version of overconvergence on the de Rham-Witt complex of a localized poly-
nomial algebra plays a crucial role. Using an idea of Monsky-Washnitzer,
we can show that any two overconvergent Witt lifts of a smooth algebra
which is finite étale over a localized polynomial algebra lead to comparison
morphisms which are rationally homotopic. We then apply this local ho-
matopy argument to derive our first main comparison result, between the
Monsky-Washnitzer cohomology of a smooth algebra over $k$ and its (ra-
tional) overconvergent de Rham-Witt cohomology. Moreover, in the case
$\dim C < \text{char } k$ we obtain an integral isomorphism induced by a quasi-
isomorphism $\Omega_{\tilde{C}^\dagger/W(k)}^e \cong W^\dagger\Omega_{C/k}$ and in general we have a bound on the
failure of our map to be a quasi-isomorphism integrally.

In the final section we prove the comparison result with rigid cohomology
for any smooth $X$. We first study again the case were $X$ is affine. We take
a slightly more general notion of Witt lifts to construct special frames in
the sense of rigid cohomology [2]. This yields a comparison morphism with
rigid cohomology. For the globalization it is essential to express the com-
parison in terms of Grosse-Klöne’s dagger spaces [7]. For suitable frames
we recover the comparison morphism with Monsky-Washnitzer cohomology
defined above. We prove that the comparison morphism is independent of
the special frame chosen. This follows by considering the product of two
special frames and to some extent replaces the homotopy argument above.

To globalize the result to quasiprojective schemes $X$ we take an affine
open covering $X_i$ and chose a special frame for each $X_i$. This gives rise to
a simplicial dagger space which computes the rigid cohomology of $X$. The
comparison morphism with the overconvergent de Rham-Witt cohomology
of $X$ is defined by this simplicial dagger space. Finally we consider the
case of an arbitrary smooth scheme $X$. As before we use simplicial methods
which rely on the rigid cohomological descent theory of Chiarellotto-Tsuzuki
[5].

0. Definition of the overconvergent de Rham-Witt complex

Let $R$ be an $\mathbb{F}_p$-algebra which is an integral domain. We consider the
polynomial algebra $A = R[T_1, \ldots, T_d]$. Before we recall the de Rham-Witt
complex, we review a few properties of the de Rham complex $\Omega_{A/R}$.

There is a natural morphism of graded rings

$$F : \Omega_{A/R} \to \Omega_{A/R},$$

which is the absolute Frobenius on $\Omega^0_{A/R}$ and such that $F dT_i = T_i^{p-1} dT_i$. As shown in [12], $\Omega_{A/R}$ has an $R$-basis of so called basic differentials. Their
definition depends on certain choices which we will fix now in a more special way than in loc. cit.

We consider functions \( k : [1, d] \to \mathbb{Z}_{\geq 0} \) called weights. On the support \( \text{Supp} k = \{ i_1, \ldots, i_r \} \) we fix an order \( i_1, \ldots, i_r \) with the following properties:

(i) \( \text{ord}_p k_{i_1} \leq \text{ord}_p k_{i_2} \leq \ldots \leq \text{ord}_p k_{i_r} \).

(ii) If \( \text{ord}_p k_{i_n} = \text{ord}_p k_{i_{n+1}} \), then \( i_n \leq i_{n+1} \).

Let \( \mathcal{P} = \{ I_0, I_1, \ldots, I_r \} \) be a partition of \( \text{Supp} k \) as in [12]. A basic differential is a differential of the form:

\[
\epsilon(k, \mathcal{P}) = T^{k_{I_0}} \left( \frac{dT^{k_{I_1}}}{p^{\text{ord}_p k_{i_1}}} \right) \cdots \left( \frac{dT^{k_{I_r}}}{p^{\text{ord}_p k_{i_r}}} \right).
\]

It is shown in [12] Proposition 2.1 that the elements (0.1) form a basis of the de Rham complex \( \Omega^*_{A/R} \) as an \( R \)-module. The de Rham-Witt complex \( W\Omega^*_{A/R} \) has a similar description, but now fractional weight functions are involved. More precisely, an element \( \omega \in W\Omega^*_{A/R} \) has a unique decomposition as a sum of basic Witt differentials [12]

\[
\omega = \sum_{k, \mathcal{P}} e(\xi_{k, \mathcal{P}, k, \mathcal{P}}),
\]

where \( k : [1, d] \to \mathbb{Z}_{\geq 0}[\frac{1}{p}] \) is any weight ([12], 2.2) and \( \mathcal{P} = \{ I_0, I_1, \ldots, I_r \} \) runs through all partitions of \( \text{Supp} k \). Moreover, the coefficients \( \xi_{k, \mathcal{P}} \in W^{\mathcal{P}}(R) \) satisfy a certain convergence condition ([12], Theorem 2.8).

For each real number \( \varepsilon > 0 \) we define the Gauss norm of \( \omega \):

\[
\gamma_\varepsilon(\omega) = \inf_{k, \mathcal{P}} \{ \text{ord}_V \xi_{k, \mathcal{P}} - \varepsilon |k| \}.
\]

We will also use the truncated Gauss norms for a natural number \( n \geq 0 \):

\[
\gamma_\varepsilon[n](\omega) = \inf_{k, \mathcal{P}} \{ \text{ord}_V \xi_{k, \mathcal{P}} - \varepsilon |k| \mid \text{ord}_V \xi_{k, \mathcal{P}} \leq n \}.
\]

The truncated Gauss norms factor over \( W_{n+1}\Omega_{A/R} \). We note that in the truncated case the inf is over a finite set.

If \( \gamma_\varepsilon(\omega) > -\infty \), we say that \( \omega \) has radius of convergence \( \varepsilon \).

We call \( \omega \) overconvergent, if there is an \( \varepsilon > 0 \) such that \( \omega \) has radius of convergence \( \varepsilon \). It follows from the definitions that

\[
\gamma_\varepsilon(\omega_1 + \omega_2) \geq \min(\gamma_\varepsilon(\omega_1), \gamma_\varepsilon(\omega_2)).
\]

This inequality shows that the overconvergent Witt differentials form a subgroup of \( W\Omega^*_{A/R} \) which is denoted by \( W^\varepsilon\Omega^*_{A/R} \). We have \( W^1\Omega_{A/R} = \bigcup_\varepsilon W^\varepsilon\Omega_{A/R} \) where \( W^\varepsilon\Omega_{A/R} \) are the overconvergent Witt differentials with radius of convergence \( \varepsilon \).

If \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \cup \{-\infty\} \), then an \( \mathbb{R} \)-valued function \( c \) on an abelian group \( M \) which satisfies (0.4), so that \( c(a + b) \geq \min\{ c(a), c(b) \} \), is called an order function.

**Definition 0.5.** We say that \( \omega \) is homogeneous of weight \( k \) if in the sum \( \omega = \sum e(\xi_{k, \mathcal{P}, k, \mathcal{P}}) \) the weight \( k \) is fixed. We write \( \text{weight}(\omega) = k \).
If \( g \in \mathbb{Q} \), then we can consider sums which are homogeneous of degree \( g \), i.e.
\[
\omega = \sum_{|k|=g, \rho} e(\xi_k, \rho, k, \rho).
\]
Then we define \( \text{deg}(\omega) = g \). If \( \omega \) is homogeneous of a fixed degree, we define
\[
\text{ord}_p \omega = \min \text{ord}_p \xi_{k, \rho}.
\]
It is easy to see that \( \gamma_\epsilon(\omega) > -\infty \) if and only if there are real constants \( C_1, C_2 \), with \( C_1 > 0 \) such that for all weights \( k \) occurring in \( \omega \) we have
\[
|k| \leq C_1 \text{ord}_p \xi_{k, \rho} + C_2.
\]
(0.6)

One can take \( C_1 = \frac{1}{\epsilon} \).

Using this equivalent definition one can show that the product of two overconvergent Witt differentials is again overconvergent, as follows: For two homogeneous forms \( \omega_1, \omega_2 \) one has \( \text{ord}_p (\omega_1 \wedge \omega_2) \geq \max (\text{ord}_p \omega_1, \text{ord}_p \omega_2) \). This follows from a (rather tedious) case by case calculation with basic Witt differentials.

We have \( \text{deg}(\omega_1 \wedge \omega_2) = \text{deg} \omega_1 + \text{deg} \omega_2 \).

Assume now that
\[
\text{deg} \omega \leq C_1 \text{ord}_p \omega + C_2
\]
and
\[
\text{deg} \omega' \leq C'_1 \text{ord}_p \omega' + C'_2
\]
for two homogeneous forms \( \omega, \omega' \) of fixed degrees. Then
\[
\text{deg}(\omega \wedge \omega') = \text{deg} \omega + \text{deg} \omega' \leq (C_1 + C'_1) \text{ord}_p (\omega_1 \wedge \omega_2) + C_2 + C'_2.
\]
This implies that if \( \omega \) and \( \omega' \) are overconvergent Witt differentials with radii of convergence \( \epsilon \) and \( \epsilon' \) then \( \omega \wedge \omega' \) is overconvergent with radius of convergence \( \frac{\epsilon + \epsilon'}{2} \). In the special case \( \epsilon = \epsilon' \) we get that \( \omega \wedge \omega' \) is overconvergent with radius of convergence \( \frac{\epsilon}{2} \) and
\[
\frac{\gamma_\epsilon(\omega) + \gamma_\epsilon(\omega')}{2} \geq \frac{\gamma_\epsilon(\omega \wedge \omega')}{2}.
\]
This shows that \( W^\dagger \Omega_{A/R} \) is a differential graded algebra over the ring \( W^\dagger (A) \) of overconvergent Witt vectors.

We recall from [6] the definition of a pseudovaluation. An order function \( c \) on a ring \( M \) is called a pseudovaluation if in addition it satisfies: (i) \( c(1) = 0 \) and \( c(0) = \infty \); (ii) \( c(m) = c(-m) \) for all \( m \in M \); (iii) \( c(m_1 m_2) \geq c(m_1) + c(m_2) \) if \( c(m_1) \neq -\infty, c(m_2) \neq -\infty \).

In general, the Gauss norms \( \gamma_\epsilon \) form a set of pseudovaluations on the ring of Witt vectors, i.e. in degree zero; however, from the formula
\[
V_{[T^{p-1}]dV[T]} = pd[T]
\]
and
\[
\text{ord}_p \left( V_{[T^{p-1}]} \right) = \text{ord}_p \left( dV[T] \right) = \text{ord}_p \left( pd[T] \right) = 1,
\]
we see that we cannot expect a formula
\[
\gamma_\epsilon(\omega_1 \wedge \omega_2) \geq \gamma_\epsilon(\omega_1) + \gamma_\epsilon(\omega_2).
\]
Hence the Gauss norms do not extend to pseudovaluations in higher degrees.
Proposition 0.7. Let $R$ be an integral domain such that $p \cdot R = 0$. Let
\[
\varphi : R[T_1, \ldots, T_d] \to R[U_1, \ldots, U_l]
\]
be a homomorphism. It induces a map
\[
\varphi^* : W\Omega_R[T_1, \ldots, T_d] / R \to W\Omega_R[U_1, \ldots, U_l] / R.
\]
Then there is a constant $\alpha > 0$, such that for any $\varepsilon > 0$ and any natural
number $n$:
\[
\gamma_{\alpha \varepsilon}[n](\varphi^* \omega) \geq \gamma_{\varepsilon}[n](\omega).
\]
The same inequality holds if $[n]$ is removed. In particular, if $\omega$ is overconvergent with radius of convergence $\varepsilon$ then $\varphi^* \omega$ is overconvergent with radius of convergence $\alpha \varepsilon$.

Proof. We set $Y_j = [U_j]$ and $X_i = [T_i]$. From Lemma 2.23 in [6] we obtain
an expansion:
\[
\varphi^*(X_i) = [Q_i(U_1, \ldots, U_l)] = \sum_{|k| < c} a_{ik} Y^k,
\]
where $a_{ik} \in W(R)$. More generally we obtain for a monomial
$X^l = X_1^{l_1} \cdots X_d^{l_d}$, $l_i \in \mathbb{Z}_{\geq 0}$ an expansion:
\[
\varphi^*(X^l) = \sum_{|k| < c|l|} b_k Y^k, \quad b_k \in W(R).
\]

Since $\varphi^*$ commutes with the action of $V$ we find for $l$ not necessarily integral
\[
\varphi^*(V^u \eta X^l) = V^u \left( \varphi^* \left( \eta X^{l \cdot p^u} \right) \right) =
= \left( \sum_{k' \leq c|l|} \eta \cdot b_{k'} \cdot Y^{k'} \right) = \sum_{|k| < c|l|} V^u(\eta b_{k'}) \cdot Y^k.
\]

From this we see immediately the following fact: Let $\omega \in W\Omega_R[T_1, \ldots, T_d] / R$
be a Witt differential which is homogeneous of degree $l$, and such that $\text{ord}_V \omega = m$. Then $\varphi^* \omega$ is a $V-$convergent sum $\sum \eta_k$ of homogeneous Witt
differentials of degree $|k| < c|l|$ and such that $\text{ord}_V \eta_k \geq m$. Assume that
$\omega = \sum \omega_l$ is a sum of homogeneous differentials such that
\[
\text{ord}_V \omega_l - \varepsilon |l| \geq D.
\]
Then $\varphi^* \omega_l = \sum \eta_{l,k}$, where $\eta_{l,k}$ is homogeneous of degree $k$, such that $|k| \leq c|l|$ and $\text{ord}_V \eta_{l,k} \geq m$. Therefore for $\delta > 0$,
\[
\text{ord}_V \eta_{l,k} - \delta |k| \geq m - \delta c|l|.
\]
If $\delta < \frac{\varepsilon}{c}$ the last expression is bounded below by $D$. This proves the proposition with $\alpha = 1/c$. \qed

By the proposition we obtain a map:
\[
W^{\varepsilon} \Omega_R[T_1, \ldots, T_d] / R \to W^{\alpha \varepsilon} \Omega_R[U_1, \ldots, U_l] / R.
\]
Proposition 0.9. Let \( \varphi : R[T_1, \ldots, T_d] \to R[U_1, \ldots, U_l] \) be an \( R \)-algebra homomorphism. Then the induced map
\[
\varphi^* : W^1\Omega_{R[T_1, \ldots, T_d]/R} \to W^1\Omega_{R[U_1, \ldots, U_l]/R}
\]
maps \( W^1\Omega_{R[T_1, \ldots, T_d]/R} \) to \( W^1\Omega_{R[U_1, \ldots, U_l]/R} \).

If, moreover, \( \varphi \) is surjective then
\[
W^1\Omega_{R[T_1, \ldots, T_d]/R} \to W^1\Omega_{R[U_1, \ldots, U_l]/R}
\]
is surjective too.

Proof. Only the last statement needs a verification. If \( \varphi \) is surjective we find a homomorphism
\[
\psi : R[U_1, \ldots, U_l] \to R[T_1, \ldots, T_d],
\]
such that \( \varphi \circ \psi = id \). Then for \( \eta \in W^1\Omega_{R[U_1, \ldots, U_l]/R} \), \( \psi \eta \) is overconvergent and therefore a preimage of \( \eta \).

We have seen that \( \gamma_c \) fails to be a pseudovaluation on the ring \( W\Omega_{A/R} \). However we will face a situation where we will need an inequality
\[
\gamma_c(f \omega) \geq \gamma_c(f) + \gamma_c(\omega)
\]
for certain \( f \in W(A) \) and \( \omega \in W\Omega_{A/R} \). For suitable \( f \) and overconvergent \( \omega \) we can even achieve equality.

From now on, let \( R = k \) be a perfect field. Let \( A = k[T_1, \ldots, T_d] \) be the polynomial ring. The Teichmüller of \( T_i \) in \( W(A) \) is denoted by \( X_i \). For a Witt differential \( \omega \in W\Omega_{A/k} \) we define:
\[
\nu_p(\omega) = \max\{a \in \mathbb{Z} \mid p^{-a} \omega \in W\Omega_{A/k}\}.
\]

Obviously we have that
\[
\nu_p(\omega_1 \omega_2) \geq \nu_p(\omega_1) + \nu_p(\omega_2)
\]
for arbitrary Witt differentials.

Let \( \omega = e(\xi, k, P) \) be a basic Witt differential. Let \( p^u \) be the denominator of the weight \( k \). Then we have:
\[
\text{ord}_V \omega = \text{ord}_V \xi = \nu_p(\omega) + u.
\]

For an arbitrary \( \omega \in W\Omega_{A/k} \) we write the expansion:
\[
\omega = \sum_{k, P} e(\xi_{k, P}, k, P).
\]

Let \( \varepsilon > 0 \). We have the Gauss norm \( \gamma_c \):
\[
\gamma_c(\omega) = \inf_{k, P} \{\text{ord}_V(e(\xi_{k, P}, k, P)) - \varepsilon |k|\}.
\]

We also define the modified Gauss norm:
\[
\gamma^*_c(\omega) = \inf_{k, P} \{\nu_p(e(\xi_{k, P}, k, P)) - \varepsilon |k|\}.
\]

We note that:
\[
\gamma_c(\omega) \geq \gamma^*_c(\omega).
\]
Consider the polynomial algebra $\hat{A} = W(k)[X_1, \ldots, X_d]$. For each real number $\varepsilon > 0$ we define on $\hat{A}$ a valuation $\gamma_\varepsilon$. We write $f \in \hat{A}$. We will use the vector notation $I = (i_1, \ldots, i_d)$ and write
\[
f = \sum c_I X^I, \quad c_I \in W(k).
\]
We write $|I| = i_1 + \ldots + i_d$. Then we set
\[
\gamma_\varepsilon(f) = \min \{\text{ord}_p(c_I) - \varepsilon |I|\}.
\]
We extend $\gamma_\varepsilon$ to the differential forms $\Omega_{\hat{A}/W(k)}$. We write a differential form as of degree $r$:
\[
\omega = \sum \alpha f_\alpha dX_{\alpha_1} \wedge \ldots \wedge dX_{\alpha_r}, \quad f_\alpha \in \hat{A},
\]
where $\alpha = (\alpha_1, \ldots, \alpha_r)$ runs over vectors with $1 \leq \alpha_1 < \ldots < \alpha_r \leq d$. Then we set:
\[
\gamma_\varepsilon(\omega) = \min_\alpha \{\gamma_\varepsilon(f_\alpha) - r\varepsilon\}.
\]
We have the following properties:
\[
(0.12) \quad \gamma_\varepsilon(f \omega) = \gamma_\varepsilon(f) + \gamma_\varepsilon(\omega), \quad f \in \hat{A}
\]
\[
\gamma_\varepsilon(\omega_1 \wedge \omega_2) \geq \gamma_\varepsilon(\omega_1) + \gamma_\varepsilon(\omega_2), \quad \omega_i \in \Omega_{\hat{A}/W(k)}.
\]
We may write $\omega$ as a sum of $p$-basic elements [12] (2.3):
\[
e(c, k, P) = cX^{k_{i_0}} \frac{dX^{k_{i_1}}}{p^{\text{ord}_p k_{i_1}}} \cdots \frac{dX^{k_{i_l}}}{p^{\text{ord}_p k_{i_l}}}.
\]
\[
\text{Lemma 0.13. Let us write } \omega \in \Omega_{\hat{A}/W(k)} \text{ as a sum of } p\text{-basic differentials:}
\]
\[
\omega = \sum e(c_k, p, k, P).
\]
Then we have:
\[
\gamma_\varepsilon(\omega) = \min_\alpha \{\text{ord}_p(c_k, p) - |k|\varepsilon\}.
\]
\[
\text{Proof. Clearly it is sufficient to consider the case where } \omega \text{ belongs to the free } W(k)\text{-module of forms of a given weight } k \text{ (compare [12] proof of Prop. 2.1). Then } \omega \text{ may be written:}
\]
\[
\omega = \sum b_{i_1 \ldots i_l} X_1^{k_1} \cdots X_n^{k_n} d\log X_{i_1} \wedge \ldots \wedge d\log X_{i_l}.
\]
The result follows because $b_{i_1 \ldots i_l}$ and $c_k, P$ are related by a unimodular matrix with coefficients in $Z_p$. [12] 2.1.

Consider the natural map $\hat{A} \to W(A)$ which sends $X_i$ to the Teichmüller representative $[T_i]$. It induces a map:
\[
(0.14) \quad \Omega_{\hat{A}/W(k)} \to W\Omega_{A/k}.
\]
The $p$-adic completion of the image of this map consists of the integral Witt differentials. From Lemma 0.13 we obtain:

\[
\text{Proposition 0.15. The map (0.14) is compatible with the Gauss norms } \gamma_\varepsilon \text{ on both sides.}
\]
Corollary 0.16. Let \( \omega, \eta \in W\Omega_{A/k} \). Then we have:
\[
\gamma_{\varepsilon}(\omega \eta) \geq \gamma_{\varepsilon}(\omega) + \gamma_{\varepsilon}(\eta) \quad \text{for } \omega \text{ integral}
\]
\[
\gamma_{\varepsilon}(\omega \eta) \geq \gamma_{\varepsilon}(\omega) + \gamma_{\varepsilon}(\eta) \quad \text{for } \omega \text{ arbitrary.}
\]
We note that for \( \omega \) integral, \( \gamma_{\varepsilon}(\omega) = \bar{\gamma}_{\varepsilon}(\omega) \). Let \( f \in A \), then we have \( \bar{\gamma}_{\varepsilon}([f]) = \gamma_{\varepsilon}([f]) \). In particular we find for arbitrary \( \omega \)
\[(0.17) \quad \gamma_{\varepsilon}([f] \omega) \geq \gamma_{\varepsilon}([f]) + \gamma_{\varepsilon}(\omega).
\]
Proof. We begin with the first inequality. If \( \eta \) is integral too, we can apply (0.12). For the general case we may assume that \( \eta = V^n \tau \) or \( \eta = d^V_v \tau \) where \( \tau \) is a primitive basic Witt differential. We note that for primitive \( \tau \):
\[
\gamma_{\varepsilon}(V^n \tau) = u + \gamma_{\varepsilon/p^n}(\tau).
\]
For integral \( \omega \) we have
\[
\gamma_{\varepsilon/p^n}(F^w \omega) = \gamma_{\varepsilon}(\omega).
\]
If \( \omega \) is not integral we have only the inequality:
\[
\gamma_{\varepsilon/p^n}(F^w \omega) \geq \gamma_{\varepsilon}(\omega) - u.
\]
Then we find using the integral case:
\[
\gamma_{\varepsilon}(\omega V^n \tau) = \gamma_{\varepsilon}((F^w \omega) V^n \tau) \geq u + \gamma_{\varepsilon/p^n}(F^w \omega V^n \tau) \geq u + \gamma_{\varepsilon/p^n}(F^w \omega) + \gamma_{\varepsilon/p^n}(\tau) = \gamma_{\varepsilon}(V^d \tau) + \gamma_{\varepsilon/p^n}(F^w \omega) \geq \gamma_{\varepsilon}(V^u \tau) + \gamma_{\varepsilon}(\omega).
\]
The case \( \eta = d^V_v \tau \) is reduced to the former case by the Leibniz rule:
\[
\omega d^V_v \tau = d(\omega V^n \tau) - (d\omega)^V_v \tau.
\]
Now we verify the second inequality. We may assume that \( \omega = V^n \tau \) or \( \omega = d^V_v \tau \) for a primitive basic Witt differential. Then we have:
\[
\bar{\gamma}_{\varepsilon}(\omega) = \gamma_{\varepsilon/p^n}(\tau), \quad \text{and}
\]
\[
\gamma_{\varepsilon}(V^n \tau \eta) = \gamma_{\varepsilon}(V^n (\tau F^u \eta)) \geq u + \gamma_{\varepsilon/p^n}(\tau F^u \eta) \geq u + \gamma_{\varepsilon/p^n}(\tau) + \gamma_{\varepsilon/p^n}(F^u \eta) = \bar{\gamma}_{\varepsilon}(\omega) + u + \gamma_{\varepsilon/p^n}(F^u \eta) \geq \bar{\gamma}_{\varepsilon}(\omega) + \gamma_{\varepsilon}(\eta).
\]
Finally we have to show that \( \gamma_{\varepsilon}([f]) = \bar{\gamma}_{\varepsilon}([f]) \). We denote by \( m = (m_1, \ldots, m_d) \) a vector of non negative integers and write:
\[(0.18) \quad f = \sum_m a_m T_1^{m_1} \cdots T_d^{m_d} = \sum_m a_m \mathcal{T}^m.
\]
Let \( g \) be the total degree of \( f \). Then we have
\[
\gamma_{\varepsilon}([f]) = -\varepsilon g.
\]
We enumerate the \( m \) with \( a_m \neq 0 \):
\[
m(1), \ldots, m(t).
\]
By Lemma 2.23 in [6] we find:
\[
[f] = \sum_{k_1+\ldots+k_t = 1} a_{k_1, \ldots, k_t} [\mathcal{T}^{m(1)k_1+\ldots+m(tk_t)}.
\]
If we take \( \bar{\gamma}_{\varepsilon} \) of one summand it is bigger than the degree of this summand times \( -\varepsilon \):
\[
\bar{\gamma}_{\varepsilon}(a_{k_1, \ldots, k_t} [\mathcal{T}^{m(1)k_1+\ldots+m(tk_t)} \geq \varepsilon(|m(1)|k_1 + \ldots + |m(t)|k_t)
\]
\[
\geq \varepsilon(gk_1 + \ldots + gk_t) = -\varepsilon g.
\]
Proposition 0.19. Let \( f \in W(A) \), \( f = (f_0, f_1, \ldots) \) be a Witt vector, such that \( f_0 \neq 0 \). Let \( \omega \in W\Omega_{A/k} \) be an element, whose decomposition into basic Witt differentials has the following form:

\[
\omega = \sum e(\xi_{k,\mathcal{P}}, k, \mathcal{P}).
\]

(0.20)

We assume that all weights \( k \) appearing in this decomposition have the same denominator \( p^u \) with \( u \geq 0 \), and the same degree \( \kappa = |k| \). Moreover we assume that only partitions \( \mathcal{P} \) with \( I_0 \neq \emptyset \) appear and that there is a weight \( k \) and a partition \( \mathcal{P} \) such that \( \text{ord}_V \xi_{k,\mathcal{P}} = u \). The last condition says that there is \( k \) and \( \mathcal{P} \) such that \( e(\xi_{k,\mathcal{P}}, k, \mathcal{P}) = V^\nu \tau \), for a primitive basic Witt differential \( \tau \).

We can write \( f\omega \) as a sum of basic Witt differentials:

\[
f\omega = \sum e(\xi'_{h,\mathcal{P}}, h, \mathcal{P}).
\]

(0.21)

Then there is a summand \( e(\xi'_{h,\mathcal{P}}, h, \mathcal{P}) \) such that \( \text{ord}_V(\xi'_{h,\mathcal{P}}) = u \), such that \( h \) has denominator \( p^u \), and such that \( I_0 \neq \emptyset \). Moreover if \( g \) is the degree of the polynomial \( f_0 \), then the degree of \( h \) is \( |h| = g + \kappa \).

In particular we have the inequality:

\[
\gamma(\omega) \leq \gamma(\omega) - \epsilon \deg f_0.
\]

Proof. We write:

\[
f = \tilde{f} + V^\rho,
\]

where \( \tilde{f} \) is a polynomial in \( X_1 = [T_1], \ldots, X_d = [T_d] \) with coefficients in \( W(k) \), which are not divisible by \( p \). The degrees of the polynomials \( f_0 \) and \( \tilde{f} \) are the same.

We set \( \omega = V^u \tau \), where \( \tau \) is an integral Witt differential with \( \nu_0(\tau) = 1 \).

Then we have:

\[
f\omega = (\tilde{f} + V^\rho)V^u \tau = V^u(\tilde{f} + p^{\nu - 1} \rho \tau).
\]

(0.22)

We write \( \tilde{f} = \sum_i \tilde{f}_i \) as a sum of homogeneous polynomials of different degree \( g_i \). The maximum of the \( g_i \) is \( g \). Then the Witt differential \( \eta_i = F^{g_i} \tilde{f}_i \tau \) is for each \( i \) an integral homogeneous Witt differential of degree \( p^u g_i + p^u \kappa \).

By assumption the reduction of this Witt polynomial in \( \Omega_{A/k} \) is not closed. The basic Witt differentials which appear in the decomposition of \( \eta_i \) have weights which are not divisible by \( p \), because the weights appearing in \( F^{g_i} \tilde{f}_i \) are divisible by \( p \) but those appearing in \( \tau \) are not divisible by \( p \). This shows that primitive basic Witt differentials appear in the decomposition of each \( \eta_i \). These can’t be destroyed by basic Witt differentials which appear in the decomposition of the last summand in the brackets of (0.22), because of the factor \( p \). If we apply \( V^u \) we obtain the desired basic Witt differential in the decomposition of \( f\omega \). \( \square \)

Corollary 0.23. With the notations of the proposition consider a Witt differential of the form \( \omega_1 = \omega + d\eta \), and write

\[
f^p \omega_1 = \sum e(\xi_{h,\mathcal{P}}, h, \mathcal{P}).
\]
Then there is a summand $e(\hat{\xi}_{h, P}, h, P)$ in the above sum, such that $\text{ord}_F \hat{\xi}_{h, P} = u$, such that $h$ has denominator $p^u$ and such that $I_0 \neq \emptyset$. Moreover the degree of $h$ is $|h| = pg + \kappa$.

**Proposition 0.24.** Let $f_0 \in A = k[T_1, \ldots, T_d]$ be a polynomial of degree $g$. Let $\omega \in W\Omega_{L/k}$. Then we have for the Gauss norm on $A$:

\[(0.25)\quad \gamma_e([f_0] \omega) = \gamma_e([f_0]) + \gamma_e(\omega).\]

**Proof.** We write $\omega$ as a sum of basic Witt differentials:

\[(0.26)\quad \omega = \sum_{i \in I} e_i.\]

By continuity we may assume that the sum is finite. By Corollary 0.16 we have the inequality:

\[(0.27)\quad \gamma_e([f_0] \omega) \geq \gamma_e([f_0]) + \gamma_e(\omega).\]

We may therefore assume that in the sum (0.26)

\[(0.28)\quad \gamma_e(\omega) = \gamma_e(e_i)\]

for all $i \in I$. We may further assume that $\nu_p(\omega) = 0$.

Let us first consider the case where there is an integral basic Witt differential $e_{i_0}$ in the sum (0.26) such that $\nu_p(e_{i_0}) = 0$. Then we decompose $\omega$ into three parts:

$$\omega = \eta + \omega' + \omega'',$$

where $\eta$ is the sum of those Witt differentials $e_i$ in (0.26) which are integral and such that $\nu_p(e_i) = 0$, where $\omega'$ is the sum of those Witt differentials $e_i$ in (0.26) which are integral and such that $\nu_p(e_i) > 0$, and where $\omega''$ is the sum of those Witt differentials in (0.26) which are not integral.

Let $e_i$ be a summand in $\eta$ and let $\kappa$ be its degree. By assumption we find:

$$\gamma_e(\omega) = \gamma_e(e_i) = \nu_p(e_i) - \varepsilon \kappa = - \varepsilon \kappa.$$

It follows that all these $e_i$ have the same degree $\kappa$.

Consider the differential $f_0 \eta \in \Omega_{A/k}$ which is the reduction of $[f_0] \eta$. If we write the reduction as a sum of basic differentials in $\Omega_{A/k}$ it must clearly contain a basic Witt differential of degree $g + \kappa$. In the decomposition of $[f_0] \eta$ appears therefore an integral basic Witt differential $\tilde{e}$ of degree $g + \kappa$ such that $\nu_p(\tilde{e}) = 0$. On the other hand all basic Witt differentials which appear in the decomposition of $[f_0](\omega' + \omega'') \in V W \Omega_{A/k} + d V W \Omega_{A/k}$ are either integral with $\nu_p > 0$ or nonintegral. Therefore they can’t destroy completely $\tilde{e}$. We found in the decomposition of $[f_0] \omega$ an integral basic Witt differential $\tilde{e}'$ of degree $g + \kappa$, such that $\nu_p(\tilde{e}') = 0$. We conclude that

$$\gamma_e([f_0] \omega) \leq \gamma_e(\tilde{e}') = - \varepsilon (g + \kappa) = \gamma_e([f_0]) + \gamma_e(\omega).$$

Since we know the opposite inequality we obtain the equation (0.25) in the first case.

Let $\omega$ be a Witt differential which doesn’t belong to the first case. Then we write:

\[(0.29)\quad \omega = \omega(u) + \omega(du) + \omega' + \omega''\]
where $\omega'$ is the sum of all $e_i$ in (0.26), such that $\nu_p(e_i) > 0$. There is a natural number $u$ such that the following holds:

$$\omega'' = V^{u+1}W\Omega_A/k + dV^{u+1}W\Omega_A/k$$

and each basic Witt differential appearing in the decomposition of $\omega(u)$ is of the form $V^u\tau$ for a primitive basic Witt differential $\tau$ and any basic Witt differential which appears in $\omega(du)$ is of the form $dV^u\tau$. By our assumption (0.28) we find that for each of these $\tau$:

$$\gamma_\epsilon(\omega) = u + \gamma_{\epsilon/p^n}(\tau) = u - \epsilon\kappa,$$

where $\kappa$ is obviously independent of $\tau$.

Before proceeding we make a general remark: It suffices to show the equality (0.25) in the case where $\kappa > 0$. As before we see that the basic Witt differential $\omega''$ contains a non-closed basic Witt differential $\varepsilon(\omega + \varepsilon(g + \kappa))$ and $\gamma_\epsilon(\omega + \varepsilon(g + \kappa)) = \gamma_\epsilon(\omega)$.

We conclude:

$$\gamma_\epsilon([f_0]) \gamma_\epsilon(\omega) \geq \gamma_\epsilon([f_0]) \omega).$$

Since we already know the opposite the inequality (0.25) follows.

We consider now the second case where $\omega(u) \neq 0$. By Proposition 0.19 the product $[f_0] \omega(u)$ contains a basic Witt differential $e(\xi, k, P)$, where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. This basic Witt differential can’t be destroyed by any basic Witt differential appearing in $[f_0] \omega'$, because $\nu_p > 0$, or by any basic Witt differential appearing in $[f_0] \omega''$, because those have reduction 0 in $W_u\Omega_A/k$. It can also not cancel with an exact basic Witt differential appearing in $[f_0] \omega(du)$. Indeed since $f_0$ is a $p$-th power those basic Witt differentials are either exact or have $\nu_p > 0$. Therefore $[f_0] \omega$ contains as a summand a basic Witt differential $e(\eta, k, P)$ where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. This proves the inequality:

$$\gamma_\epsilon([f_0] \omega) \leq u - \varepsilon(g + \kappa) = \gamma_\epsilon([f_0]) + \gamma_\epsilon(\omega).$$

This gives the desired equality in the second case.

Let us now consider the third and last case, where $\omega(u) = 0$ in (0.29). Then we rewrite (0.29) in the form:

$$\omega = dV^u\sigma + \omega' + \omega''$$

where $\sigma$ is a sum of primitive basic Witt differentials of the same degree $p^n\kappa$, where $\gamma_\epsilon(\omega) = u - \epsilon\kappa$. We assume as above that $f_0 = g_0^p$. We find:

$$[f_0]dV^u\sigma = d([g_0]^{pV^u}\sigma) - p[h_0]^{p-1}(d[h_0])V^u\sigma.$$

By Proposition 0.19 we know that $[h_0]^{pV^u}\sigma$ contains a non-closed basic Witt differential $e(\xi, k, P)$, where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. As before we see that the basic Witt differential $de(\xi, k, P)$ can’t be destroyed by any basic Witt differential which appears in $[f_0] \omega'$ or $[f_0] \omega''$. It can’t also be destroyed by a basic Witt differential

$$\gamma_\epsilon([f_0]) \gamma_\epsilon(\omega) \geq \gamma_\epsilon([f_0]) \omega).$$

Since we already know the opposite the inequality (0.25) follows.

We conclude:

$$\gamma_\epsilon([f_0]) \gamma_\epsilon(\omega) \geq \gamma_\epsilon([f_0]) \omega).$$

Since we already know the opposite the inequality (0.25) follows.

We consider now the second case where $\omega(u) \neq 0$. By Proposition 0.19 the product $[f_0] \omega(u)$ contains a basic Witt differential $e(\xi, k, P)$, where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. This basic Witt differential can’t be destroyed by any basic Witt differential appearing in $[f_0] \omega'$, because $\nu_p > 0$, or by any basic Witt differential appearing in $[f_0] \omega''$, because those have reduction 0 in $W_u\Omega_A/k$. It can also not cancel with an exact basic Witt differential appearing in $[f_0] \omega(du)$. Indeed since $f_0$ is a $p$-th power those basic Witt differentials are either exact or have $\nu_p > 0$. Therefore $[f_0] \omega$ contains as a summand a basic Witt differential $e(\eta, k, P)$ where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. This proves the inequality:

$$\gamma_\epsilon([f_0] \omega) \leq u - \varepsilon(g + \kappa) = \gamma_\epsilon([f_0]) + \gamma_\epsilon(\omega).$$

This gives the desired equality in the second case.

Let us now consider the third and last case, where $\omega(u) = 0$ in (0.29). Then we rewrite (0.29) in the form:

$$\omega = dV^u\sigma + \omega' + \omega''$$

where $\sigma$ is a sum of primitive basic Witt differentials of the same degree $p^n\kappa$, where $\gamma_\epsilon(\omega) = u - \epsilon\kappa$. We assume as above that $f_0 = g_0^p$. We find:

$$[f_0]dV^u\sigma = d([g_0]^{pV^u}\sigma) - p[h_0]^{p-1}(d[h_0])V^u\sigma.$$

By Proposition 0.19 we know that $[h_0]^{pV^u}\sigma$ contains a non-closed basic Witt differential $e(\xi, k, P)$, where $k$ is a weight of denominator $u > 0$, such that $|k| = g + \kappa$ and ord$_V\xi = u$. As before we see that the basic Witt differential $de(\xi, k, P)$ can’t be destroyed by any basic Witt differential which appears in $[f_0] \omega'$ or $[f_0] \omega''$. It can’t also be destroyed by a basic Witt differential
which appears in the last summand of (0.31), because for them $\nu_p$ is positive. From this we conclude as before the desired equality (0.25).

**Corollary 0.32.** Let $\tilde{f} \in W(k[T_1, \ldots, T_d]) = W(A)$ be an integral Witt vector with radius of convergence $\varepsilon$. Let $\omega \in W\Omega_{A/k}$ be an arbitrary Witt differential of radius of convergence $\varepsilon$. Then we have:

$$\gamma_{\varepsilon}(\tilde{f}\omega) = \gamma_{\varepsilon}(\tilde{f}) + \gamma_{\varepsilon}(\omega).$$

**Proof.** By Corollary 0.16 we have the inequality:

$$\gamma_{\varepsilon}(\tilde{f}\omega) \geq \gamma_{\varepsilon}(\tilde{f}) + \gamma_{\varepsilon}(\omega).$$

For the opposite inequality we may assume that $\tilde{f}$ is a polynomial by considering the truncations in $W_n\Omega_{A/k}$. We write $\tilde{f} = \sum_i \tilde{f}_i$ as a sum of homogeneous polynomials $\tilde{f}_i$ of different degrees $g_i$. By the inequality (0.33) we may assume that $\gamma_{\varepsilon}(\tilde{f}) = \gamma_{\varepsilon}(\tilde{f}_i)$ for each $i$. Moreover we may clearly assume that $\nu_p(\tilde{f}) = 0$. With these remarks the proof works in the same way as above. □

### 1. Sheaf properties of the overconvergent de Rham-Witt complex

Let $A = k[t_1, \ldots, t_r]$ be a smooth finitely generated $k$-algebra, $S = k[T_1, \ldots, T_r]$ a polynomial algebra. Then $S \to A$, $T_i \to t_i$ induces a canonical epimorphism

$$\lambda : W\Omega^\bullet_{A/k} \to W\Omega^\bullet_{A/k}$$

of de Rham-Witt complexes.

**Definition 1.1.** We set $W^!\Omega^\bullet_{A/k} = \text{image} \left(W^!\Omega^\bullet_{S/k}\right)$ under $\lambda$.

We have seen in Proposition 0.9 that this definition is independent from the choice of generators and the representation $S \to A$. The same proposition shows that the assignment $A \mapsto W^!\Omega_{A/k}$ is functorial. Indeed, given smooth finitely generated $k$-algebras $A, B$ as above, and a presentation $k[T_1, \ldots, T_r] \to A$, we extend this to a presentation $k[T_1, \ldots, T_r, U_1, \ldots, U_l] \to B$ such that the following diagram commutes:

$$
\begin{array}{c}
A \\
\downarrow \quad \downarrow \\
B \\
\end{array}
\quad
\begin{array}{c}
k[T_1, \ldots, T_r] \\
\quad
k[T_1, \ldots, T_r, U_1, \ldots, U_l].
\end{array}
$$

Then it is clear that the induced map $W\Omega_{A/k} \to W\Omega_{B/k}$ sends $W^!\Omega_{A/k} \to W^!\Omega_{B/k}$.

For $\omega \in W\Omega_{A/k}$ a convergent sum of images of basic Witt differentials in $W\Omega^\bullet_{S/k}$, so

$$\omega = \sum_{(k,p)} e(\xi_{k,p}, k, p),$$

where $e(\xi_{k,p})$ is the exponential part, and $\xi_{k,p}$ is a Witt vector of radius of convergence $\varepsilon$.
we know that \( \omega \) is overconvergent iff there exist constants \( C_1 > 0, C_2 \in \mathbb{R} \) such that
\[
|k| \leq C_1 \text{ord}_p \xi, p + C_2 \quad \text{for all } (k, \mathcal{P}).
\]
We can also express overconvergence on \( W \Omega^\bullet_{A/k} \) by using the Gauss norms \( \{ \omega_\varepsilon \}_{\varepsilon > 0} \) obtained as quotient norms of the canonical Gauss norms on \( W \Omega^\bullet_{S/k} \) that we defined before. An \( \omega \in W \Omega^\bullet_{A/k} \) is overconvergent if there exist \( \varepsilon > 0, C \in \mathbb{R} \) such that \( \gamma_\varepsilon(\omega) \geq C \). If we use another presentation \( S' = k[U_1, \ldots, U_r] \rightarrow A \), then the associated set of quotient norms \( \{ \delta_\varepsilon \}_{\varepsilon > 0} \) on \( W \Omega^\bullet_{A/k} \) is equivalent to the set \( \{ \omega_\varepsilon \}_{\varepsilon > 0} \). Here, the notion of equivalence is defined in the same way as for Witt vectors ([6] Definition 2.12).

**Proposition 1.2.** (a) We denote by \( f \in A \) an arbitrary element. Let \( d \in \mathbb{Z} \) be nonnegative. The presheaf
\[
W^d \Omega^\bullet_{\text{Spec } A/k}(\text{Spec } A_f) := W^d \Omega^\bullet_{A_f/k}
\]
is a sheaf for the Zariski topology on \( \text{Spec } A \) (compare [9] 0, 3.2.2).

(b) The Zariski cohomology of these sheaves vanishes in degrees \( j > 0 \), i.e.
\[
H^j_{\text{Zar}}(\text{Spec } A, W^d \Omega^\bullet_{\text{Spec } A/k}) = 0.
\]

We fix generators \( t_1, \ldots, t_r \) of \( A \) and denote by \( [t_1], \ldots, [t_r] \) the Teichmüller representatives in \( W(A) \). An elementary Witt differential in the variables \( [t_1], \ldots, [t_r] \) is the image of a basic Witt differential in variables \( [T_1], \ldots, [T_r] \) under the map \( \lambda \).

Before we prove the proposition, we need a special description of an overconvergent element \( z \) in \( W^d \Omega^\bullet_{A_f/k} \). Let \( [f] \in W(A) \) be the Teichmüller representative. Hence \( \frac{1}{[f]} = \begin{pmatrix} 1 & \vdots & 1 \end{pmatrix} \) is the Teichmüller of \( \frac{1}{f} \) in \( W(A_f) \). For the element \( z \) we have the following description.

**Proposition 1.3.** The element \( z \in W^d \Omega^\bullet_{A_f/k} \) can be written as a convergent series
\[
z = \sum_{l=0}^{\infty} \frac{1}{[f]^r} \eta_l
\]
where \( \eta_l \) is a finite sum of elementary Witt differentials \( \eta_l^{(t)} \) in the variables \([t_1], \ldots, [t_r]); \) images of basic Witt differentials \( \eta_l^{(t)} \) in variables \([T_1], \ldots, [T_r]) \) with weights \( k_l^r \) satisfying the following growth condition:
\[
\exists C_1 > 0, C_2 \in \mathbb{R} \text{ such that for each summand } \eta_l^{(t)} \text{ we have } r_l + |k_l^r| \leq C_1 \text{ord}_p \eta_l^{(t)} + C_2.
\]
Furthermore we require that for a given \( K > 0 \),
\[
\min_l \text{ord}_p \eta_l^{(t)} > K \text{ for almost all } l.
\]

**Proof.** We use here an extended version of basic Witt differentials to the localized polynomial algebra \( k[T_1, \ldots, T_r, Y, Y^{-1}] \) (compare [10]). A basic Witt differential \( \alpha \) in \( W \Omega^\bullet_{k[T_1, \ldots, T_r, Y, Y^{-1}]/k} \) has one of the following shapes:

1) \( \alpha \) is a classical basic Witt differential in variables \([T_1], \ldots, [T_r], [Y]).
II) Let \( e(\xi, p, k, P) \) be a basic Witt differential in variables \([T_1], \ldots, [T_r]\). Then

\[
\begin{align*}
\text{II 1)} &\quad \alpha = e(\xi, p, k, P) d \log[Y] \\
\text{II 2)} &\quad \alpha = [Y]^{-r} e(\xi, p, k, P) \text{ for some } r > 0, r \in \mathbb{N} \\
\text{II 3)} &\quad \alpha = F^s d[Y]^{-r} e(\xi, p, k, P) \text{ for some } l > 0, p \nmid l, s \geq 0.
\end{align*}
\]

III) \( \alpha = V_u(\xi[Y]^{p^n k_Y} [T]^{p^n k_{I_0}}) d V_u(I_1) [T]^{p^n(I_1) k_{I_1}} \ldots F^{-l(I_d)} d[T]^{p^{-l(I_d)} k_{I_d}} \) (compare [12], (2.15)).

In particular, for each such \( \alpha \) we have a weight function \( k \) on variables \([T_1], \ldots, [T_r]\) with partition \( I_0 \cup \ldots \cup I_d = P, u > 0, k_Y \in \mathbb{Z}[1/p], u < 0, u(I_0) \leq u = \max\{u(I_0), u(k_Y)\} \) (notations as in [12]).

If \( I_0 = \emptyset \), we require \( u = \max\{u(I_1), u(k_Y)\} \).

IV) \( \alpha = d\alpha' \) when \( \alpha' \) is as in III).

It follows from loc. cit. that each \( \omega \in W \Omega^*_k[T_1, \ldots, T_r, Y, Y^{-1}]/k \) is in a unique way a convergent sum of basic Witt differentials. Here convergent is meant with respect to the canonical filtration on the de Rham-Witt complex.

It is straightforward to show that \( \omega \) is overconvergent iff there exists \( \tilde{C}_1 > 0, \tilde{C}_2 \in \mathbb{R} \), such that the basic Witt differentials \( \alpha \) appearing in the decomposition of \( \omega \) have the following properties.

- If \( \alpha \) of type I) or of type II 1) occurs as a summand in \( \omega \), we require

  \[ |k| \leq \tilde{C}_1 \text{ord}_p(\xi, p) + \tilde{C}_2. \]

- If \( \alpha \) of type II 2) or II 3) occurs as a summand in \( \omega \) then

  \[ r + |k| \leq \tilde{C}_1 \text{ord}_p(\xi, p) + \tilde{C}_2 \text{ (with } r = l \cdot p^s \text{ in case II 3).} \]

- If \( \alpha \) of type III) or IV), then

  \[ |k_Y| + \sum_{j=0}^{d} |k_{I_j}| \leq \tilde{C}_1 \text{ord}_p(V^u \xi) + \tilde{C}_2 \]

  (here, \( |k_Y| = -k_Y, |k_{I_j}| = \sum_{i \in I_j} k_i \)).

We have a surjective map of complexes:

\[
W^\dagger \Omega^*_k[T_1, \ldots, T_r, Y, Y^{-1}]/k \rightarrow W^\dagger \Omega^*_A/k.
\]

We may represent the \( z \) of the proposition as the image of an overconvergent \( \omega \), which is a sum of basic Witt differentials as described above. To obtain the representation of \( z \) in the proposition, we expand the images of the basic Witt differentials \( \alpha \) separately.
In case of condition III) we consider the first factor \( V^u \left( \xi[Y]^{p^k Y} [T]^{p^k l_0} \right) \).
For simplicity we assume \( l_0 = \emptyset \); this does not affect the following calculations. Let \( -k_Y = \frac{r}{p^u} \) and \( l \leq \frac{r}{p^u} < l + 1 \) for an integer \( l \). We have
\[
V^u \left( \xi[Y]^{p^k Y} \right) = V^u \left( \frac{1}{[Y]^r} \right) = V^u \left( \frac{1}{[Y]^l} \frac{1}{[Y]^{r-lp^u}} \right) = \frac{1}{[Y]^{l+1}} V^u \left( \xi[Y]^{(l+1)p^u-r} \right).
\]
Now consider the image of \( \alpha \) in \( \Omega^d_{A_f/k} \) where
\[
[Y] \rightarrow [f], \quad [Y^{-1}] \rightarrow [f^{-1}], \quad [T_i] \rightarrow [t_i].
\]
The factor \( \frac{1}{[Y]^{l+1}} V^u \left( \xi[Y]^{(l+1)p^u-r} \right) \) is mapped to \( \frac{1}{[f]^{l+1}} V^u \left( \xi[f]^{(l+1)p^u-r} \right) \).

Represent \( f \) as a polynomial of degree \( g \) in \( t_1, \ldots, t_r \). Then it is easy to see that the image of \( \alpha \) in \( \Omega^d_{A_f/k} \) is of the form \( \frac{1}{[f]^{l+1}} \widetilde{\eta} \) where \( \widetilde{\eta} \) is a (possibly infinite) sum of images of basic Witt differentials \( \tilde{\eta}^i \) in variables \( [T_1], \ldots, [T_r] \) with weights \( k^i \) satisfying
\[
|k^i| \leq g \left( l + 1 - \frac{r}{p^u} \right) + \sum_{j=0}^d |k_{I_j}|
\]
\[
\leq g + \sum_{j=0}^d |k_{I_j}|.
\]
The case \( d \alpha \) (type IV) is deduced from the case III by applying \( d \) to \( \alpha \) and the Leibniz rule to the image of \( d \alpha \) in \( \Omega^d_{A_f/k} \). So if the image of \( \alpha \) as above is \( \frac{1}{[f]^{l+1}} \widetilde{\eta} \) then the image of \( d \alpha \) is
\[
\frac{1}{[f]^{l+1}} d \widetilde{\eta} - \frac{1}{[f]^{l+2}} \cdot l d[f] \widetilde{\eta} = \frac{1}{[f]^{l+2}} \left( [f] d \widetilde{\eta} - l d[f] \widetilde{\eta} \right) = \frac{1}{[f]^{l+2}} \widetilde{\eta},
\]
where \( \widetilde{\eta} \) is a sum of images of basic Witt differentials \( \tilde{\eta}^i \) in variables \( [T_1], \ldots, [T_r] \) with weights \( k^i \) satisfying
\[
|k^i| \leq 2g + \sum_{j=0}^d |k_{I_j}|.
\]
We can also compute the images of \( \alpha \) in \( \Omega^d_{A_f/k} \) where \( \alpha \) is of type I or II and obtain again a representation
\[
\frac{1}{[f]^{r}} \widetilde{\eta} \text{ for } r \geq 0.
\]
These cases are easier and omitted.
Now we return to the original element $z \in W^1 \Omega^d_{A_f/k}$. We may write $z$ as a convergent sum

$$z = \sum_{m=0}^{\infty} \tilde{\omega}_m,$$

where $\tilde{\omega}_m$ is an elementary Witt differential being the image of a basic Witt differential $\alpha_m$ in $W \Omega_k[T_1, \ldots, T_r, Y, Y^{-1}]/k$ of type I, II, III or IV.

In all cases we have a representation

$$\tilde{\omega}_m = \frac{1}{[f]} \tilde{\eta}_m$$

where $\tilde{\eta}_m$ is the sum of images of basic Witt differentials $\eta^t_m$ in variables $[T_1], \ldots, [T_r]$ such that

$$r_m + |k^t_m| \leq C_1 \text{ord}_p(\eta^t_m) + C_2 + 2(g+1).$$

Now consider - for a given integer $N$ - the element $z$ modulo $\text{Fil}^N$, so the image $z(N)$ of $z$ in $W^1 \Omega^d_{A_f/k} = W^1 \Omega^d_{A/k} \bigotimes_{W^1_1(A)} W_N(A)$.

One then finds a lifting $z^{(N)}$ of $z(N)$ in $W^1 \Omega^d_{A_f/k}$ such that $z^{(N)} = \sum_{m=0}^{b(N)} \omega_m$ is a finite sum, i.e.

$$\omega_m = \frac{1}{[f]} \eta^t_m$$

where now $\eta^t_m$ is a finite sum of images of basic Witt differentials $\eta^t_m$ in variables $[T_1], \ldots, [T_r]$ satisfying the growth condition

$$r_m + |k^t_m| \leq C_1 \text{ord}_p(\eta^t_m) + C_2$$

with $C_1 := C_1', C_2 = C_2' + 2(g+1)$.

The elements $z^{(N)}$ can be chosen to be compatible for varying $N$ and we have $z = \lim z^{(N)}$. It is clear that the second condition of the lemma is also satisfied, this finishes the proof of Proposition 1.3.

\[ \square \]

**Remark.** It will later be convenient to express the assertion in Proposition 1.3 using Gauss norms. Let $\{\gamma_\varepsilon\}_{\varepsilon > 0}$ be the set of Gauss norms on $W^1 \Omega^d_{A/k}$ obtained as quotient norms from the canonical Gauss norms on $W \Omega^d_{S/k}$ using the presentation $S \to A$. Let $\{\delta_\varepsilon\}_{\varepsilon > 0}$ be the set of Gauss norms on $W^1 \Omega^d_{A_f/k}$ obtained as quotient norms using the presentation $S := k[T_1, \ldots, T_r, U] \to A_f, T_i \mapsto t_i, U \mapsto \frac{1}{f}$. We now define another set of Gauss norms as follows. For $\omega \in W^1 \Omega^d_{A_f/k}$ we consider the collection of all possible representations

$$\omega = \sum_{t \geq 0} [f]^{-t} \eta_t, \text{ for } \eta_t \in W \Omega^d_{A/k},$$

such that for a given $t$, almost all $\eta_t$ are zero in $W^{t+1} \Omega^d_{A/k}$. We set

$$\gamma_\varepsilon^{\text{quot}}(\omega) = \sup \{ \inf \{ \gamma_\varepsilon(\eta_t) - t \varepsilon \} \}$$

where the sup is taken over all possible representations $\omega$. Then Proposition 1.3 is equivalent to the assertion that the set $\{\gamma_\varepsilon^{\text{quot}}\}_{\varepsilon > 0}$ is equivalent.
to the set $\{\delta_r\}_{r>0}$. Equally, we will obtain an equivalent set of Gauss norms $\{\gamma'_r\}_{r>0}$ if in the above definition we only allow representations such that the exponents of $f$ are all divisible by $p$.

Now we are ready to prove Proposition 1.2.

As $W^*\Omega$ is a complex of Zariski sheaves we need to show--in order to prove part (a) of the proposition--the following claim:

Let $z \in W^d\Omega^d_{A/k}$ for some fixed $d$, let $\{f_i\}_i$ be a collection of finitely many elements in $A$ that generate $A$ as an ideal. Assume that for each $i$ the image $z_i$ of $z$ in $W^d\Omega^d_{A_{f_i}/k}$ is already in $W^\dagger\Omega^d_{A_{f_i}/k}$. Then $z \in W^\dagger\Omega^d_{A/k}$.

Let $[f_i]$ be the Teichmüller of $f_i$ with inverse $\frac{1}{[f_i]} = [\frac{1}{f_i}]$.

**Lemma 1.4.** There are elements $r_i \in W^\dagger(A)$ such that $\sum_{i=1}^n r_i[f_i] = 1$.

**Proof.** Consider a relation $\sum_{i=1}^n a_i f_i = 1$ in $A$. Then $\sum_{i=1}^n [a_i][f_i] = 1 + V\eta \in W^\dagger(A)$. By Lemma 2.25 in [6],

$$(1 + V\eta)^{-1} \in W^\dagger(A).$$

Define $r_i = (1 + V\eta)^{-1} \cdot [a_i]$. □

**Lemma 1.5.** For each $t$ there are polynomials $Q_{i,t}[T_1, \ldots, T_{2n}]$ in $2n$ variables such that

1. degree $Q_{i,t} \leq 3 \cdot nt$
2. $\sum_{i=1}^n Q_{i,t} ([f_1], \ldots, [f_n], r_1, \ldots, r_n) [f_i]^t = 1$.

For the proof of this lemma, compare [15].

We know that $Spec A = \cup_{i=1}^n D(f_i)$. For a tuple $1 \leq i_1 < \cdots < i_m \leq n$, let $U_{i_1, \ldots, i_m} = \cap_{j=1}^m D(f_{i_j})$. Fix $d \in \mathbb{N}$ and let

$$C^m = C^m(Spec A, W^\dagger\Omega^d_{A/k})$$

$$= \oplus_{1 \leq i_1 < \cdots < i_m \leq n} W^\dagger\Omega^d_{A_{f_{i_1} \cdots f_{i_m}}/k}$$

$$= \oplus_{1 \leq i_1 < \cdots < i_m \leq n} \Gamma(U_{i_1, \ldots, i_m}, W^\dagger\Omega^d_{A/k}).$$

Then consider the Čech complex

$$0 \to C^0 \to C^1 \to C^2 \to \cdots.$$  

We have $C^0 = W^\dagger\Omega^d_{A/k}$ and $C^0 \to C^1$ is the restriction map $W^\dagger\Omega^d_{A/k} \to W^\dagger\Omega^d_{A_{f_i}/k}$ for all $i$. It is then clear that Proposition 1.2 follows from the following.

**Proposition 1.6.** The complex $C^*$ is exact.

**Proof.** The proof is very similar to the proof of Lemma 7 in [15]. We fix as before $k$-algebra generators $t_1, \ldots, t_r$ of $A$. Suppose $\sigma \in C^m, m \geq 2$, is a cocycle. Then $\sigma$ has components

$$\sigma_{i_1 \cdots i_m} \in \Gamma(U_{i_1, \ldots, i_m}, W^\dagger\Omega^d_{Spec A/k}) = W^\dagger\Omega^d_{A_{f_{i_1} \cdots f_{i_m}}/k}.$$
Applying Proposition 1.3 we see that \( \sigma_{i_1, \ldots, i_m} \) has a representation as an overconvergent sum of Witt differentials as follows: \( \sigma_{i_1, \ldots, i_m} = \sum_{l=0}^{\infty} M^{i_1, \ldots, i_m}_l \) with
\[
M^{i_1, \ldots, i_m}_l = \sum_j \frac{1}{[f_{i_1, \ldots, i_m}]^j} \eta^{(j)}_{l1, \ldots, l_m} \text{ a finite sum}
\]
where \([f_{i_1, \ldots, i_m}]^j := [f_{i_1}]^j \cdots [f_{i_m}]^j\), \( \eta^{(j)}_{l1, \ldots, l_m} \) is a sum of images of basic Witt differentials \( \eta^{(j)}_{l1, \ldots, l_m} \) in variables \([T_1], \ldots, [T_r], (T_i \to t_i)\) and weights \( k^{(j)}_{l1, \ldots, l_m} \)

satisfying

1. \( j + |k^{(j)}_{l1, \ldots, l_m}| \leq C(\text{ord}_p \eta^{(j)}_{l1, \ldots, l_m} + 1) \)
2. \( l \geq \text{ord}_p \eta^{(j)}_{l1, \ldots, l_m} \geq l - 1. \)

Notation: We say that \( M^{i_1, \ldots, i_m}_l \) has degree \( \leq C(l + 1) \).

We shall construct a cochain \( \tau \) so that \( \partial \tau = \sigma \). The reduced complex
\[
C^* / \text{Fil}^n C^* = C^* \{ \{ D(f_i) \} \}, W^n \Omega^*_{A/k}
\]
is exact. We will inductively construct a sequence of cochains
\[
\tau_k = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq n} \tau_{ki_1, \ldots, i_{m-1}}
\]
such that the sum
\[
\sum_{k=0}^{\infty} \tau_k
\]
converges in \( C^{m-1} \) to a coboundary of \( \sigma \). The \( \tau_k \) are chosen to satisfy the following properties:

1. \( \partial(\sum_{k=0}^{l-1} \tau_k) = \sigma \) modulo \( \text{Fil}^{2l-1} C^m \)
2. \( \tau_{ki_1, \ldots, i_{m-1}} \in W^1 \Omega^*_{A_{f_{i_1} \cdots f_{i_m}/k}} \) and \( \tau_{ki_1, \ldots, i_{m-1}} \in W^1 \Omega^*_{A_{f_{i_1} \cdots f_{i_m}/k}} \) for \( k \geq 1. \)
3. \( \tau_{ki_1, \ldots, i_{m-1}} \in W^1 \Omega^*_{A/k} \left[ [f_1], \ldots, [f_n], r_1, \ldots, r_n, \frac{1}{[f_{i_1, \ldots, i_{m-1}]}] \right] \) to be understood as a polynomial in the “variables” \([f_1], \ldots, [f_n], r_1, \ldots, r_n\) and \( \frac{1}{[f_{i_1, \ldots, i_{m-1}]}] \) with the coefficients being finite sums of elementary Witt differentials in \([t_1], \ldots, [t_r]\) such that the total degree (with \([t_1], \ldots, [t_r]\) contributing to the degree via possibly fractional weights) is bounded by \( 24nC^2 k \). We write degree \( \tau_{ki_1, \ldots, i_{m-1}} \leq 24nC^2 k \).
4. \( [f_{i_1}] [W^{2k+1}] \tau_{ki_1, \ldots, i_{m-1}} \in W^1 \Omega^*_{A/k} \left[ [f_1], \ldots, [f_n], r_1, \ldots, r_n, \frac{1}{[f_{i_1, \ldots, i_{m-1}]}] \right] \)

with degree \( [f_{i_1}] [W^{2k+1}] \tau_{ki_1, \ldots, i_{m-1}} \leq C^{2k+1} + 24nC^2 k \).

Then (2) implies that all the coefficients \( \eta \) of the polynomial representation (3) satisfy \( \text{ord}_p \eta \geq 2k - 1 \). Also (1) implies that \( \partial(\sum_{k=0}^{\infty} \tau_k) = \sigma \). Using (2) and (3) we will show that \( \sum_{k=0}^{\infty} \tau_k \in C^{m-1} \), i.e. is overconvergent.

Define elements \( \sigma_{s_{i_1, \ldots, i_m}} \in W^1 \Omega^*_{A_{f_{i_1} \cdots f_{i_m}}/k} \) for \( n \geq 0 \) by
\[
\sigma_{s_{i_1, \ldots, i_m}} = \sum_{\alpha=0}^{2^{2k+1}-1} M^1_{i_1, \ldots, i_m}.
\]
Then \( \sigma_{s_{i_1, \ldots, i_m}} \equiv \sigma_{i_1, \ldots, i_m} \mod \text{Fil}^{2^{k+1}} \) and degree \( \sigma_{s_{i_1, \ldots, i_m}} \leq C^{2k+1} \).
Define the cochain $\tau_0 \in C^{m-1}$ by

$$\tau_{0i_1\ldots i_{m-1}} = \sum_{i=1}^{n} \alpha_{i} 2C[f_i]^{2C} \sigma_{0i_1\ldots i_m}.$$ 

Suppose we have constructed, for some integer $s > 0$, cochains $\tau_k \in C^{m-1}$ for $0 \leq k < s$ satisfying (1) – (4). Then we construct $\tau_s$ as follows: Let $\gamma_{s1\ldots i_m} = \sigma_{s1\ldots i_m} - \partial(\sum_{k=0}^{s-1} \gamma_k)_{i_1\ldots i_m}$. We see that $\gamma_{s1\ldots i_m} \in \text{Fil}^{2s-1} C^m$ is a cocycle modulo $\text{Fil}^{2s-1} C^m$ and degree $\gamma_{s1\ldots i_m} \leq 24nC^{2s-1}$.

Define

$$\tau_{si_1\ldots i_{m-1}} = \sum_{i=1}^{n} Q_{s,i,C}^{2s+1}[f_i]^{2s+1} \gamma_{si_1\ldots i_{m-1}}.$$ 

Then $\sum_{k=0}^{s} \gamma_k$ satisfies (1) by ([EGA], III.1.2.4.). We have

$$[f_i]^{C^2s+1} \gamma_{si_1\ldots i_{m-1}} \in W^{1} \Omega_{A_{i_1\ldots i_{m-1}}} \cap \text{Fil}^{2s-1} W^{1} \Omega_{A_{i_1\ldots i_{m-1}}/k}$$

$$= \text{Fil}^{2s-1} W^{1} \Omega_{A_{i_1\ldots i_{m-1}}}$$

and therefore $\tau_{si_1\ldots i_{m-1}}$ satisfies (2) (we have used (4) for $\tau_k, k < s$). Moreover, $\tau_{si_1\ldots i_{m-1}}$ has total degree bounded by

$$24nC^{2s-1} + 3nC^{2s+1} + C^{2s+1} \leq 24nC^s$$

and $\tau_s$ satisfies (3). It is straightforward to show property (4) for $\tau_s$. Therefore it remains to show that $\sum_{k=0}^{\infty} \tau_k$ is overconvergent. This will be derived from properties (2) and (3) as follows.

It follows from (3) that $\tau_{si_1\ldots i_{m-1}}$ can be written as a finite sum $\tau_{si_1\ldots i_{m-1}} = \sum_{I} r^I M_{s,I}$, where $I$ runs through a finite set of multi-indices in $\mathbb{N}_0^n$, $r^I = r^{\lambda_1} \cdots r^{\lambda_n}$ for $I = (\lambda_1, \ldots, \lambda_n)$ and $M_{s,I}$ is a finite sum of images of basic Witt differentials $\omega^I_s$ in variables $[T_1], \ldots, [T_r], [Y_1], \ldots, [Y_n], [Z]$ with

$$[T_j] \rightarrow [t_j], [Y_j] \rightarrow [f_j], [Z] \rightarrow \prod_{j=1}^{n-1} \frac{1}{|f_{ij}|}$$

with weights $k^I_s$ satisfying

$$|I| + |k^I_s| \leq 24nC^{2s} = C''C^s$$

($C'' := 24nC$) and

$$\text{ord}_p \omega^I_s \geq 2^s - 1 = \frac{1}{C'} (C''C^s) - 1$$

(*)

$$\geq \frac{1}{C'} (|I| + |k^I_s|) - 1.$$ 

For fixed $I$ and varying $s$ we get a sum

$$\sum_{s} r^I M_{s,I} = r^I \sum_{s} M_{s,I}.$$ 

Because of the condition (*), $\omega_I = \sum_{s} M_{s,I}$ is overconvergent with radius of convergence $\varepsilon = \frac{1}{C'}$ and

$$\gamma_{\omega_I} (\omega_I) \geq \frac{1}{C'} |I| - 1.$$
Here $\hat{\gamma}_\varepsilon$ is the quotient norm of the canonical $\gamma_\varepsilon$ on $W\Omega_k[T_1, \ldots, T_r, Y_1, \ldots, Y_n, Z]/k$.

We now look again at the definition of $r_i$. There exist liftings $\tilde{\eta}, \tilde{r}_i$ of $\eta, r_i$ in $W^1(S)$ and $\tilde{a}_i$ of $a_i$ in $S$ where $\tilde{\eta}$ is a finite sum of homogeneous elements such that

$$\tilde{r}_i = (1 + V_{\tilde{\eta}})^{-1}[\tilde{a}_i].$$

For $\delta := \frac{1}{C^r}$, there exists $\varepsilon > 0$, $\frac{1}{C^r} > \varepsilon$ such that

$$\hat{\gamma}_\varepsilon(V_{\tilde{\eta}}) \geq -\delta,$$

because we have a finite sum of homogeneous elements. By [6] Lemma 2.25, $\hat{\gamma}_\varepsilon(\tilde{r}_i) \geq -\delta$ as well.

Let $\tilde{\omega}_I$ be a lifting of $\omega_I$ in $W^1\Omega_k[T_1, \ldots, T_r, Y_1, \ldots, Y_n, Z]/k$ such that $\hat{\gamma}_\varepsilon(\omega_I) = \hat{\gamma}_\varepsilon(\tilde{\omega}_I)$. Then we obtain by Corollary 0.16,

$$\hat{\gamma}_\varepsilon(r^I \omega_I) \geq \hat{\gamma}_\varepsilon(\tilde{r}^I \tilde{\omega}_I) \geq \hat{\gamma}_\varepsilon(\tilde{\omega}_I) + \hat{\gamma}_\varepsilon(\tilde{r}^I) = \hat{\gamma}_\varepsilon(\omega_I) + \hat{\gamma}_\varepsilon(\tilde{r}^I) \geq \hat{\gamma}_\varepsilon(\omega_I) + \hat{\gamma}_\varepsilon(\tilde{r}_I) \geq \delta |I| - 1 + |I|(-\delta) = -1.$$

As this holds for all $I$, we see that $\sum_{s=0}^\infty \tau_{s_1 \ldots i_m} - 1$ is overconvergent with radius of convergence $\varepsilon$, and hence Proposition 1.6 follows, and so does Proposition 1.2.

**Remark.** The above final arguments in the proof of Proposition 1.2 are very similar to the proof that $W^1(A)$ is weakly complete in the sense of Monsky-Washnitzer (compare [16] and Proposition 2.28 of [6]). Hence $W^1\Omega^d_{A/k}$ satisfies a certain property of weak completeness in positive degrees as well.

**Corollary 1.7.** The complex $W^1\Omega_{\text{Spec} A/k}$, defined for each affine scheme as above, extends to a complex of Zariski sheaves $W^1\Omega_X/k$ on any variety $X/k$.

In the remainder of this section and the next, we prove the following.

**Theorem 1.8.** Let $X$ be a smooth variety. Then $W^1\Omega^d_X/k$ defines a complex of étale sheaves on $X$.

**Proof.** As $W^1\Omega^d_X/k$ is a complex of Zariski sheaves on $X$, the problem of being a sheaf on the étale site is local on $X$. By a result of Kedlaya [11] any smooth variety $X$ has a covering by affine smooth schemes $\text{Spec} A$ which are finite étale over distinguished opens in an affine space $A^d_n$. It therefore suffices to show that if $A$ is a finite étale extension over a localized polynomial algebra, $A'$ a standard étale extension of $A$, then an element $z$ in $W\Omega^d_{A'/k}$ that becomes overconvergent in $W\Omega^d_{A'/k}$ is already overconvergent over $A$.

By localizing further we may assume first that there is an element $f$ in $A$ such that $A'_f$ is finite étale over $A_f$, of the form $A'_f = A_f[X]/(p(X))$ for
some monic irreducible polynomial \( p(X) \). The following proposition reduces the argument to the case \( A_f = A_f' \); hence we will need to show

\[
W \Omega^d_{A/k} \cap W^! \Omega^d_{A_f/k} = W^! \Omega^d_{A_f/k}.
\]

**Proposition 1.9.** Let \( B \) be a finite étale and monogenic \( A \)-algebra, where \( A \) is smooth over a perfect field of char \( p > 0 \). Let \( B = A[X]/(f(X)) \) for a monic irreducible polynomial \( f(X) \) of degree \( m = [B : A] \) such that \( f'(X) \) is invertible in \( B \). Let \( [x] \) be the Teichmüller of the element \( X \) mod \( f(X) \) in \( W(B) \). Then we have for each \( d \geq 0 \) a direct sum decomposition of \( W^!(A) \)-modules

\[
W^d \Omega_{B/k}^d = W^d \Omega_{A/k}^d + W^d \Omega_{A_f/k}^d[x] + \ldots + W^d \Omega_{A_f/k}^d[x]^{m-1}.
\]

**Proof.** From Corollary 2.46 in [6] we know that this proposition is true for \( d = 0 \):

\( W^!(B) \) is a finite \( W^!(A) \)-module with basis \( 1, [x], \ldots, [x]^{m-1} \). There is a unique lifting \( \tilde{f}(X) \in W^!(A)[X] \) of \( f(X) \) such that \( W^!(B) = W^!(A)[X]/\tilde{f}(X) \) and \( \tilde{f}'([x]) \) is invertible in \( W^!(B) \). In particular \( W^!(B) \) étale over \( W^!(A) \).

Let \( \tilde{f}(X) = X^m + a_{m-1}X^{m-1} + \ldots + a_1X + a_0 \), with \( a_i \in W^!(A) \) and

\[
\frac{1}{\tilde{f}'([x])} = c_{m-1}[x]^{m-1} + \ldots + c_1[x] + c_0,
\]

with \( c_i \in W^!(A) \).

When we consider an element \( z \) in \( W^d \Omega_{B/k}^d \) with radius of convergence \( \varepsilon > 0 \) we will always assume that \( \varepsilon \) is small enough such that all \( a_j, c_j, j = 0, \ldots, m - 1 \) are in \( W^\varepsilon(A) \).

The equation

\[
\tilde{f}'([x]) = [x]^m + a_{m-1}[x]^{m-1} + \ldots + a_1[x] + a_0 = 0
\]

(note that \( \tilde{f}(X) \) is the minimal polynomial of \([x]\) over \( W^!(A)\)) implies that

\[
d \tilde{f}'([x]) = 0.
\]

Hence we get

\[
\tilde{f}'([x])d[x] + da_{m-1}[x]^{m-1} + \ldots + da_1[x] + da_0 = 0.
\]

As \( \tilde{f}'([x])^{-1} \) has coefficients in \( W^\varepsilon(A) \) and \( W^\varepsilon(A) \) is a ring we see that

\[
d[x] = -\frac{1}{\tilde{f}'([x])} (da_{m-1}[x]^{m-1} + \ldots + da_1[x] + da_0)
\]

\[
= \sum_{t_j=0}^{m-1} \lambda_t da_t[x]^t \text{ with } \lambda_t, a_t \in W^\varepsilon(A).
\]

The elements \( a_t \in W^\varepsilon(A) \) are homogeneous as they are elementary symmetric function in the \([t_i] \), where \([t_i], i = 1, \ldots, m \) are the roots of \( \tilde{f} \), lifting the roots \( t_i \) of \( f \).

We have \( \lambda_t da_t = d(a_t \lambda_t) - a_t d\lambda_t \) by the Leibniz rule. The elements \( a_t \lambda_t \) are in \( W^\varepsilon(A) \), hence \( d(a_t \lambda_t) \in W^\varepsilon \Omega^1_{A/k} \). As \( a_t \) is homogeneous, the element \( a_t d\lambda_t \) is in \( W^\varepsilon \Omega^1_{A/k} \) as well (Corollary 0.16). So we get

\[
d[x] \in W^\varepsilon \Omega^1_{A/k} \oplus \ldots \oplus W^\varepsilon \Omega^1_{A/k} [x]^{m-1}.
\]
One proves similarly that
\[ d[x]^i \in W^e \Omega^1_{A/k} \oplus \cdots \oplus W^e \Omega^i_{A/k}[x]^{m-1}. \]
for all \( i, \ 1 \leq i \leq m - 1. \)

Let \( b_1, \ldots, b_r \) be generators of the \( k \)-algebra \( A \) and \( z \in W^1 \Omega^d_{B/k} \) be an overconvergent sum of elementary Witt differentials \( z_i \) in variables \([b_1], \ldots, [b_r], [x] \) with \( \gamma_\epsilon(z_i) > C \) for all \( i \). If \( z_i \) the variable \([x]\) occurs with integral weight \( k_x \) we may assume \( 1 \leq k_x \leq m - 1 \). If \([x] \) belongs to the interval \( I_f \) with underlying partition \( P \) corresponding to \( z_i \), then evidently \( z_i = \eta_i[x]^{k_x} \) with \( \eta_i \) an elementary Witt differential in the variables \([b_1], \ldots, [b_r] \) with \( \gamma_\epsilon(\eta_i) > C \). If \([x] \) occurs with integral weight \( k_x, 1 \leq k_x \leq m - 1 \) and belongs to the interval \( I_j, j \geq l \), then after applying the Leibniz rule and the previous case we see that
\[ z_i = \omega_i + \eta_i d[x]^{k_x} \]
with \( \omega_i \in W^e \Omega^d_{A/k} \oplus \cdots \oplus W^e \Omega^{d-1}_A \) and \( \eta_i \in W^e \Omega^{d-1}_A \) with \( \gamma_\epsilon(\eta_i) > C \).

In addition, all coefficients \( \omega_i^{(j)} \) in \( W^e \Omega^d_{A/k} \) satisfy \( \gamma_\epsilon(\omega_i^{(j)}) > C \). We may also assume that all coefficients \( \beta_i^{(j)} \) of \( d[x] \) in \( W^e \Omega^d_{A/k} \) for all \( 1 \leq i \leq m - 1 \) satisfy \( \gamma_\epsilon(\beta_i^{(j)}) > C \). Then,
\[ \eta_i d[x]^{k_x} \in W^e \Omega^d_{A/k} \oplus \cdots \oplus W^e \Omega^{d-1}_A \]
and we have for all coefficients \( \delta_i^{(j)} \in W^e \Omega^d_{A/k} \) that occur in this representation of \( \eta_i d[x]^{k_x} \) that
\[ \gamma_\epsilon(\delta_i^{(j)}) > C. \]

Now we use [6] Corollary 2.46. If \( \alpha = \sum_{i=0}^{m-1} \xi_i x^i \in W^e(B) \) satisfies \( \gamma_\epsilon(\alpha) > C \) then \( \xi_i \in W^e(A) \) with \( \gamma_\epsilon(\xi_i) > C' \) and \( C' \) only depends on \( C \) and \( \epsilon \); wlog \( C' < C \).

Assume that in an elementary Witt differential \( z_i \) occurring in the overconvergent \( z \) we have
\[ z_i = V^r \eta \cdot d\omega \]
and \([x]\) occurs in \( \eta \) with fractional weight \( k_x, k_x = \frac{i}{p^t}, 1 \leq i \leq m - 1 \). Then applying the above fact we see that
\[ z_i \in W^e \Omega^d_{A/k} \oplus \cdots \oplus W^e \Omega^{d-1}_A \]
and the coefficients \( z_i^{(j)} \) satisfy \( \gamma_\epsilon(z_i^{(j)}) > C' \).

If \([x]\) occurs with fractional weight \( k_x \) in an interval \( I_j, j \geq 1 \) of the underlying partition of \( z_i \), then by combining the previous cases we see that
\[ z_i \in W^e \Omega^d_{A/k} \oplus \cdots \oplus W^e \Omega^{d-1}_A \]
and all coefficients \( z_i^{(j)} \) satisfy \( \gamma_\epsilon(z_i^{(j)}) > C' \).
This implies that the original \( z \in W^d \Omega^d_{B/k} \) with \( \gamma(z) > C \) has a representation

\[
z = \sum_{i=0}^{m-1} \sigma_i [x]^i \in W^d \Omega^d_{A/k} \oplus \ldots \oplus W^d \Omega^d_{A/k}[x]^{m-1}
\]

with \( \gamma(z)(\sigma_i) > C' \) for all \( i = 0, \ldots, m-1 \).

On the other hand, by possibly applying the Leibniz rule repeatedly, it is clear that an element in

\[
W^d \Omega^d_{A/k} \oplus \ldots \oplus W^d \Omega^d_{A/k}[x]^{m-1}
\]

can be represented as an overconvergent sum of elementary Witt differentials in variables \([b_1], \ldots, [b_r], [x]\), and hence lies in \( W^d \Omega^d_{B/k} \). This finishes the proof of the proposition. \( \square \)

**Remark.** Note that the isomorphism in the proposition is a restriction of the isomorphism

\[
W \Omega^d_{B/k} \cong W(B) \otimes_{W(A)} W \Omega^d_{A/k} \cong \bigoplus_{i=0}^{m-1} W \Omega^d_{A/k}[x]^i
\]

for the completed de Rham-Witt complex. As \( W(B) \) is finite étale over \( W(A) \) if \( B \) is finite étale over \( A \), this latter isomorphism is a consequence of étale base change for the de Rham-Witt complex of finite level, by passing to the inverse limit (compare [12] Proposition 1.7 and Corollary 2.46 in [6]).

To prove the theorem, it remains to show that

\[(1.10) \quad W \Omega^d_{B/k} \cap W^d \Omega^d_{B/g/k} = W^d \Omega^d_{B/k} \]

for a \( k \)-algebra \( B \) which is a finite étale extension over a localization \( A_f \) of a polynomial algebra \( A = k[T_1, \ldots, T_d] \), and some \( g \in B \). After possibly localizing again, we may assume wlog that \( g \) itself is in the polynomial algebra. After applying Proposition 1.9 again, we reduce the proof of the étale sheaf property to the case where \( B = A_f \). That is, we need to prove (1.10) in the special case \( B = A_f \) and \( g \in A \). This will follow from a further careful study of the Gauss norm properties on the de Rham-Witt complex of the polynomial algebra \( A \) and a localization \( A_f \), done in the next section.

### 2. GAUSS NORM PROPERTIES ON THE DE RHAM-WITT COMPLEX OF LOCALIZED POLYNOMIAL ALGEBRAS

We will consider the Gauss norms on the truncated de Rham-Witt complexes \( W_{t+1} \Omega_{A/k} \) and \( W_{t+1} \Omega_{A_f/k} \) (and also \( W_{t+1} \Omega_{A<f/k} \)) and describe overconvergence on the completed de Rham-Witt complexes via these truncated Gauss norms. Before we can do this, we need to review a few more properties of the de Rham complex \( \Omega_{A/k} \) for the polynomial algebra \( A = k[T_1, \ldots, T_d] \) over a perfect field \( k \) of characteristic \( p > 0 \).

We recall the basic differentials \( \varepsilon(k; \ell) \) from (0.1):

\[
(2.1) \quad \varepsilon(k; \ell) = T^{k_{t_0}} \left( \frac{dT^{k_{t_1}}}{p^{\ord_p k_{t_1}}} \right) \cdots \left( \frac{dT^{k_{t_r}}}{p^{\ord_p k_{t_r}}} \right).
\]
A basic differential is called primitive if $I_0 \neq 0$ and if the function $k$ is not divisible by $p$.

**Proposition 2.2.** Let $\mathfrak{e}(k, P)$ be a primitive basic differential. Then for all $1 \leq j \leq d$

$$T^p_j \mathfrak{e}(k, P)$$

is a linear combination of primitive basic differentials with values in $k$.

**Proof.** Let $I_0 = \{i_1, \ldots, i_t\}$. Let $I'_0 = \{i_1, \ldots, i_s\} \subset I_0$ be the subset of all indices $i_m$, such that $\text{ord}_p k_{i_m} = 0$. Let $I''_0$ be the complement of $I'_0$ in $I_0$. We have $I'_0 \neq \emptyset$ but possibly $I''_0 = \emptyset$.

Consider the case where $j = i_m \in I'_0$. We define $k'$ such that $k'_{i_m} = k_{i_m} + p$ and $k'_j = k_j$ for all other indices. Then $\text{Supp} k = \text{Supp} k'$ and the chosen order on these sets is the same. From this we see that

$$T^p_j \mathfrak{e}(k, P) = \mathfrak{e}(k', P).$$

Now we consider the case where $j$ doesn’t belong to $I'_0$. We write

$$T^p_j T^{k''}_{I'_0} \left( \frac{dT^{k_{i_1}}}{p^{\text{ord}_p k_{i_1}}} \right) \cdots \left( \frac{dT^{k_{i_t}}}{p^{\text{ord}_p k_{i_t}}} \right)$$

as a linear combination of basic differentials $\mathfrak{e}(h, Q)$ for possibly different partitions $Q$. Let $\iota$ be the weight such that $\iota(j) = p$ and such that $\iota$ vanishes on the remaining indices. Then $h = k + \iota$.

Consider the subcase where $\text{ord}_p k_j = 0$. Then $j$ must belong to one of the sets $I_0, \ldots, I_r$ and therefore $j$ must be bigger than any of the indices appearing in $I'_0$. Then

$$T^{k''}_{I'_0} \mathfrak{e}(h, Q)$$

is a primitive basic differential for each partition $Q$. Its weight function $k''$ is the sum of $k''_{I'_0}$ (the restriction of $k$ to $I'_0$) and $h$. That we obtain a basic differential follows from the fact that for the order given by $k''$ any element of $I'_0$ precedes any element in $\text{Supp} h$.

This last sentence is still true in the subcase $\text{ord}_p k_j > 0$, because this implies $\text{ord}_p h_j > 0$. This finishes the proof. \hfill $\square$

We consider $\Omega^l_{A/k}$ throughout this section as an $A$-module via restriction of scalars by $F : A \to A$. We will say that we consider $\Omega^l_{A/k}$ as an $A - F$-module.

**Proposition 2.3.** Let $P^l \subset \Omega^l_{A/k}$ be the $k$-subvector space generated by primitive basic differentials. We have a direct decomposition:

$$(2.4) \quad \Omega^l_{A/k} = P^l \oplus dP^{l-1} \oplus F\Omega^l_{A/k}.$$  

Each summand on the right hand side is a free $A - F$-module which has a basis consisting of basic differentials

**Proof.** The decomposition (2.4) is direct because the second $k$-vector space is generated by basic differentials whose weights are not divisible by $p$ and such that we have $I_0 = \emptyset$ in the partition while $F\Omega^l_{A/k}$ is generated by basic differentials whose weights are divisible by $p$.  

It follows from Proposition 2.2 that $P^l$ is an $A-F$-module. Then the other two summands of (2.4) are clearly $A-F$-modules. Therefore all summands are projective $A-F$-modules. All summands are graded by the absolute value of weights and are therefore graded $A-F$-modules. Let $a$ be the ideal of $A$ generated by $T_1, \ldots, T_d$. A basis of the $A-F$-module $P^l$ is obtained by lifting a basis of the (graded) $k$-vector space $P^l/FaP^l$. This proves the last sentence of the proposition. □

Next we consider the de Rham-Witt complex $W \Omega A/k$. We denote by $\text{Fil}^n$ the kernel of the canonical map $W \Omega A/k \rightarrow W_n \Omega A/k$. It is an abelian group generated by the basic Witt differentials $e(\xi,k,P)$ such that $\text{ord}_V \xi \geq n$ (compare [12]). We set:

$G^n_0 = \text{Fil}^n W \Omega^l A/k / \text{Fil}^{n+1} W \Omega^l A/k$.

We consider it as a $W(A)-F$-module. Clearly the module structure factors via $W(A) \rightarrow A$. We consider throughout this $A$-module structure on $G^n_0$. On $G^n_0 = \Omega^l A/k$ it agrees with the $A-F$-module structure considered above.

The $A$-module $G^n_0$ has a direct decomposition into free $A$-modules:

$$G^n_0 = V^n P^l \oplus pV^{n-1} P^l \oplus \cdots \oplus p^n P^l \oplus dV^n P^{l-1} \oplus pdV^{n-1} P^{l-1} \oplus \cdots \oplus p^n dP^l \oplus p^n F \Omega^l A/k. \tag{2.5}$$

This follows from Proposition 2.3 and the decomposition of $W \Omega A/k$ defined by basic Witt differentials. It is clear that each summand has a basis consisting of basic Witt differentials.

**Proposition 2.6.** For each $n \geq 0$ there is a family $\omega_i^{(n)} \in \text{Fil}^n W \Omega^l A/k$ of basic Witt differentials, where $i$ runs through some finite index set $J_n$, satisfying the following:

For each $n$ the elements $\omega_i^{(n)}$ for $i \in J_n$ form a basis of the $A$-module $G^n_0$.

A Witt differential $\omega \in W_{t+1} \Omega A/k$ has a unique expression

$$\omega = \sum_{n=0}^t \sum_{i \in J_n} F[a_i^{(n)}] \omega_i^{(n)}, \tag{2.7}$$

where $a_i^{(n)} \in A$.

Moreover the truncated Gauss norm $\gamma_\varepsilon[t]$ is given by the following formula:

$$\gamma_\varepsilon[t](\omega) = \min_{n,i \in J_n} \{ p \gamma_\varepsilon(a_i^{(n)}) + \gamma_\varepsilon(\omega_i^{(n)}) \}. \tag{2.8}$$

**Proof.** For a fixed $n$ and each of the summands of (2.5) we choose basic Witt differentials in $\text{Fil}^n$ which form a basis of this summand as an $A$-module. Therefore we obtain a basis $\omega_i^{(n)}$. Then we write:

$$\omega = \sum_{i \in J_0} F[a_i^{(0)}] \omega_i^{(0)} \mod \text{Fil}^1.$$

Then we consider the Witt differential
\[ \omega(1) = \omega - \sum_{i \in J_0} F[a_i^{(0)}] \omega_i^{(0)} \in \text{Fil}^1. \]

Then we consider \( \omega(1) \in G^{1,l} \) and express it by the chosen basis of this \( A \)-module. This process may be continued to obtain the expression (2.7).

Finally we have to prove the assertion about the Gauss norm. We consider first the case of a differential \( \omega \in G^{n,l} \subset W_{n+1} \Omega_A/k. \) We decompose \( \omega \) according to the decomposition (2.5):
\[ \omega = \sum \omega_m. \]

Since the decomposition (2.5) is defined by a partition of the set of basic Witt differentials we deduce the formula:
\[ \gamma_\varepsilon[n](\sum \omega_m) = \min_m \{ \gamma_\varepsilon(\omega_m) \}. \]

Let us denote by \( S \) an arbitrary summand of the decomposition (2.5). All nonzero elements \( \sigma \in S \) have the same order \( \text{ord}_V \sigma = o_S \). As explained, \( S \) is a free graded module over \( A \):
\[ S = \bigoplus S_t, \]

such that \( S_t \) has a basis of basic Witt differentials whose weights have absolute value \( t \). We find that for \( z \in S_t \), such that \( z \neq 0 \):
\[ \gamma_\varepsilon[n](z) = o_S - \varepsilon t. \]

Now we assume that \( z = \sum F[a_i^{(n)}] \omega_i^{(n)} \). Since \( S \) is free we deduce from this the formula:
\[ \gamma_\varepsilon[n](\sum F[a_i^{(n)}] \omega_i^{(n)}) = \min \{ \gamma_\varepsilon(F[a_i^{(n)}]) + \gamma_\varepsilon(\omega_i^{(n)}) \}. \]

Now we consider the element \( \omega \in W_{t+1} \Omega_A/k \) with the expansion (2.7). We set \( \gamma_\varepsilon[t](\omega) = C. \) Then we have:
\[ C \leq \gamma_\varepsilon[0](\omega) = \gamma_\varepsilon[0](\sum F[a_i^{(0)}] \omega_i^{(0)}) = \min \{ \gamma_\varepsilon(F[a_i^{(0)}]) + \gamma_\varepsilon(\omega_i^{(0)}) \}. \]

On the other hand we have the inequality:
\[ \gamma_\varepsilon[t](\Omega A/k) \geq \min \{ \gamma_\varepsilon(F[a_i^{(0)}]) + \gamma_\varepsilon(\omega_i^{(0)}) \}. \]

We obtain that
\[ \gamma_\varepsilon[t](\omega - \sum F[a_i^{(0)}] \omega_i^{(0)}) \geq \gamma_\varepsilon[t](\omega) = C. \]

Applying the same argument to \( \omega(1) = \omega - \sum F[a_i^{(0)}] \omega_i^{(0)} \in \text{Fil}^1 \) we find that in the decomposition (2.7) the following inequality holds:
\[ \gamma_\varepsilon(F[a_i^{(n)}]) + \gamma_\varepsilon(\omega_i^{(n)}) \geq C. \]

But on the other hand we have:
\[ C = \gamma_\varepsilon \left( \sum_{n=0}^{t} \sum_{i \in J_n} F[a_i^{(n)}] \omega_i^{(n)} \right) \geq \min \{ \gamma_\varepsilon(F[a_i^{(n)}]) + \gamma_\varepsilon(\omega_i^{(n)}) \}. \]

This proves the last assertion. \( \square \)
Remark. Let \( f = \sum \alpha_k T^k \in A \), where \( \alpha_k \in k \) is a polynomial. We set \( \bar{f} = \sum [a_k][T]^k \in W(A) \). This is an integral Witt vector which lifts \( f \).
We can replace in the proof the Teichmüller representatives \( \{a_i^{(n)}\} \) by \( \bar{a}_i^{(n)} \), and the element \( F[a_i^{(n)}] \) by the element \( F\bar{a}_i^{(n)} \). Then we obtain a unique expression:

\[
\omega = \sum_{n=0}^{t} \sum_{\epsilon \in J_n} F\bar{a}_i^{(n)} \omega_i^{(n)},
\]

The Gauss norm is given by the formula (2.8).

Our next aim is to prove a similar proposition for the localization \( A_f \) of the polynomial algebra \( A = k[T_1, \ldots, T_d] \) for an element \( f \in A \). We write \( \delta = \deg f \).

Let \( \omega \in W_{t+1} \Omega_{A_f/k} \). We have seen that an admissible pseudovaluation \( \gamma'_\varepsilon[t] \) on this de Rham-Witt complex is obtained as follows. We consider all possible representations:

\[
(2.10) \quad \omega = \sum_l (\eta_l/|f|^{l_p}), \quad \text{where } \eta_l \in W_{t+1} \Omega_{A/k}.
\]

Then \( \gamma'_\varepsilon[t](\omega) \) is the maximum over all possible numbers

\[
\min \{ \gamma_\varepsilon[t]\eta_l - \varepsilon l_p \}.
\]

There is always a representation where this maximum is taken. Such representations will be called optimal. The following inequalities are immediate:

\[
\begin{align*}
\gamma'_\varepsilon[t](\omega) & \leq \gamma'_\varepsilon[t-1](\omega) \\
\gamma'_\varepsilon[t](\omega) & \leq \gamma'_\varepsilon[t](\omega) \quad \text{for } \varepsilon \geq \delta.
\end{align*}
\]

We could also consider all representations of the form \( \omega = \sum_l (\eta_l/|f|^{l_p}) \) without the extra factor \( p \). Then we denote by \( \bar{\gamma}'_{\varepsilon}(\omega) \) the maximum of the numbers \( \min \{ \gamma_\varepsilon[t]\eta_l - \varepsilon l_p \} \). We will use this Gauss norm only for the Witt ring.

We write \( \text{Fil}^n_f = \text{Fil}^m W\Omega_{A_f/k} \). By étale base change \( \text{Fil}^n_f \) is obtained from \( \text{Fil}^m \) by localizing with respect to \( [f] \).

**Lemma 2.11.** Let \( \omega \in \text{Fil}^m_f \). Then there is an optimal representation (2.10) of \( \omega \) such that \( \eta_l \in \text{Fil}^m \).

**Proof.** The case \( m = 0 \) is trivial. We assume by induction that there is an optimal representation such that \( \eta_l \in \text{Fil}^{m-1} \). Consider the residual classes of \( \bar{\eta}_l \) of \( \eta_l \) in \( G^{m-1} = \oplus_l G^{m-1,l} = \text{Fil}^{m-1}/\text{Fil}^m \). We use the abbreviation \( \delta_\varepsilon(\bar{\eta}_l) = \gamma_\varepsilon[i-1](\eta_l) \). Clearly we have that \( \delta_\varepsilon(\bar{\eta}_l) \geq \gamma_\varepsilon[i](\eta_l) \). Then we have in \( G^{m-1} \) the relation:

\[
(2.12) \quad \sum_{l=0}^{M} (\bar{\eta}_l/|f|^{l_p}) = 0.
\]

We may assume that \( \bar{\eta}_M \neq 0 \) and that \( M \) is the minimal possible value for all optimal representations. Then we have to show that \( M \geq 1 \) is impossible.
We see that \( \bar{\eta}_M \) is divisible by \( [f] \). Then we write:

\[
\bar{\eta}_M = [f] \hat{\tau}.
\]
We obtain that \( \delta_\varepsilon(\bar{\tau}) - \varepsilon \delta = \delta_\varepsilon(\bar{\eta}_M) \). We may lift \( \bar{\tau} \) to an element \( \tau \in \mathrm{Fil}^{m-1} \) such that \( \gamma_\varepsilon(t) \tau = \delta_\varepsilon(\bar{\tau}) \). We write:

\[
\eta_M = [f] \tau + \rho, \quad \text{where} \quad \rho \in \mathrm{Fil}^m.
\]

Since \( \gamma_\varepsilon(t) ([f] \tau) = \gamma_\varepsilon(t) - \varepsilon \delta = \delta_\varepsilon(\bar{\eta}_m) \geq \gamma_\varepsilon(\eta_M) \) we conclude that \( \gamma_\varepsilon(\rho) \geq \gamma_\varepsilon(\eta_M) \).

Now we consider the equation:

\[
(\eta_M/\lfloor f \rfloor^M) = (\tau/\lfloor f \rfloor^{M-1}) + (\rho/\lfloor f \rfloor^M).
\]

Inserting this in (2.10) we obtain again an optimal expression, since:

\[
\gamma_\varepsilon(\rho) - M \varepsilon \geq \gamma_\varepsilon(\eta_M) - M \varepsilon.
\]

Reducing this modulo \( \mathrm{Fil}^m \) we see that the number \( M \) became smaller.

**Lemma 2.13.** Let \( \omega \in G^t \subset W_{t+1} \Omega_{A_f/k} \). Then \( \omega \) has a unique expression:

\[
\omega = \sum F[c_i] \omega_i(t), \quad c_i \in A_f.
\]

Then we have:

\[
\gamma'_\varepsilon(t)(\omega) = \min_{\substack{i \in J_n \{p \gamma_\varepsilon(a_i) + \gamma_\varepsilon(\omega_i(t))\}.}}
\]

**Proof.** Since \( G^t \) is a free \( A - F \)-module it is clear that the localization is a free \( A_f - F \)-module with the same basis. From this it follows that such a decomposition exists.

We choose an optimal representation:

(2.14)

\[
\omega = \sum_l (\eta_l/\lfloor f \rfloor^l)
\]

By the last lemma we may assume that \( \eta_l \in G^t \). Then we find for \( \eta_l \) an expression:

\[
\eta_l = \sum F a_{il} \omega_i^{(l)}, \quad a_{il} \in A.
\]

Therefore we obtain by definition and Proposition 2.6:

(2.15)

\[
\gamma'_\varepsilon(t)(\omega) = \min_{\substack{i \in J_n \{\sum F a_{il} \omega_i^{(l)} - \varepsilon lp\} = \min_{\substack{i \in J_n \{p \gamma_\varepsilon(a_{il}) + \gamma_\varepsilon(\omega_i^{(l)}) - \varepsilon lp\}.
\]

We set

\[
c_i^{(n)} = \sum_l (a_{il}^{(n)}/f^l).
\]

We can assume that this expression is optimal for \( \gamma'_\varepsilon \). Because in the other case we could insert the optimal expression in the equation:

(2.16)

\[
\omega = \sum_i (\sum_l F a_{il}/f^l) \omega_i^{(l)}.
\]

This would make the right hand side of (2.15) bigger. But then (2.16) would again be an optimal expression of the form (2.14).

We obtain \( \gamma'_\varepsilon(c_i^{(n)}) = \min_i \{\gamma_\varepsilon(a_{il}^{(n)}) - \varepsilon l\} \). This shows the last formula of the lemma. \( \square \)
Let \( c \in A_f \) be an element. We choose an optimal representation:
\[
c = \sum (a_l / f^l).
\]
We set:
\[
(2.17) \quad \hat{c} = \sum [a_l] / [f]^l \in W_{t+1}(A_f).
\]
We find
\[
\hat{\gamma}_\epsilon'[t](\hat{c}) \geq \hat{\gamma}_\epsilon'(c).
\]
But the other inequality is obvious since \( \hat{\gamma}_\epsilon'[t](\hat{c}) \leq \hat{\gamma}_\epsilon'[1](\hat{c}) \).
Therefore we have an equation:
\[
(2.18) \quad \hat{\gamma}_\epsilon'[t](\hat{c}) = \hat{\gamma}_\epsilon'(c).
\]
In the same way we obtain:
\[
\gamma_\epsilon'[t](\hat{c}) = \gamma_\epsilon'(c).
\]
Indeed we have:
\[
(2.19) \quad \gamma_\epsilon'(Fc) = p\hat{\gamma}_\epsilon'(c) \quad \text{for } c \in A_f.
\]
To see this we can reduce to the case, where \( f \) is regular with respect to one variable. Then one uses that reduced representations are optimal.

**Proposition 2.20.** With the same notation as in Proposition 2.6 consider a Witt differential \( \eta \in W_{t+1}\Omega_{A_f/k} \). Then there is a unique decomposition:
\[
\eta = \sum_{i,n} F_{c_i}^{(n)} \omega_i^{(n)}, \quad c_i^{(n)} \in A_f.
\]
The truncated Gauss norm is given by the formula:
\[
\gamma_\epsilon'[t](\eta) = \min_{i,n} \{ p\gamma_\epsilon'(c_i^{(n)}) + \gamma_\epsilon(\omega_i^{(m)}) \}.
\]

**Proof.** Since \( t \) is fixed we will set \( \gamma_\epsilon' = \gamma_\epsilon'[t] \) Consider an expression in \( \text{Fil}^m \):
\[
z = \sum_i F_{c_i}^{(m)} \omega_i^{(m)}.
\]
We claim that:
\[
(2.21) \quad \gamma_\epsilon'(z) = \gamma_\epsilon'[m](z) = \min_i \{ \gamma_\epsilon'(F_{c_i}^{(m)}) + \gamma_\epsilon(\omega_i^{(m)}) \}.
\]
Indeed, the second equality follows from Lemma 2.13. We see easily that \( \gamma_\epsilon'(z) \) is greater than the right hand side of (2.21). Indeed, we choose optimal representations for \( c_i^{(m)} \):
\[
c_i^{(m)} = \sum_l a_{il} / f^l.
\]
We obtain:
\[
z = \sum_{l,i} [a_{il}] p\omega_i^{(m)}/[f]^l p = \sum_l (\sum_i [a_{il}] p\omega_i^{(m)})/[f]^l p.
\]
This shows that
\[
\gamma_\epsilon'(z) \geq \min_i \{ p\gamma_\epsilon(\sum_i [a_{il}] p\omega_i^{(m)}) - lp\epsilon \} = \min_i \{ \min_i \{ p\gamma_\epsilon(a_{il}) + \gamma_\epsilon(\omega_i^{(m)}) \} - lp\epsilon \}.
\]
The last equation follows from Proposition 2.6. By definition we have the equation:

\[ p\gamma_\varepsilon(c_i^{(m)}) = \min\{\gamma_\varepsilon(a_d) - l\varepsilon\}. \]

This shows the inequality:

\[ \gamma'_\varepsilon(z) \geq \min\{\gamma'_\varepsilon(Fc_i^{(m)}) + \gamma_\varepsilon(\omega_i^{(m)})\} = \gamma'_\varepsilon[m](z). \]

On the other hand we have \( \gamma'_\varepsilon(z) \leq \gamma'_\varepsilon[m](z) \), and this proves the equality (2.21).

As in the proof of Proposition 2.6 we find an expansion with the desired properties. \( \square \)

Remark. Consider the natural map \( B = k[T_1, \ldots, T_d, S] \to A_f \), which maps \( S \) to \( f^{-1} \). We have defined the overconvergent Witt vectors \( W^1\Omega_{A_f/k} \) as the image of \( W^1\Omega_{B/k} \) by the canonical map:

\[ W^1\Omega_{B/k} \to W^1\Omega_{A_f/k}. \]

Assume that we are given \( \omega \in W^1\Omega_{A_f/k} \), such that there is a constant \( C \) with (2.23)

\[ \gamma'_\varepsilon[t](\omega) \geq C \]

for all \( t \geq 0 \). We claim that \( \omega \in W^1\Omega_{A_f/k} \). By the unicity statement of the last proposition we have an infinite expansion:

\[ \omega = \sum_{i,n} Fc_i^{(n)}\omega_i^{(n)}. \]

As in the proof above we take optimal representations:

\[ c_i^{(n)} = \sum_l a_{il}^{(n)}/f^l. \]

Then we find a convergent sum in the Fil-topology:

\[ \omega = \sum_l \left( \sum_i [a_{il}^{(n)}/f^l][S^{(n)}]^{lp} \right) \]

where \( p\gamma_\varepsilon(a_{il}^{(n)}) - \varepsilon lp + \gamma_\varepsilon(\omega_i^{(n)}) \geq C \). But then

\[ \sum_l \left( \sum_i [a_{il}^{(n)}/f^l][S^{(n)}]^{lp} \right) \in W^1\Omega_{B/k} \]

is clearly an overconvergent Witt differential which lifts \( \omega \). Conversely the condition (2.23) is clearly fulfilled for an overconvergent \( \omega \), because \( \gamma'_\varepsilon \) is equivalent to the quotient norm induced by (2.22).

Corollary 2.24. For \( \eta \in W_{t+1}\Omega_{A_f/k} \) we have the equation:

\[ \gamma'_\varepsilon[t + 1](p\eta) = 1 + \gamma'_\varepsilon[t](\eta). \]

Proof. We note that the proposition holds for each set \( \omega_i^{(n)} \in W^1\Omega_{A_f/k} \) of basic Witt differentials which for each given \( n \) induce a basis of \( G^n \) as \( A - F \)-module. But clearly \( p\omega_i^{(n)} \) is part of a basis of \( G^{n+1} \) consisting of basic Witt differentials. This gives with the notations of the proposition:

\[ \gamma'_\varepsilon[t + 1](\eta) = \gamma'_\varepsilon(\sum_{i,n} Fc_i^{(n)}(p\omega_i^{(n)})) = \min\{p\gamma_\varepsilon(c_i^{(n)}) + \gamma_\varepsilon(p\omega_i^{(n)})\}. \]

This proves the result. \( \square \)
Proposition 2.25. Let \( f, g \in A \) be two non-zero elements without common divisors. There is a constant \( Q > 1 \) with the following property. Let \( t \) be a rational number and let \( \varepsilon > 0 \) a real number. We denote by \( \gamma'_\varepsilon = \gamma'_\varepsilon[t] \) the natural Gauss norm on \( W_{t+1} \Omega_{A_f/k} \) and by \( \gamma''_\varepsilon \) the natural Gauss norm on \( W_{t+1} \Omega_{A_{fg}/k} \).

We denote the image of a Witt differential \( \omega \in W_{t+1} \Omega_{A_f/k} \) in \( W_{t+1} \Omega_{A_{fg}/k} \) by the same letter. Then the following inequality holds:

\[
\gamma'_\varepsilon/Q(\omega) \geq \gamma''_\varepsilon(\omega) \\
\gamma''_\varepsilon/Q(\omega) \geq \gamma'_\varepsilon(\omega).
\]

Proof. We begin with the proof of the first inequality, which is the nontrivial one. We may extend the ground field \( k \) and assume that \( k \) is infinite. After a coordinate change we may assume that \( f \) and \( g \) are regular with respect to \( T_1 \). Consider an element \( c \in A_f \) with the reduced representation

\[
c = \sum a_i/f^i.
\]

If we regard \( c \) as an element of \( A_{fg} \) it has the reduced representation:

\[
c = \sum (a_i g^i)/(fg)^i.
\]

We have defined a lifting \( \hat{c} \in W(A_f) \) of \( c \) (2.17). This coincides with the lifting \( \hat{c} \in W(A_{fg}) \):

\[
\sum [a_i]/[f]^i = \sum ([a_i g^i])/(fg)^i.
\]

We set \( C = \gamma''_\varepsilon(\omega) \). By Proposition 2.20 we have the expansion:

\[
\omega = \sum_{i,n} \hat{c}_i^{(n)} \omega_i^{(n)}, \quad c_i^{(n)} \in A_f.
\]

Since the \( \hat{c} \) with respect to \( A_f \) and with respect to \( A_{fg} \) means the same (2.27) is also the expansion of \( \omega \) with respect to \( A_{fg} \) according to Proposition 2.20.

Therefore we conclude that:

\[
C = \min\{p^{(n)}_{\gamma''_\varepsilon(c_i^{(n)})} + \gamma'_\varepsilon(\omega_i^{(n)})\}.
\]

By Proposition 1.30 of [6] there are constants which depend only on \( \deg f \) and \( \deg g \), such that the pseudovaluation \( \hat{\gamma}'_\varepsilon \) on \( A_f \) (respectively \( \hat{\gamma}''_\varepsilon \) on \( A_{fg} \)) compare to the \( \mu \)-functions:

\[
Q_1 \mu'(c) \leq \hat{\gamma}'_\varepsilon(c) \leq Q_2 \mu'(c) \quad \text{for } c \in A_f, \\
Q_1 \mu''(d) \leq \hat{\gamma}''_\varepsilon(d) \leq Q_2 \mu''(d) \quad \text{for } d \in A_{fg}.
\]

If \( c \in A_f \) has denominator \( f^n \), then \( c \) regarded as an element of \( A_{fg} \) has denominator \((fg)^n\). This shows the equality

\[
\mu'(c) = \mu''(c).
\]

We find the inequalities:

\[
\hat{\gamma}''_\varepsilon(c) \leq Q_2 \mu''(c) = Q_2 \mu'(c) \leq (Q_2/Q_1) \hat{\gamma}'_\varepsilon(c).
\]

We set \( Q = \max\{1, (Q_1/Q_2)\} \) and rewrite the above inequality:

\[
\hat{\gamma}''_\varepsilon(c) \leq \gamma'_\varepsilon/Q(c), \quad \text{for } c \in A_f.
\]
From this we find:
\[ p\gamma'_{\varepsilon/Q}(c_i^{(n)}) + \gamma_{\varepsilon/Q}(\omega^{(n)}_i) \geq p\gamma''_{\varepsilon/Q}(c_i^{(n)}) + \gamma_{\varepsilon}(\omega^{(n)}_i) \geq C. \]
Using Proposition 2.20 this implies the first inequality (2.26).
The second inequality is straightforward: We choose an optimal representation of \( \omega \in W_{t+1}\Omega_{A_f/k} \) with respect to \( \varepsilon \)
\[ \omega = \eta_l/[f]^p, \quad \eta_l \in W_{t+1}\Omega_{A/k}. \]
From the representation
\[ \omega = \eta_l g^p/[f g]^p, \quad \eta_l \in W_{t+1}\Omega_{A/k} \]
we obtain that:
\[ \gamma''_{\varepsilon/Q}(\omega) \geq \gamma_{\varepsilon/Q}(\eta_l) - lp\varepsilon/Q \geq \gamma_{\varepsilon}(\eta_l) - \varepsilon lp = \gamma'_{\varepsilon}(\omega). \]
□
Using the remark before Corollary 2.24, we see that Proposition 2.25 implies the claim in (1.10) and finishes the proof of Theorem 1.8.
□

**Corollary 2.28.** With the notations of the proposition we have the inequality:
\[ (2.29) \quad \gamma'_{\varepsilon}([g]^p\omega) \leq \gamma'_{\varepsilon/Q^2}(\omega) + p\varepsilon/Q. \]
Let \( c \in A_f \), such that \( c \neq 0 \). Then there are constant \( C, Q \in \mathbb{R}, Q > 1 \) such that for every \( \omega \in W_{t+1}\Omega_{A_f/k} \)
\[ \gamma'_{\varepsilon}([c]\omega) \leq \gamma'_{\varepsilon/Q^2}(\omega) + C\varepsilon. \]
This shows in particular that an element \( \omega \in W\Omega_{A_f/k} \) is overconvergent if for some \( c \in A_f, c \neq 0 \) the element \([c] \omega \) is overconvergent.

**Proof.** We begin to show the inequalities:
\[ (2.30) \quad \gamma'_{\varepsilon}(\frac{1}{[f]^p}\omega) \geq \gamma_{\varepsilon}(\omega) - p\varepsilon \]
\[ \gamma'_{\varepsilon}([f]^p]\omega) \leq \gamma'_{\varepsilon}(\omega) + p\varepsilon. \]
To verify the first of these inequalities we choose an optimal representation:
\[ (2.31) \quad \omega = \sum_l \eta_l/[f]^p. \]
After dividing by \([f]^p\) we conclude:
\[ \gamma'_{\varepsilon}(\frac{1}{[f]^p}\omega) \geq \min_l \{\gamma_{\varepsilon}(\eta_l) - (l + 1)p\varepsilon\} = \gamma_{\varepsilon}(\omega) - p\varepsilon. \]
From this we deduce formally the second inequality:
\[ \gamma'_{\varepsilon}(\omega) = \gamma'_{\varepsilon}(\frac{1}{[f]^p}[f]^p\omega) \geq \gamma'_{\varepsilon}([f]^p\omega) - p\varepsilon. \]
Let \( h \in A \) be arbitrary. If we multiply (2.31) by \([h]\) we obtain the inequality:
\[ (2.32) \quad \gamma_{\varepsilon}([h]\omega) \geq \gamma_{\varepsilon}(h) + \gamma_{\varepsilon}(\omega). \]
As above we obtain from this formally:

\[(2.33) \quad \gamma'_e(\frac{1}{[fp]} \omega) \leq \gamma'_e(\omega) - \gamma_e([f]^p)).\]

Using (2.30) for \(\gamma''_e\) and the proposition we obtain:

\[\gamma''_e/\omega(p \omega) \leq \gamma''_e/\omega(\omega) + p\epsilon/Q \leq \gamma'_e/\omega(\omega) + p\epsilon/Q.\]

But on the other hand the proposition shows:

\[\gamma''_e/\omega(p \omega) \geq \gamma'_e/\omega(\omega).\]

This shows (2.29).

For the last statement we remark that it is true for \([c]\) if there is an \(h\) such that the statement is true for \([hc]\). Indeed this follows from (2.32).

Therefore it suffices to assume that \(c = f^m g\), where \(g\) has no common divisor with \(f\). This case is easily deduced from (2.29) and (2.30).

\[\square\]

3. Comparison with Monsky-Washnitzer cohomology

Let \(B/k\) be a finitely generated, smooth algebra over a perfect field \(k\) of \(\text{char} \ p > 0\). Let \(\tilde{B}^\dagger\) be the weak completion (in the sense of [16]) of a smooth finitely generated \(W(k)\)-algebra \(\tilde{B}\) lifting \(B\). To begin this section we prove the existence of a map \(\sigma : \tilde{B}^\dagger \to W^\dagger(B)\) which we call an overconvergent Witt lift. It depends on a choice of Frobenius lift \(F\) and is the same as the map \(t_F : \tilde{B} \to W(B)\) described in [10]. We must prove that this map has image in \(W^\dagger(B)\). We do this first for the case of a polynomial algebra (and any choice of Frobenius lift), and deduce the general result easily by functoriality.

**Proposition 3.1.** Let \(A = k[T_1, \ldots, T_d]\) and \(\tilde{A}^\dagger = W(k)(T_1, \ldots, T_d)^\dagger\). Fix a Frobenius lift \(F\) on \(\tilde{A}^\dagger\). Then the map \(t_F\) defined in [10] p. 509 (and recalled below) has image in \(W^\dagger(A)\).

**Proof.** Let \(a \in \tilde{A}^\dagger\) have the form

\[\sum_{k \in \mathbb{N}^d} \alpha_k T_1^{k_1} \cdots T_d^{k_d}.\]

For \(\epsilon > 0\), we define a Gauss norm on \(\tilde{A}^\dagger\) by

\[\gamma_\epsilon(a) = \inf_k \{\text{ord}_p \alpha_k - \epsilon|k|\}.\]

We define

\[W^\dagger(\tilde{A}^\dagger) := \{(a_0, a_1, \ldots) \in W(\tilde{A}^\dagger) \mid m + \gamma_\epsilon^\dagger(a_m) \geq C, \text{ for some } \epsilon > 0, C \in \mathbb{R}\}.\]

The projection map \(pr : W(\tilde{A}^\dagger) \to W(A)\) induces a map \(W^\dagger(\tilde{A}^\dagger) \to W^\dagger(A)\).

For \(x \in W(\tilde{A}^\dagger)\), write \(x = (a_0, a_1, \ldots)\) and let \(w_m(x) \in \tilde{A}^\dagger\) denote the \(m\)th ghost component. Then we find

\[m + \gamma_\epsilon^\dagger(a_m) \geq C \iff \gamma_\epsilon^\dagger(w_m(x)) \geq C.\]

The map \(t_F\) is defined as the composition

\[\tilde{A}^\dagger \xrightarrow{t_F} W(\tilde{A}^\dagger) \xrightarrow{pr} W(A),\]
Proposition 3.2.

Remark in our first paragraph, γε(ε) follows by induction. We compute C where for any a

Proof. Take a surjective map from a polynomial algebra W has image in B overconvergent Witt lift W we assume that B admits a Witt lift, which we will call the underlying Witt lift associated to σ. Conversely, if we assume that B admits a Witt lift, σ : B → W(B) such that image(σ) ⊆ W⊥(B), then σ extends canonically to the weak completion of B, i.e. to an overconvergent Witt lift

(3.3) σ : B → W⊥(B)
because $W^1(B)$ is weakly complete (Proposition 2.28 in [6]). We derive from this a map of complexes, also denoted by $\sigma$

\begin{equation}
\Omega_{B^1/W(k)} \to W^1\Omega^*_{B/k} \subset W\Omega^*_{B/k}.
\end{equation}

If $\hat{B}$ denotes the $p$-adic completion of $B$ we also have a map

$$
\lim_{\longrightarrow} \Omega^*_{B_n/W_n(k)} =: \Omega^*_{\hat{B}/W(k)} \to W\Omega^*_{B/k}.
$$

In the following we show that $\sigma$ in (3.4) is a quasi-isomorphism if $B$ is finite étale and monogenic over a localized polynomial algebra $A_f = k[T_1, \ldots, T_n]_f$.

Let $\tilde{f} \in \tilde{A} := W(k)[T_1, \ldots, T_n]$ be a lifting of $f$ and $\tilde{A}_f := W(k)[T_1, \ldots, T_n]_{\tilde{f}}$. $B$ lifts to a finite étale extension $\tilde{B}$ over $\tilde{A}_f$. If $B = A_f[x]$, then $\tilde{B} = \tilde{A}_f[x]$.

We write $u = [x]$ for the Teichmüller representative of $x$ in $W(B)$. Consider the canonical map

$$
\sigma : \tilde{B} \to W^1(B) = W^1(A_f)[u]
$$

which extends the canonical map $\tilde{A}_f \to W^1(A_f)$. The existence of $\sigma$ is derived from Hensel’s lemma [6] Proposition 2.30. Hence $B$ has a canonical overconvergent Witt lift. Let $\hat{B}_f, \hat{A}_f$ be the weak completions of $B, A_f$.

Then $\hat{B}_f = \hat{A}_f[x]$ is finite étale over $\hat{A}_f$. Using Proposition 1.9 we see that $\sigma$ extends to a comparison map

\begin{equation}
\sigma : \Omega_{\hat{B}^1/W(k)} = \hat{B}^1 \bigotimes_{\hat{A}_f} \Omega^*_{\hat{A}_f/W(k)} \to W^1\Omega^*_{B/k} = \bigoplus_{i=0}^{m-1} W^1\Omega^*_{A_f/k}x^i
\end{equation}

(here $m = [B : A_f]$).

We want to show that $\sigma$ is a quasi-isomorphism. First we treat the special case $B = A_f = k[T_1, \ldots, T_n]_f$. So we need to show:

$$
\sigma : \Omega_{\hat{A}_f^1/W(k)} \to W^1\Omega^*_{A_f/k} is a quasi-isomorphism,
$$

We also consider $\tilde{f}_l = \text{image} (\tilde{f})$ in $W_l(k)[T_1, \ldots, T_n] =: \tilde{A}_l$. The $\tilde{A}_l$-module structure in $W_l\Omega^*_{A_f/k}$ respects the decomposition

$$
W_l\Omega^*_{A_f/k} = W_l\Omega^*_{A_f/k}^{\text{int}} \oplus W_l\Omega^*_{A_f/k}^{\text{frac}}
$$

into integral and fractional part. This follows from [13] Lemma 4.

Hence we have a direct sum decomposition

\begin{equation}
W_l\Omega^*_{A_f/k} \cong \tilde{A}_l \left[ \frac{1}{\tilde{f}_l} \right] \bigotimes_{\tilde{A}_l} W_l\Omega^*_{A_f/k}
\end{equation}

\begin{equation}
\cong \tilde{A}_l \left[ \frac{1}{\tilde{f}_l} \right] \bigotimes_{\tilde{A}_l} W_l\Omega^*_{A_f/k}^{\text{int}} \bigoplus \tilde{A}_l \left[ \frac{1}{\tilde{f}_l} \right] \bigotimes_{\tilde{A}_l} W_l\Omega^*_{A_f/k}^{\text{frac}}
\end{equation}

where the first isomorphism follows from the étale base change and the isomorphism

$$
W_l(A) \bigotimes_{\tilde{A}_l} \tilde{A}_l \left[ \frac{1}{\tilde{f}_l} \right] \cong W_l(A_f).
$$
When taking inverse limits, we put
\[
\lim_{\leftarrow} \tilde{A}_f \left[ \frac{1}{\tilde{f}} \right] \bigotimes_{\tilde{A}_f} W_l \Omega^\bullet_{A/k} = \Omega^\bullet_{\tilde{A}_f},
\]
where \( \tilde{A}_f \) is the \( p \)-adic completion of \( A_f \). Then (3.6) yields a direct sum decomposition
\[
W\Omega^\bullet_{A_f/k} \cong W\Omega^\bullet_{A_f/k}^{\text{int}} \oplus W\Omega^\bullet_{A_f/k}^{\text{frac}}
\]
into two parts which we denote again by the integral and fractional part. We can identify \( W\Omega^\bullet_{A_f/k} \) with \( \Omega^\bullet_{\tilde{A}_f} \) and we know that \( W\Omega^\bullet_{A_f/k}^{\text{frac}} \) is acyclic.

With regards to \( W^\dagger \Omega^\bullet_{A_f/k} \) we apply Proposition 1.3 and the remark after Proposition 1.3:

Any \( z \in W^\dagger \Omega^\bullet_{A_f/k} \) can be written as a convergent series
\[
z = \sum_{l=0}^{\infty} \frac{1}{\tilde{f}^l} \eta_l
\]
where \( \eta_l \) is a finite sum of basic Witt differentials \( \eta_l^{(l)} \), such that there are real numbers \( C \) and \( \epsilon > 0 \) with
\[
\gamma_{\epsilon}(\eta_l) - \epsilon r_l \geq C.
\]
The supremum over all \( C \) for all possible representations of \( z \) is by definition \( \gamma_{\epsilon}(z) \), the Gauss norm on the localization.

We can also define an order function on \( W\Omega^\bullet_{A_f/k} \) by considering representations of \( z \) of the form
\[
z = \sum_{l=0}^{\infty} \frac{1}{\tilde{f}^l} \tau_l.
\]
We call \( z \) convergent with radius \( \epsilon \) with respect to \( \tilde{f} \) if there is a representation and a constant \( C \in \mathbb{R} \), such that
\[
\gamma_{\epsilon}(\tau_l) - \epsilon r_l \geq C.
\]
We denote the supremum over all \( C \) for all possible representations by \( \gamma_{\epsilon}(z) \). We will also express the last condition of convergence a little differently: We extend the function \( \gamma_{\epsilon} \) to \( W\Omega^\bullet_{A_f/k}[1/\tilde{f}] \) as follows:
\[
\tilde{\gamma}_{\epsilon}(\omega/\tilde{f}^k) = \gamma_{\epsilon}(\omega) - k \gamma_{\epsilon}(\tilde{f})
\]
If \( z = \sum z_l \) with \( z_l \in W\Omega^\bullet_{A_f/k}[1/\tilde{f}] \), and if we denote by \( k_l \) the denominator of \( z_l \) in this localization, it is easy to see that \( \gamma_{\epsilon}(\tilde{f}) \) is the supremum over all constants \( C \) such that for a suitable representation \( z = \sum z_\alpha \) we have
\[
k_\alpha \leq \frac{1}{\epsilon (1 + \deg \tilde{f})} (\tilde{\gamma}_{\epsilon}(z_\alpha) + C).
\]
We will prove that the notions of overconvergence and overconvergence with respect to \( \tilde{f} \) are the same. We start with representations (3.8) such
The Gauss norms $\gamma$ integral and fractional part of $\gamma$ just seen that $\epsilon$

Let $f$ \[ \epsilon \]

Finally we obtain

The last inequality was explained at the end of the proof of Corollary 0.13.

We give an estimation for each summand separately:

If we interchange the roles of $f$ and $\tilde{f}$ in the argument above we see that:

The Gauss norms $\gamma^{(\tilde{f})}$ are appropriate to study overconvergence on the integral and fractional part of $W\Omega_{A_f/k}$ separately. More precisely let $z \in W\Omega_{A_f/k}$ and let $z = z_1 + z_2$ according to the decomposition (3.7). We have just seen that $\gamma_\epsilon(z) > -\infty \iff \gamma^{(\tilde{f})}_\epsilon(z) > -\infty$ for small $\epsilon$. We claim that

Let $\gamma^{(\tilde{f})}_\epsilon(z) \geq C$ then there exists a representation

such that

\[ \gamma_\epsilon(\eta) - \epsilon r_l \geq C. \]
Let $\tau_1 = \tau_1^1 + \tau_1^2$ be the decomposition in integral and fractional part. Then

$$z_1 = \sum_{l=0}^{\infty} \frac{1}{f_l} \tau_1^1$$

and

$$z_2 = \sum_{l=0}^{\infty} \frac{1}{f_l} \tau_1^2.$$  

As $\gamma(\tau_1) = \min\{\gamma(\tau_1^1), \gamma(\tau_1^2)\}$ the claim follows. Hence we obtain a direct sum decomposition

$$W_{t+1}^1 \Omega_{A_f/k}^* = W_{t+1}^1 \Omega_{A_f/k}^{\text{int}} \bigoplus W_{t+1}^1 \Omega_{A_f/k}^{\text{frac}}$$

We will also consider the truncated Gauss norms $\gamma_{\epsilon}^{(f)}[t]$ on

$$\tilde{A}_{t+1} \left[ \begin{array}{c} 1 \\ f \end{array} \right] \otimes \tilde{A}_{t+1} W_{t+1} \Omega_{A/k}^*$$

as the set of finite sums $\sum_k \eta_k \frac{f^k}{f_k} \in W_{t+1} \Omega_{A_f/k}^{\text{frac}} = \tilde{A}_{t+1} \left[ \begin{array}{c} 1 \\ f \end{array} \right] \otimes \tilde{A}_{t+1} W_{t+1} \Omega_{A/k}^{\text{frac}}$ satisfying the following. Let $K_0$ be the largest integer divisible by $p$ such that

$$K_0 \leq \frac{1}{\epsilon(1 + \deg f)} \left( \tilde{\gamma}_{\epsilon} \left( \frac{w}{f^k} \right) + C \right).$$

Then we require the following two conditions:

(i) $K_0 \geq 0$

(ii) $k \leq K_0$.

We know that the complex $\tilde{A}_{t+1} \left[ \begin{array}{c} 1 \\ f \end{array} \right] \otimes W_{t+1} \Omega_{A/k}^{\text{frac}}$ is acyclic. We show that for $\epsilon > 0$ sufficiently small $W_{t+1} \Omega_{A_f/k}^{\text{frac},e,C}$ is acyclic.

Let us assume that $f$ is regular in the variable $T_1$. Let $c \in A_f$. Then $c$ has a unique reduced representation:

$$c = \sum_l a_l f^l,$$

where $a_l \in A$. We write $a = \sum \alpha_k T^k \in A$, with $\alpha_k \in k$, and we set $\tilde{a} = \sum [\alpha_k] T^k \in W(A)$. Then we define

$$\tilde{c} = \sum_l \tilde{a}_l f^l.$$  

This is an integral element in $W(A_f)$. In the following we consider still another admissible Gauss norm on $W_{t+1} \Omega_{A_f/k}$. Let $\omega \in W_{t+1} \Omega_{A_f/k}$. Then we consider all possible expression of the type:

$$\omega = \sum_l \eta_l f^l,$$

where $\eta_l \in W_{t+1} \Omega_{A_f}$. 

We forget our old notation and denote by $\gamma'[t](\omega)$ the maximum over all possible numbers

$$
\min\{\gamma[t](\eta) - \varepsilon lp\}.
$$

It is easy to see that the condition $\gamma'[t](\omega) \geq C$ for $\omega \in W_{t+1}\Omega_{A_f/k}^{\text{frac}}$ is equivalent to condition $\omega \in W_{t+1}\Omega_{A_f/k}^{\text{frac},C}$.

We should remark that $\gamma'[1]$ coincides with the formerly defined function.

As before we define a modified $\hat{\gamma}'[t]$. Then we have $\hat{\gamma}'[t] = \gamma'[f][t]$

We find the equalities:

$$
\hat{\gamma}'[t](\hat{c}) = \hat{\gamma}'(c), \quad \gamma'[t](\hat{c}) = \gamma'(c).
$$

Indeed we verify the first equation as follows: By the representation (3.13) we find:

$$
\gamma'[t](\hat{c}) \geq \min\{\gamma[c](\hat{a}) - \varepsilon l\} = \min\{\gamma[c](a) - \varepsilon l\} = \gamma'(c) = \gamma'[1](\hat{c}).
$$

The other inequality is obvious.

**Lemma 3.14.** Each $\omega \in W_{t+1}\Omega_{A_f/k}$ has a unique representation:

$$
\omega = \sum Fc_i^{(n)}(\omega_i^{(n)}).
$$

This decomposition respects the non integral and the integral part, i.e. if $\omega$ is integral (resp. non integral) then all $\omega_i^{(n)}$ are integral (respectively non integral). For the Gauss norm we have:

$$
\gamma'[t](\omega) = \min\{p\hat{\gamma}'(\epsilon_i^n) + \gamma(\omega_i^{(n)})\}.
$$

**Proof.** The same as that of Proposition 2.20: The Lemmas 2.11 and 2.13 continue to hold with $F\hat{c}_i^{(n)}$ in place of $F[c_i^{(n)}]$, because the action of both elements is the same on the graded part $G^n$. We need to verify that for fixed $n$:

$$
\gamma'[t]\left(\sum_i Fc_i^{(n)}(\omega_i^{(n)})\right) = \min\{\gamma'[t](\sum_i F\hat{c}_i^{(n)}(\omega_i^{(n)}) + \gamma(\omega_i^{(n)}))\}
$$

$$
= \min\{p\hat{\gamma}(c_i^{(n)}) + \gamma(\omega_i^{(n)})\}.
$$

It is clear from Lemma 2.13 that this is true for $\gamma'[n]$ in place of $\gamma'[t]$. We choose reduced representations:

$$
c_i^{(n)} = \sum_l a_i^{(n)} / f^l.
$$

Then we find:

$$
\gamma'[t]\left(\sum_i Fc_i^{(n)}(\omega_i^{(n)})\right) = \gamma'[t]\left(\sum_i (\sum_l F\hat{a}_i^{(n)}(\omega_i^{(n)}) / \hat{f}^l)\right).
$$

From this we see that:

$$
\gamma'[t]\left(\sum_i Fc_i^{(n)}(\omega_i^{(n)})\right) \geq \min\{\gamma[c](\sum_l (F\hat{a}_i^{(n)}(\omega_i^{(n)}))) - \varepsilon lp\}
$$

$$
= \min\{\gamma[c](\sum_l (F\hat{a}_i^{(n)}(\omega_i^{(n)}))) + \gamma(\omega_i^{(n)}) - \varepsilon lp\}
$$

$$
= \min\{\gamma'[t](\sum_l (F\hat{c}_i^{(n)}(\omega_i^{(n)}))) + \gamma(\omega_i^{(n)})\}.
$$

This shows the equation (3.16) because $\gamma'[t] \leq \gamma'[n]$. The rest of the proof of the lemma is the same. \qed
Proposition 3.17. Let \( \varepsilon \in \mathbb{R} \) be sufficiently small. Let \( \omega \in W_{t+1} \Omega_{A_f/k} \) be a closed Witt differential in the non integral part such that \( \gamma'_\varepsilon(\omega) \geq C \). Then \( \omega = d\eta \), where \( \eta \in W_{t+1} \Omega_{A_f/k} \) is a Witt differential in the non integral part, such that \( \gamma'_\varepsilon(\eta) \geq C \).

Proof. The problem does not change if we make a finite extension of the base field \( k \), Therefore we may assume that \( f \) is regular in \( T_1 \) as above.

Consider the residue class \( \bar{\omega} \in W_2 \Omega_{A_f/k} \) of \( \omega \). This is a closed form in the fractional part, i.e. is contained in the module:

\[
(dV^n P^{l-1} f) \oplus (pdV^{n-1} P^{l-1} f) \oplus \cdots \oplus (p^{n-1} dVP^1 f)
\]

for \( n = 2 \). This means that all basic Witt differentials \( \omega_i^{(1)} \), which appear in the decomposition (3.15) must be of the form \( \omega_i^{(1)} = d\eta_i^{(1)} \) for some primitive basic Witt differential \( \eta_i^{(1)} \), such that \( \gamma'_\varepsilon(\omega_i^{(1)}) = \gamma'_\varepsilon(\eta_i^{(1)}) \). We set:

\[
\eta(1) = \sum F c_i^{(1)} \eta_i^{(1)}.
\]

Clearly \( \gamma'_\varepsilon(\eta(1)) = \min\{p\gamma'_\varepsilon(c_i^{(1)}) + \gamma'_\varepsilon(\omega_i^{(1)})\} \geq \gamma'_\varepsilon(\omega) \).

We will verify that for small \( \varepsilon \):

\[
(3.18) \quad \gamma'_\varepsilon(d\eta(1)) \geq \gamma'_\varepsilon(\omega).
\]

Then we consider \( \omega(1) = \omega - d\eta(1) \). We conclude that \( \gamma'_\varepsilon(\omega(1)) \geq \gamma'_\varepsilon(\omega) \) and that \( \omega(1) \in \tilde{F}^2 W_{t+1} \Omega_{A_f/k} \). Then we expand \( \omega(1) \) in the form (3.15) and consider the reduction in \( W_2 \Omega_{A_f/k} \). We apply the same argument and find \( \eta(2) \) with \( \gamma'_\varepsilon(\eta(2)) \geq \gamma'_\varepsilon(\omega(1)) \) and \( \gamma'_\varepsilon(d\eta(2)) \geq \gamma'_\varepsilon(\omega(1)) \). Continuing we obtain:

\[
\omega = d\eta(1) + d\eta(2) + d\eta(3) + \ldots.
\]

This proves the result if we verify (3.18).

We set \( C = \gamma'_\varepsilon(\omega) \). By definition \( F c_i^{(n)} \) is a sum of expressions \( [u]^p / \tilde{f}^l \) such that:

\[
p\gamma'_\varepsilon([u]) - \varepsilon lp + \gamma'_\varepsilon(\eta_i^{(1)}) \geq C.
\]

Here \( u \) is a monomial in the variables \( T \). We have to verify that

\[
\gamma'_\varepsilon(d([u]^p \eta_i^{(1)}/\tilde{f}^l \tilde{f})) \geq C.
\]

We write:

\[
d([u]^p \eta_i^{(1)}/\tilde{f}^l \tilde{f}) = (d([u]^p \eta_i^{(1)})/\tilde{f}^l \tilde{f}^l) + lp([u]^p \eta_i^{(1)} \tilde{f}^l \tilde{f}^l - lp([u]^p \eta_i^{(1)} \tilde{f}^{l-1} \tilde{f})/\tilde{f}^{(l+1)p}.
\]

Clearly \( \gamma'_\varepsilon \) of the first summand is greater than \( C \). We have:

\[
\gamma'_\varepsilon(p[u]^p \eta_i^{(1)} \tilde{f}^{l-1} \tilde{f}) \geq p\gamma'_\varepsilon([u]) - \varepsilon (l+1)p + \gamma'_\varepsilon(\omega_i^{(1)}) + p\gamma'_\varepsilon(\tilde{f}) + 1.
\]

The last expression is bigger \( C \) if

\[
p\gamma'_\varepsilon(\tilde{f}) + 1 - p\varepsilon \geq 0.
\]

But this is clearly fulfilled for small \( \varepsilon \).  

Hence \( W_{t+1} \Omega_{A_f/k}^{\text{frac}, \varepsilon,C} \) is acyclic. As the notions of overconvergence on \( W \Omega_{A_f/k} \) and overconvergence with respect to \( \tilde{f} \) are the same we can apply the remark preceding Corollary 2.24. We see that the complex \( W^{1} \Omega_{A_f/k}^{\text{frac}, \varepsilon} \)
consisting of elements $\omega \in \Omega_{A_f/k}^{\text{frac}}$ satisfying $\gamma'_\epsilon[t](\omega) \geq C$ for some $C$ independently of $t$ is exact as well. Hence

$$W^1\Omega_{A_f/k}^{\text{frac}} = \lim_{\epsilon \to 0} W^1\Omega_{A_f/k}^{\text{frac},\epsilon}$$

is exact, as desired.

Now we can prove the following comparison result.

**Theorem 3.19.** Let $f \in k[T_1, \ldots, T_d] = A$. Let $B$ be finite étale and monogenic over $A_f$.

Then the map $\sigma$, explicitly given in 3.5, of complexes

$$\sigma : \Omega_{B/W(k)} \cong W^1\Omega_{B/k}$$

is a quasi-isomorphism.

**Proof.** We consider a lift $\tilde{A}_f$ of $A_f$ over $W(k)$ and a finite monogenic étale algebra $\tilde{B}$ over $\tilde{A}_f$ which lifts $B$. We write $\tilde{B} = \tilde{A}_f[x]$.

We have the isomorphism of modules (not of complexes):

$$W\Omega_{B/k} = \tilde{B} \otimes_{\tilde{A}_f} W\Omega_{A_f/k} = \bigoplus_{i=0}^{m-1} x^i W\Omega_{A_f/k}.$$  

Let $\gamma'_\epsilon$ be the of Gauss norms on $W\Omega_{A_f/k}$ considered in Lemma 3.14. We consider the product norms on the right hand side of 3.20. We write $\omega \in W\Omega_{B/k}$:

$$\omega = \sum \eta_j x^{jp}.$$  

Then we set

$$\gamma(\omega) = \min\{\gamma'(\eta_j)\}.$$  

According to (3.20) we find:

$$dx^{jp} = pix^{ip-1}dx = \sum_{j=0}^{m-1} x^{jp} \partial_{ij},$$

where the $\partial_{ij} \in \Omega_{A_f/W(k)} \subset W\Omega_{A_f/k}$ are integral differentials. We restrict our attention to small $\epsilon$. Then we may assume that

$$\gamma'(\partial_{ij}) > 0.$$  

This is possible because the $\partial_{ij}$ are divisible by $p$ and $\gamma(\epsilon)p = 1$. The last assumption ensures that

$$\gamma(\partial \omega) \geq \gamma(\omega).$$
We define the fractional part of $W\Omega_{B/k}$:

$$W\Omega_{B/k}^{\text{frac}} = \tilde{B} \otimes_{\tilde{A}} W\Omega_{A_{J/k}}^{\text{frac}}.$$  

This is a subcomplex of $W\Omega_{B/k}$. We denote by $W^{\dagger}\Omega_{B/k}^{\text{frac}}$ the overconvergent differentials in $W\Omega_{B/k}^{\text{frac}}$. By the decompositions (3.5), (3.7), and (3.11), it remains to show that this complex of overconvergent fractional differentials is acyclic.

From (3.20) we obtain decompositions for the filtrations:

(3.22) $\text{Fil}^{n} W\Omega_{B/k}^{\text{frac}} = \bigoplus_{j=0}^{m-1} x^{jp} \text{Fil}^{n} W\Omega_{A_{J/k}}^{\text{frac}}$.

Consider a closed overconvergent Witt differential $\omega \in W\Omega_{B/k}^{\text{frac}}$:

$$d\omega = 0, \quad \gamma_{\epsilon}(\omega) \geq -C.$$  

We will show that $\omega = d\eta$ for $\eta \in W\Omega_{B/k}^{\text{frac}}$ with $\gamma_{\epsilon}(\eta) \geq -C$. This implies that the complex $W^{\dagger}\Omega_{B/k}^{\text{frac}}$ is acyclic.

We note that $\omega \in \text{Fil}^{1} W\Omega_{B/k}^{\text{frac}} = W\Omega_{B/k}^{\text{frac}}$. We set $\omega_{1} = \omega$. We construct inductively fractional differentials $\omega_{i}, \eta_{i} \in \text{Fil}^{i} W\Omega_{B/k}^{\text{frac}}$, such that $\gamma_{\epsilon}(\omega_{i}) \geq -C, \gamma_{\epsilon}(\eta_{i}) \geq -C$ and

$$\omega_{i} = \omega_{i+1} + d\eta_{i}.$$  

We consider $\omega_{i}$ modulo $\text{Fil}^{i+1} W\Omega_{B/k}^{\text{frac}}$, i.e. as an element of $\text{gr}^{i} W\Omega_{B/k}^{\text{frac}} \subset W_{i+1} \Omega_{B/k}^{\text{frac}}$. Then, using (3.22), we may write:

$$\omega_{i} = \sum x^{jp}(V^{i} \sigma_{j} + dV^{i} \rho_{j}).$$  

Since $\text{gr}^{i} W\Omega_{B/k}^{\text{frac}}$ is annihilated by $p$ we have

$$0 = d\omega_{i} = \sum x^{jp} dV^{i} \sigma_{j}.$$  

This shows that $V^{i} \sigma_{j} = 0$, for $j = 0, \ldots, m-1$. We find for the truncated norms:

$$\min\{\gamma'_{\epsilon}[i](dV^{i} \rho_{j})\} = \gamma_{\epsilon}[i](\omega_{i}) \geq -C.$$  

Using Proposition 3.17 we may assume after a possible modification of the $\rho_{j}$ that $\gamma'_{\epsilon}[i](V^{i} \rho_{j}) \geq -C$. We choose liftings $V^{i} \tilde{\rho}_{j} \in W\Omega_{A_{J/k}}^{\text{frac}}$, such that

$$\gamma'_{\epsilon}(V^{i} \tilde{\rho}_{j}) = \gamma'_{\epsilon}[i](V^{i} \rho_{j}) \geq -C.$$  

Since $d$ increases the product norm we find

$$\gamma(\sum x^{jp} V^{i} \tilde{\rho}_{j}) \geq \gamma(\sum x^{jp} V^{i} \tilde{\rho}_{j}) \geq -C.$$  

We set

$$\eta_{i} = \sum x^{jp} V^{i} \tilde{\rho}_{j}, \quad \omega_{i+1} = \omega_{i} - d\eta_{i}.$$  

This ends the induction and the proof of the proposition. □
For an arbitrary smooth algebra $A$, consider an overconvergent Witt lift
\[(3.23) \quad \psi : \tilde{A}^\dagger \to W^\dagger(A)\]
which is uniquely determined by a lifting of the Frobenius to $\tilde{A}^\dagger$. (Compare Proposition 3.2.) It induces a map of complexes, also denoted by $\psi$,
\[\psi : \Omega_{\tilde{A}/W(k)} \to W^\dagger \Omega_A/k.\]
Passing to cohomology we will prove the following comparison result.

**Proposition 3.24.** Let $\kappa = \lfloor \log_p \dim A \rfloor$. Then the kernel and cokernel of the induced homomorphism
\[\psi_* : H^i(\Omega_{\tilde{A}/W(k)}) \to H^i(W^\dagger \Omega_A/k)\]
are annihilated by $p^{2\kappa}$.

**Corollary 3.25.** (a) Let $\dim A < p$. Then $\psi_*$ is an isomorphism.
(b) In general, there is a (rational) isomorphism
\[H^*_\text{MW}(A/K) \cong H^*(W^\dagger \Omega_A/k \otimes_{W(k)} K)\]
between Monsky-Washnitzer cohomology and overconvergent de Rham-Witt cohomology. (Here $K = W(k)[1/p]$.)

We will reduce the proof of the proposition to a local homotopy argument. The map $\psi$ induces a map of complexes of Zariski sheaves on $\text{Spec } A$:
\[\tilde{\psi} : \tilde{\Omega}_{\tilde{A}/W(k)} \to W^\dagger \Omega_{\text{Spec } A/k}.\]
As $H^i_{\text{Zar}}(\text{Spec } A, \tilde{\Omega}_{\tilde{A}/W(k)}^d) = H^i_{\text{Zar}}(W^\dagger \Omega_{\text{Spec } A/k}^d) = 0$ for all $d \geq 0$ and all $i > 0$ (Proposition 1.2 and [15] Lemma 7), we have
\[R\Gamma(\text{Spec } A, \tilde{\Omega}^*_{\tilde{A}/W(k)}) = \Omega^*_{\tilde{A}/W(k)} \quad \text{and} \quad R\Gamma(\text{Spec } A, W^\dagger \Omega_{\text{Spec } A/k}) = W^\dagger \Omega_A/k,\]
hence we can reconstruct $\psi$ from $\tilde{\psi}$ by applying $R\Gamma(\text{Spec } A, .)$. Let $\{U_j\}_j$ be a finite affine covering of $\text{Spec } A$ such that each $U_j$ is finite étale and monogenic over a localized polynomial algebra. By a result of Kedlaya [11], such a covering always exists. Let $U_j = \text{Spec } B_j$ and $\tilde{B}_j^\dagger$ the Monsky-Washnitzer lift of $B_j$. Then we consider the “localization” $\psi_j$ of $\psi$ to $U_j$:
\[\psi_j : \Omega_{B_j^\dagger/W(k)} \to W^\dagger \Omega_{B_j/k}.\]

We compare the map $\psi_j$ with the explicitly given comparison map $\sigma$ in (3.5) from which we know it is a quasi-isomorphism and show the following.

**Proposition 3.26.** The maps $p^r \psi_j$ and $p^r \sigma$ are homotopic, hence induce the same map on cohomology.

Before proving the proposition we finish the proof of Proposition 3.24. We know that the kernel and cokernel of $(p^r \psi_j)_*$ are annihilated by $p^r$. As $\text{Ker}(\psi_j)_* \subseteq \text{Ker}(p^r \psi_j)_*$ and $\text{Coker}(\psi_j)_*$ is a subquotient of $\text{Coker}(p^r \psi_j)_*$, $\text{Ker}(\psi_j)_*$ and $\text{Coker}(\psi_j)_*$ are annihilated by $p^r$ as well.
Define $C^\bullet$ as the complex of Zariski sheaves obtained by taking the cokernel of $\tilde{\psi}$. Then one has an exact sequence of complexes of Zariski sheaves

$$0 \to \tilde{\Omega}^\bullet_{\text{Spec} \to A/k} \to W^1 \Omega^\bullet_{\text{Spec} \to C} \to C^\bullet \to 0.$$ 

The cohomology sheaves $H^i(C^\bullet)$ are annihilated by $p^{2\kappa}$. Hence the map $C^\bullet \overset{p^{2\kappa}}{\to} C^\bullet$ induces the zero map on cohomology. Therefore it is zero in the derived category. Applying the functor $R\Gamma$ we see that $R^i \Gamma(Spec \to A, C^\bullet) \overset{p^{2\kappa}}{\to} R^i \Gamma(Spec \to A, C^\bullet)$ is the zero map. This finishes the proof of Proposition 3.24.

We now prove Proposition 3.26. It is implied by the following more general result. Let $B, C$ denote smooth $k$-algebras which are finite and étale over localized polynomial algebras, with smooth lifts $\tilde{B}, \tilde{C}$ and corresponding weak completions $\tilde{B}^\dagger, \tilde{C}^\dagger$.

**Proposition 3.27.** Let $\tilde{\phi}_1, \tilde{\phi}_2 : \tilde{B}^\dagger \to W^1(C)$ denote two lifts of a map $\phi : B \to C$. Then the induced maps

$$p^\kappa \tilde{\phi}_1, p^\kappa \tilde{\phi}_2 : \Omega^e_{\tilde{B}} \to W^1 \Omega^e_{C}$$

are chain homotopic, where $\kappa = [\log_p \dim B]$.

We will closely follow the argument on pages 205-206 of [16].

**Proof.** The chain homotopy we produce will factor through the following algebra.

**Definition 3.28.** Denote by $D^\nu(C)$ the differential graded algebra with ith graded piece

$$D^\nu(C)^i = W^i \Omega_{C/k}^i \oplus W^i \Omega_{C/k}^{i-1} [U] \wedge dU.$$ 

Denote by $D^\nu(C)$ the sub-differential graded algebra of $D^\nu(C)$ generated in degree zero by terms

$$f = \sum_{i=0}^\infty U^i \omega_i$$

for which $\omega_i \in p^{i-1} V W^1(C)$ for $i \geq 1$ and such that there exist $\varepsilon, G$ with $\gamma_\varepsilon(\omega_i) \geq G$ for all $i$. For such a term $f$, we define

$$\gamma_\varepsilon(f) = \inf_i \{\gamma_\varepsilon(\omega_i)\}.$$ 

Note that $D^\nu(C)^0$ is an algebra. The only non-obvious fact is that it is closed under multiplication, and this follows from the property $V(w_a) V(w_b) = p V(w_c)$.

We now define a map

$$\varphi : \Omega_{\tilde{B}/W(k)} \to D^\nu(C)$$

as follows. Fix a presentation

$$\tilde{B}^\dagger = W(k)[x_1, \ldots, x_n, \frac{1}{g}]^\dagger[z]/(P(z)).$$

Our map will send

$$\varphi : x_i \mapsto \tilde{\phi}_1(x_i) + U(\tilde{\phi}_2(x_i) - \tilde{\phi}_1(x_i)).$$
Because we have for \( a, b \in D'(C)^0 \), \( \gamma_e(ab) \geq \gamma_e(a) + \gamma_e(b) \) and \( \gamma_e(a + b) \geq \min(\gamma_e(a), \gamma_e(b)) \), the proof of Proposition 2.28 in [6] can be mimicked to show that \( D'(C)^0 \) is weakly complete. This immediately shows that \( \varphi \) extends to \( W(k)\langle x_1, \ldots, x_n \rangle \).

As \( g \in W(k)\langle x_1, \ldots, x_n \rangle \), we have just shown \( \varphi(g) \in D'(C) \), and we must show this element is invertible. Write \( \varphi(g) = \tilde{\phi}_1(g) + Uf \), some \( f \) such that \( Uf \in D'(C) \). Because \( \tilde{\phi}_1(g) \) is invertible in \( W(1) \),

\[
\frac{1}{\varphi(g)} = \frac{\tilde{\phi}_1(g)^{-1}}{1 - U(-\tilde{\phi}_1(g)^{-1}f)},
\]

so to show \( \varphi(g) \) is invertible it suffices to show that any \( 1 - U\tilde{g} \in D'(C)^0 \) is invertible. Write

\[
\tilde{g} = Vw_0 + UpVw_1 + U^2p^2Vw_2 + \cdots.
\]

It follows by a simple induction on \( k \), starting with the base case \( k = 1 \), that

\[
\tilde{g}^k = \sum_{i=0}^{\infty} U^i p^{k+i-1} Vw_i,
\]

with \( \gamma_e(p^{k+i-1} Vw_i) \geq 0 \), same \( \varepsilon \) as above. Hence

\[
1 + U\tilde{g} + U^2\tilde{g}^2 + \cdots \in D'(C),
\]

as required.

Next we prove that \( \varphi \) extends to \( z \).

**Lemma 3.29.** There exists

\[
\sum_{i=0}^{\infty} U^i c_i \in D'(C)
\]

which is a root of \( \varphi(P)(z) = z^r + \varphi(f_1)z^{r-1} + \cdots + \varphi(f_r) \).

**Proof.** Because \( D'(C) \) is weakly complete (with respect to \( (p) \)), by Hensel’s Lemma (Proposition 2.30 in [6]) it suffices to find a root modulo \( p \). Because the ideal \( (U^2) \subseteq (p) \), it will suffice for us to find a root modulo \( U^2 \). Thus we need only find the terms \( c_0 \) and \( c_1 \). As usual, \( c_0 = \tilde{\phi}_1(z) \). For \( c_1 \), we simply set \( z = \sum_{i=0}^{\infty} U^i c_i \) in \( \varphi(P)(z) = z^r + \varphi(f_1)z^{r-1} + \cdots + \varphi(f_r) = 0 \) and check that this forces

\[
c_1 = -\left(\tilde{\phi}_1(P)'(z)\right)^{-1} \left( (\tilde{\phi}_2(f_1) - \tilde{\phi}_1(f_1))c_0^{r-1} + \cdots + \tilde{\phi}_2(f_r) - \tilde{\phi}_1(f_r) \right).
\]

\( \square \)

We have now shown the existence of a map \( \varphi : \overline{B}^l \to D'(C)^0 \). We extend it to a map, also denoted by \( \varphi \), of complexes,

\[
\varphi : \Omega_{\overline{B}^l/W(k)} \to D'(C).
\]

The chain homotopy promised in our proposition will factor through its image. This motivates the following.

**Definition 3.30.** Let \( D(C) \subseteq D'(C) \) denote the image of \( \varphi \).

We give now a more explicit description of what terms in \( D(C) \) look like.
Lemma 3.31. (i) Let \( x \) denote some element of \( \Omega^d_{B^1/W(k)} \). Write
\[
\varphi(x) = \cdots + U^{i+1}w' + U^i dU w'' + \cdots
\]
where \( i \geq 0 \). Then we may write \( w' = p_{\max(i-d,0)} \mu_i \) and \( w'' = p_{\max(i-d+1,0)} \eta_i \) with \( \mu_i, \eta_i \in \fil^1 W^1 \Omega_{C/k} \).

(ii) We may find \( \epsilon, G \) depending only on \( x \) such that \( \gamma_{\epsilon}(w) \geq G \) for each coefficient \( w \).

Proof. (i) We prove this by induction on \( d \). The base case \( d = 0 \) has already been shown.

Inductively assume the result for \( x \) of degree \( d - 1 \).

A term \( x \) in degree \( d \) may be written as a finite sum of terms \( bdx_{i_1} \cdots dx_{i_d} \) with \( b \in \tilde{B}^1 \) and \( x_{i_j} \) one of the generators of the polynomial algebra of which we have taken an étale extension. We will show the result for \( bdx_1 \cdots dx_d \). Extending to other index sets is trivial, and extending to finite sums is easy.

We are assuming the result for \( \varphi(bdx_1 \cdots dx_{d-1}) \), which is possibly just \( \varphi(b) \). And we know
\[
\varphi(dx_d) = d\tilde{\phi}_1(x_d) + dUV(w_d) + UdV(w_d).
\]
The result concerning the form of the coefficients now follows easily.

(ii) We again may restrict to the case of a term \( bdx_{i_1} \cdots dx_{i_d} \). Concerning \( \varphi(b) \), we already know the result. There are only finitely many nonzero terms of the form \( dx_{i_1} \cdots dx_{i_d} \) (varying \( d \) allowed). Thus we can find \( \epsilon', G' \) such that every coefficient \( w \) appearing in some term \( \varphi(dx_{i_1} \cdots dx_{i_d}) \) satisfies \( \gamma_{\epsilon'}(w) \geq G' \). The result now follows from the fact that there exist \( \epsilon'', G'' \) such that for any \( \gamma_{\epsilon}(\eta) \geq G, \gamma_{\epsilon'}(w) \geq G' \) we have \( \gamma_{\epsilon''}(\eta \land w) \geq G'' \).

Let \( h_0, h_1 \) denote the maps of differential graded algebras \( D'(C) \to W^1 \Omega_{C/k} \) which send \( U \mapsto 0 \) and \( U \mapsto 1 \), respectively. Our definition of \( D'(C)^0 \) immediately implies that the image in degree zero really does land in \( W^1(C) \), and hence the image lands there in every degree. We also let \( h_0, h_1 \) denote their restrictions to \( D(C) \).

Clearly we have \( h_0 \circ \varphi = \tilde{\phi}_1 \) and \( h_1 \circ \varphi = \tilde{\phi}_2 \), because both sides agree in degree zero. We define \( p^\kappa L : D(C)^{\bullet} \to W^1 \Omega_{C/k}^{\ast-1} \) by setting
\[
p^\kappa L(U^j \omega_j) = 0 \quad \text{and} \quad p^\kappa L(U^j dU \land \omega_j) = \frac{p^\kappa \omega_j}{j+1},
\]
and then extending to all of \( D(C) \) in the obvious way. Of course, it is not at all clear that our map has image where we claim.

Lemma 3.32. The map \( p^\kappa L \) has image in \( W^1 \Omega_{C/k} \).

Proof. We first show it maps to \( W \Omega_{C/k} \), and then establish overconvergence. For an arbitrary \( x \in \Omega_{B^1/W(k)}^{\ast} \), write
\[
\varphi(x) = \cdots + U^j dU \land \omega_j + \cdots
\]
as in the previous lemma. From the lemma, it suffices that
\[
\kappa + \max(j - \dim B + 1, 0) \geq \lceil \log_2(j + 1) \rceil.
\]
For the case \( j - \dim B + 1 > 0 \), check the specific case \( j = \dim B \), then note that the left hand side grows faster with \( j \) than the right hand side. For the
Proof. It suffices to prove this for the equivalent norm $j\leq \dim B-1$, we want to prove $[\log_p \dim B] \geq [\log_p(j+1)]$, which in this case is obvious.

Now we must check overconvergence. We are done if we verify the existence of $\varepsilon', G'$ independent of $j$ such that $\gamma_{\varepsilon'}(\frac{\omega_{\varepsilon'}(x)}{x+1}) \geq G'$. For arbitrary $\omega_j \in W^1 \Omega_{C/k}$ with $\gamma_{\varepsilon}(\omega_j) \geq G$ this is not true. But as before we know that

$$p^{m'} \mid \frac{p^{\omega_j}}{j+1},$$

where $m' \geq j - \dim B + \kappa + 1 - [\log_p(j+1)]$.

There exists $N$ depending only on $\dim B$ such that for $j \geq N$, $m' \geq [\log_p(j+1)]$. So the following claim applies to all but finitely many terms in $\varphi(x)$.

**Claim.** Let $\omega_j \in W^1 \Omega_{C/k}$. If $p^{[\log_p(j+1)]} \mid \frac{p^{\omega_j}}{j+1}$ and $\gamma_{\varepsilon}(p^{\nu}(\omega_j)) \geq G$, then there exist $\varepsilon', G'$ depending only on $\varepsilon, G$ with $\gamma_{\varepsilon'}(\frac{\omega_{\varepsilon'}(x)}{x+1}) \geq G'$.

**Proof.** It suffices to prove this for the equivalent norm $\gamma'$ of page 28. We shall prove the result for $(\varepsilon', G') = \left(\frac{\varepsilon}{2}, \frac{G}{2}\right)$. Let $l := \log_p(j+1)$. Pick an $\eta$ such that $p^{2l} \eta = p^{\nu}(\omega_j)$. Write $C:= \gamma_{\varepsilon'}(\eta)$. From Corollary 2.24 or rather its evident generalisation to finite étale extensions over $A_f$, we know $\gamma_{\varepsilon'}(p^{2l} \eta) = C + 2l$, so from our assumption $C + 2l \geq G$. We also have $\gamma_{\varepsilon} (\eta) \geq \frac{G}{2}$, and so

$$\gamma_{\varepsilon'}(\frac{\omega_{\varepsilon'}(x)}{x+1}) \geq \frac{C}{2} + l \geq \frac{G}{2},$$

as claimed. $\square$

This proves that for all but finitely many terms $a$ in $p^{\nu}L(\varphi(x))$, $\gamma_{\varepsilon'}(a) \geq \frac{G}{2}$. For the other terms $b$ in $p^{\nu}L(\varphi(x))$, we know $\gamma_{\varepsilon}(b(j+1)) \geq G$, with $j + 1 \leq N + 1$. Thus we can find $\varepsilon''$, $G''$ with $\gamma_{\varepsilon''}(a) \geq G''$ and $\gamma_{\varepsilon''}(b) \geq G''$ for all $a, b$ as above, which covers everything. This completes the proof that $p^{\nu}L(\varphi(x))$ is indeed overconvergent. $\square$

Now we are basically done. It is trivial to check that $p^{\nu}L$ is a homotopy between $p^{\nu}h_0$ and $p^{\nu}h_1$. Thus $p^{\nu}L \circ \varphi$ is a homotopy between $p^{\nu}h_0 \circ \varphi = \varphi_1$ and $p^{\nu}h_1 \varphi = \varphi_2$. For the convenience of the reader, we state explicitly the sign convention:

$$d(\omega \land \eta) = d\omega \land \eta + (-1)^i \omega \land d\eta,$$

where $\omega$ is in degree $i$. $\square$

### 4. Comparison with rigid cohomology

We will first define the comparison morphism with rigid cohomology in a local situation. Let $k$ be a perfect field, $W = W(k)$ the ring of Witt vectors and $K = W \otimes \mathbb{Q}$. Let $X = \text{Spec } A$ be a smooth affine scheme over $k$. Let $F = \text{Spec } B$ be a smooth scheme over $W(k)$ and

$$(4.1) \quad X \rightarrow F$$

be a closed immersion with comorphism $B \rightarrow A$. We call $(X, F)$ a special frame.

Assume we are given a ring homomorphism $\varphi : B \rightarrow W(A)$ which lifts the comorphism. A special frame with this extra structure is called a Witt frame.
Let $F^\wedge$ be the formal completion of $F$ in the special fibre. We denote by $|X|_{F^\wedge}$ the tubular neighbourhood in the sense of Berthelot [2]. We define a natural morphism:

\[
\Gamma(|X|_{F^\wedge},\mathcal{O}|_{X_{F^\wedge}}) \to W(A) \otimes \mathbb{Q}.
\]

Let $\hat{F}/X$ be the formal completion of $F$ along $X$. By [2] 1.1.4 (ii) the tubular neighbourhood $|X|_{F^\wedge}$ coincides with the rigid analytic space associated with the formal scheme $\hat{F}/X$. Let $I$ be the kernel of the homomorphism $B \to A$. We denote by $R$ the completion of $B$ in the ideal $I$. We have $\hat{F}/X = \text{Spf } R$.

The associated rigid analytic space is defined as follows: We choose a set of generators $f_1, \ldots, f_m$ of $I$. For a natural number $n$ we denote by $R^\wedge_n$ the $p$-adic completion of $R_n = R[T_1, \ldots, T_m]/(f_1^n - pT_1, \ldots, f_m^n - pT_m)$.

Then $R^\wedge_n \otimes \mathbb{Q}$ is a Tate algebra and we have by definiton

\[
\Gamma(|X|_{F^\wedge},\mathcal{O}^{rig}) = \varprojlim R^\wedge_n \otimes \mathbb{Q}.
\]

To define (4.2) it suffices to define a compatible system of maps

\[
R_n \to W(A).
\]

for $n$ large enough. The homomorphism $\varkappa$ maps $I$ to $VW A$. Since $W(A)$ is complete in the ideal $VW(A)$ the homomorphism $\varkappa$ extends to a morphism

\[
R \to W(A).
\]

Since $\varkappa(f_i) \in VW(A)$ for $i = 1, \ldots, m$ we obtain for $n \geq 2$:

\[
\varkappa(f_i^n) \in p^{n-1}VW(A).
\]

Since $p$ is not a zero divisor in $W(A)$ the element $(1/p)\varkappa(f_i^n) \in W(A)$ is well defined. Mapping $T_i$ to this element we obtain the desired compatible system of maps (4.3). This finishes the definition of (4.2).

This construction is clearly functorial in the following sense: Assume we have a second special frame (4.1)

\[
X_1 \to F_1 = \text{Spec } B_1,
\]

where $X_1 = \text{Spec } A_1$ is an affine scheme over $k$. Let $\varkappa_1 : B_1 \to W(A_1)$ be a Witt frame structure. Assume that we are given a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & F \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & F_1,
\end{array}
\]

and a compatible commutative diagram of Witt frames

\[
\begin{array}{ccc}
B_1 & \longrightarrow & W(A_1) \\
\downarrow & & \downarrow \\
B & \longrightarrow & W(A).
\end{array}
\]

\[
\varkappa_1 : B_1 \to W(A_1) \to W(A)
\]
The diagram (4.4) induces a morphism of formal schemes \( \hat{F}_1/X_1 \rightarrow \hat{F}/X \) and therefore a morphism of the tubular neighbourhoods \( |X_1[\hat{F}_1]| \rightarrow |X[\hat{F}]| \). Our construction gives a commutative diagram

\[
\begin{array}{ccc}
\Gamma(|X_1[\hat{F}_1], \mathcal{O})|X_1[\hat{F}_1]| & \longrightarrow & W(A) \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
\Gamma(|X[\hat{F}], \mathcal{O})|X[\hat{F}]| & \longrightarrow & W(A) \otimes \mathbb{Q}.
\end{array}
\]

(4.5)

Next we consider overconvergent Witt frames. Let \( X \) be an affine smooth scheme over \( k \). Let \( X = \text{Spec} \, A \), where \( A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \). This presentation induces a closed immersion \( X \rightarrow \mathbb{A}^n_k \). We set \( E = \mathbb{A}^n_W \) and \( P = \mathbb{P}^n_W \). We denote by \( Y \) the closure of \( X \) in \( P \). We obtain a frame for the definition of rigid cohomology:

(4.6)

\[
X \rightarrow Y \rightarrow \hat{P}.
\]

Here \( \hat{P} \) is the formal completion of \( P \) in the ideal generated by \( p \). We consider the tubular neighbourhood \( |X| \subseteq X[p] \) in the rigid analytic space \( \hat{P}_K \). In the following we omit the index \( \hat{P} \) because all tubular neighbourhoods are with respect to this formal scheme.

Assume we are given a homomorphism

(4.7)

\[
W[X_1, \ldots, X_n] \rightarrow W(A)
\]

which lifts the natural surjection \( k[X_1, \ldots, X_n] \rightarrow A \). We have seen that (4.7) induces a map:

(4.8)

\[
\Gamma(|X|, \mathcal{O}|X|) \rightarrow W(A) \otimes \mathbb{Q}.
\]

**Proposition 4.9.** Let us assume that the image of (4.7) is contained in \( W^\dagger(A) \). Let \( V \subseteq |X| \) be a strict tubular neighbourhood with respect to the frame (4.6). Then the map (4.8) induces a map:

\[
\Gamma(V, \mathcal{O}_V) \rightarrow W^\dagger(A) \otimes \mathbb{Q}.
\]

**Proof.** We choose liftings \( f_j \in W[X_1, \ldots, X_n] \) for \( j = 1, \ldots, m \) of the polynomials \( f_j \), such that \( \deg f_j = \deg f_j \). We set \( d_j = \deg f_j \).

We take homogeneous coordinates \( X_i = T_i/T_0 \) for \( i = 1, \ldots, n \). Consider the homogeneous polynomials for \( j = 1, \ldots, m \):

\[
F_j(T_0, \ldots, T_n) = T_0^{d_j} f_j(T_1/T_0, \ldots, T_n/T_0).
\]

We denote by \( \bar{F}_j \) the residue class modulo \( p \). Then \( Y \subseteq \mathbb{P}^n_k \) is given by the equations:

\[
\bar{F}_j(T_0, \ldots, T_n) = 0
\]

We write a point of \( (t_0, \ldots, t_n) \) of \( \hat{P}_K = P_K^\text{an} \) always in such a way that \( |t_i| \leq 1 \) for all \( i = 1, \ldots, n \) and such that we have equality for at least one index. The tubular neighbourhood of of \( Y \) is:

\[
|Y| = \{ (t_0, \ldots, t_n) \in \hat{P}_K \mid |F_j(t_0, \ldots, t_n)| < 1 \}.
\]

Let \( Z \subseteq Y \) denote the intersection of \( Y \) with the hyperplane \( \{ T_0 = 0 \} \). We have disjoint decompositions

\[
Y = X \sqcup Z, \quad |Y| = |X| \cup |Z|.
\]
We follow the notations of [2] 1.2. For $\lambda < 1$ we have

$$|Z| = |Y| \setminus \{t_0\} < \lambda.$$ 

Then $U_\lambda = Y[\{Z\}]$ is a strict neighbourhood of $X$. Let $E^{an} \subset P^{an} = (\hat{\mathbb{P}}_K)^{an}$ be the analytic variety associated to $\hat{\mathbb{P}}_K$. We have $U_\lambda \subset E^{an}$. If $B(0, 1/\lambda)$ denotes the closed ball of radius $1/\lambda$ around 0 in $E^{an}$ we can write

$$(4.10) \quad U_\lambda = Y[\cap B(0, 1/\lambda)].$$

We describe $Y \cap E^{an}$ in affine coordinates. Consider a point $(t_0, \ldots, t_n) \in P^{an}$ with $t_0 \neq 0$ and let $(x_1, \ldots, x_n)$ be the affine coordinates. We find:

$$1/|t_0| = \max\{1, |x_1|, \ldots, |x_n|\}.$$ 

Therefore the defining inequalities for $Y$ become

$$|f_j(x_1, \ldots, x_n)| < \max\{1, |x_1|^{d_j}, \ldots, |x_n|^{d_j}\},$$

for $j = 1, \ldots, m$.

We set

$$U'_\lambda = \{(x_1, \ldots, x_n) \in B(0, 1/\lambda) \mid |f(x_1, \ldots, x_n)| < 1\}.$$ 

We find $U'_\lambda \subset U_\lambda$. For $\eta < 1$ we set $U_{\lambda, \eta} = U_\lambda \cap \{Y\}_\eta ([2] (1.2.4)).$ We consider the following affinoid subset of $U_{\lambda, \eta}$:

$$U'_{\lambda, \eta} = \{(x_1, \ldots, x_n) \in B(0, 1/\lambda) \mid |f(x_1, \ldots, x_n)| \leq \eta\}.$$ 

We normalize the absolute value such that $|p| = 1/p$. For $\eta = p^{-1/r}$ the affinoid algebra of $U'_{\lambda, \eta}$ is

$$T = K\langle \lambda X_1, \ldots, \lambda X_n, T_1, \ldots, T_m \rangle / (f_1^{r} - pT_1, \ldots, f_m^{r} - pT_m).$$

It consists of all power series

$$p = \sum a_{I,J}X^{I}T^{J}, \quad a_{I,J} \in K,$$

such that $\lim_{|I| + |J| \to \infty} |a_{I,J}|(1/\lambda)^{|I|} = 0$. We have seen that there is a homomorphism $T \to W(A) \otimes \mathbb{Q}$ for large $r$. It maps the $X_i$ to $\eta_i \in W(A)$ according to (4.7). Clearly we have $f_j(\eta_1, \ldots, \eta_n) \in VW(A)$. We set

$$f_j(\eta_1, \ldots, \eta_n) = V\rho_j, \quad \text{for } j = 1, \ldots, m.$$ 

For $r \geq 3$ the variable $T_j$ is mapped to

$$(V\rho_j)^{r/p} = p^{r-2} V(\rho_j^r).$$

Then the power series $p$ is mapped to

$$(4.11) \quad \sum a_{I,J}p^{(r-2)\mid I\mid} (V(\rho_j^r))^{J}.$$ 

We have to show that this power series converges to an element in $W^r(A) \otimes \mathbb{Q}$. Almost all coefficients $a_{I,J}$ are in $W$. Therefore we may assume that all these coefficients are in $W$. Since $W^r(A)$ is a weakly complete $W(k)$-algebra we see immediately that the series (4.11) represents an element of $W^r(A)$.

Altogether we find a homomorphism

$$(4.12) \quad \Gamma(U_{\lambda, \eta}, \mathcal{O}_{U_{\lambda, \eta}}) \to \Gamma(U'_{\lambda, \eta}, \mathcal{O}_{U'_{\lambda, \eta}}) \to W^r(A) \otimes \mathbb{Q}.$$
Each strict neighbourhood $V$ of $|X|$ contains some $U_{\lambda,\eta}$ for $\eta$ sufficiently close to 1. Therefore the morphism (4.8) induces a map

$$\Gamma(V, \mathcal{O}_V) \to W^\dagger(A) \otimes \mathbb{Q}.$$ 

By the universality of the de Rham complex we obtain a map

$$(4.13) \quad \Gamma(\hat{P}_K, j^\dagger \Omega_{\hat{P}_K}) \to W^\dagger \Omega_{A/k} \otimes \mathbb{Q}.$$ 

We may generalize the comparison morphism to the following situation:

Let $B$ be a smooth algebra over $W$ and let $B \to A$ be an epimorphism. Assume we are given a lift of the epimorphism to

$$\gamma : B \to W^\dagger(A).$$

We call $(A, B, \gamma)$ an overconvergent Witt frame. We set $X = \text{Spec} A$, $F = \text{Spec} B$ and write the overconvergent Witt frame also in the form $(X, F, \gamma)$.

We choose a representation

$$(4.14) \quad W[X_1, \ldots, X_n] \to B.$$ 

We write $F \to E$ for the corresponding closed immersion of affine schemes. Let $Q$ be the closure of $F$ in $P$. We obtain a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Q \\
\downarrow & & \downarrow \\
Y & \longrightarrow & P \\
\end{array}
$$

We have a closed immersion

$$|Y|_Q = \hat{Q}_K \cap |Y|_P \to |Y|_P.$$ 

Then $U_{\lambda,\eta} \cap \hat{Q}_K$ are exactly the neighbourhoods “$U_{\lambda,\eta}$” with respect to the frame $X \to Y \to Q$. We obtain a closed immersion of affinoids

$$U_{\lambda,\eta} \cap \hat{Q}_K \to U_{\lambda,\eta},$$

and therefore an epimorphism

$$\Gamma(U_{\lambda,\eta}, \mathcal{O}_{U_{\lambda,\eta}}) \to \Gamma(U_{\lambda,\eta} \cap \hat{Q}_K, \mathcal{O}_{U_{\lambda,\eta} \cap \hat{Q}_K})$$

whose kernel is generated by the elements in the kernel of (4.14). Therefore the morphism

$$\Gamma(U_{\lambda,\eta}, \mathcal{O}_{U_{\lambda,\eta}}) \to W^\dagger(A) \otimes \mathbb{Q}$$

factors through a morphism

$$\Gamma(U_{\lambda,\eta} \cap \hat{Q}_K, \mathcal{O}_{U_{\lambda,\eta} \cap \hat{Q}_K}) \to W^\dagger(A) \otimes \mathbb{Q}.$$ 

We conclude as above that for each strict neighbourhood $V$ of $|X|_{\hat{Q}_K}$ we obtain a morphism

$$(4.15) \quad \Gamma(V, \mathcal{O}_V) \to W^\dagger(A) \otimes \mathbb{Q}.$$ 

As before we obtain the comparison morphisms for the frame $X \to Y \to \hat{Q}$

$$(4.16) \quad \Gamma(\hat{Q}_K, j^\dagger \Omega_{\hat{Q}_K}) \to W^\dagger \Omega_{A/k} \otimes \mathbb{Q}.$$
In the case where $B$ is a smooth lift of $A$ we obtain from [3] Cor. 1.7 that
\[
\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}) \cong R\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K})
\]
coincides with the Monsky-Washnitzer complex of the weak completion of $B$. In the general case we find always a smooth lifting $B'$ of $A$ and a commutative diagram
\[
\begin{array}{ccc}
B & \longrightarrow & B' \\
\downarrow & & \downarrow \\
A & \longrightarrow &
\end{array}
\]
With the obvious notation, this induces a commutative diagram
\[
\begin{array}{ccc}
\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}) & \longrightarrow & R\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}) \\
\downarrow & & \downarrow \\
\Gamma'(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}) & \longrightarrow & R\Gamma'(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}).
\end{array}
\]
We have seen that the vertical arrow on the right hand side is a quasi-isomorphism. We conclude that
\[
\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K}) \to R\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K})
\]
is an epimorphism in the derived category. We note that the object
\[
R\Gamma(\hat{Q}_K, j^!\Omega_{\hat{Q}_K})
\]
is well defined in the derived category of abelian groups. We denote it by
\[
R\Gamma_{\text{rig}}(X).
\]
The comparison morphism (4.16) commutes with base change in the following sense. With the notations of (4.14) assume that we are given a commutative diagram:
\[
\begin{array}{ccc}
X_1 & \longrightarrow & F_1 & \longrightarrow & E_1 & \longrightarrow & P_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_2 & \longrightarrow & F_2 & \longrightarrow & E_2 & \longrightarrow & P_2.
\end{array}
\]
We assume that $(X_1, F_1) \to (X_2, F_2)$ is a morphism of overconvergent Witt frames
\[
\begin{array}{ccc}
B_2 & \longrightarrow & W^\dagger(A_2) \\
\downarrow & & \downarrow \\
B_1 & \longrightarrow & W^\dagger(A_1),
\end{array}
\]
where $X_i = \text{Spec } A_i$ and $F_i = \text{Spec } B_i$ for $i = 1, 2$. We obtain a map of the affine analytic spaces $E_1^{an} \to E_2^{an}$. Let $U_{1,\lambda} \subset E_1^{an}$ and $U_{2,\lambda} \subset E_2^{an}$ be the sets defined before (4.10). By continuity we find for each $\lambda_2 > 0$ a $\lambda_1$ such that (4.17) induces a map $U_{1,\lambda_1} \to U_{2,\lambda_2}$. This implies that the inverse image of each strict neighbourhood $V_2$ of $|X_2|_{E_2}$ is a strict neighbourhood $V_1$ of $|X_1|_{E_1}$. 
We deduce a diagram

\[
\begin{align*}
\Gamma(V_2, O_{V_2}) & \longrightarrow W^\dagger(A_2) \\
\downarrow & \quad & \downarrow \\
\Gamma(V_1, O_{V_1}) & \longrightarrow W^\dagger(A_1).
\end{align*}
\]

(4.18)

The commutativity of this diagram is clear from (4.5). We obtain a commutative diagram of comparison morphisms (4.13):

\[
\begin{align*}
\Gamma(\hat{P}_2, J^1\Omega_{P_2,K}) & \longrightarrow W^\dagger\Omega_{A_2/k} \otimes \mathbb{Q} \\
\downarrow & \quad & \downarrow \\
\Gamma(\hat{P}_1, J^1\Omega_{P_1,K}) & \longrightarrow W^\dagger\Omega_{A_1/k} \otimes \mathbb{Q}.
\end{align*}
\]

(4.19)

The same functoriality follows then for the comparison morphism (4.16):

\[
\begin{align*}
\Gamma(\hat{Q}_2, J^1\Omega_{Q_2,K}) & \longrightarrow W^\dagger\Omega_{A_2/k} \otimes \mathbb{Q} \\
\downarrow & \quad & \downarrow \\
\Gamma(\hat{Q}_1, J^1\Omega_{Q_1,K}) & \longrightarrow W^\dagger\Omega_{A_1/k} \otimes \mathbb{Q}.
\end{align*}
\]

(4.20)

Let \( A \) be a smooth algebra over \( k \) as before. Let \( B \) be a smooth \( W(k) \)-algebra which lifts \( A \). Let \( \varphi : B \to W^\dagger(A) \) be a homomorphism which lifts \( B \to A \). Then we say that the overconvergent Witt frame \( (A, B, \varphi) \) is an overconvergent Witt lift. We set \( F = \text{Spec} B \) and choose an embedding in an affine space \( E \)

\[
X \to F \to E.
\]

Then we have defined a comparison morphism (4.16)

\[
R\Gamma_{\text{rig}}(X) \to W^\dagger\Omega_{A/k} \otimes \mathbb{Q}.
\]

(4.21)

**Proposition 4.22.** The comparison morphism (4.21) for overconvergent Witt lifts is an isomorphism in the derived category. It is independent of the embeddings \( X \to F \to E \) and of the overconvergent Witt lift \( B \to W^\dagger(A) \).

**Proof.** Let \( (A, B', \kappa') \) be a second overconvergent Witt lift. We choose an embedding \( F' = \tilde{B}' \to E' \) in an affine space. We set \( B" = B \otimes_{W(k)} B' \) and with the embedding \( F" = F \times F' \to E" = E \times E' \). We obtain a overconvergent Witt frame \( B" \to W^\dagger(A) \) by taking the product of the overconvergent Witt lifts for \( B \) and \( B' \).

The independence of (4.21) follows from the commutative diagram

\[
\begin{align*}
\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) & \quad R\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) \quad W^\dagger\Omega_{A/k} \otimes \mathbb{Q} \\
\downarrow & \quad \downarrow & \quad \downarrow \\
\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) & \quad R\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) \quad W^\dagger\Omega_{A/k} \otimes \mathbb{Q} \\
\downarrow & \quad \downarrow & \quad \downarrow \\
\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) & \quad R\Gamma(\hat{Q}_K, J^1\Omega_{Q_K}) \quad W^\dagger\Omega_{A/k} \otimes \mathbb{Q}
\end{align*}
\]

(4.23)

where the vertical arrows are isomorphisms.
To prove that (4.21) is an isomorphism we may take a special overconvergent Witt lift. Therefore the proposition follows from Corollary 3.25. □

We give now a localized form of the comparison morphism (4.16). For this we will use the dagger spaces introduced by Grosse-Klönne.

Let $A$ be a smooth finitely generated $k$-algebra. Let $B$ be a smooth $W$-algebra. Assume we are given a surjection $B \to A$. Let $X \to F$ the corresponding closed immersion of affine schemes. We will call the pair $(X, F)$ a special frame.

We will endow $]X[\hat{\mathcal{F}}_K$ with the structure of a dagger space [7]. This means that the completion of this dagger space is the rigid analytic space $]X[\hat{\mathcal{F}}_K$.

The construction will be functorial with respect to morphisms of special frames.

We choose an arbitrary closed immersion

(4.24) \[ F \to \mathbb{A}^n_W \subset \mathbb{P}^n_W, \]

and denote by $Q$ the closure of $F$ in $\mathbb{P}^n_W$. We obtain an associated frame

(4.25) \[ X \to Y \to \hat{Q}. \]

The inclusion $F \subset Q$ induces an open immersion of rigid analytic spaces $F^n_K \subset Q^n_K = \hat{Q}_K$. Therefore we can regard strict tubular neighbourhoods as open subsets in $F^n_K$.

**Proposition 4.26.** The frame (4.25) is up to isomorphism independent of the closed immersion $F \to \mathbb{A}^n_W$. In particular the structure of the dagger space on $]X[\hat{\mathcal{F}}_K$ obtained by the frame (4.25) is independent of this closed immersion. Therefore the notion of a strict tubular neighbourhood of $]X[\hat{\mathcal{F}}_K$ in $F^n_K$ has an intrinsic meaning.

A commutative diagram

\[ \begin{array}{ccc}
X & \longrightarrow & F \\
\downarrow & & \downarrow \\
X' & \longrightarrow & F'
\end{array} \]

induces a morphism of dagger spaces $]X[\hat{\mathcal{F}}_K \to X'[\hat{\mathcal{F}}'_K].$

**Proof.** We begin with a special case. Consider a second closed immersion $F \to \mathbb{A}^{n+1}$ which is obtained from (4.24) by a closed immersion $\mathbb{A}^n \to \mathbb{A}^{n+1}$ whose comorphism is of the form

\[ W[X_1, \ldots, X_n, Z] \to W[X_1, \ldots, X_n], \]

where $Z$ is mapped to a polynomial $g(X_1, \ldots, X_n)$. In this case the closed immersion $\mathbb{A}^n \to \mathbb{A}^{n+1}$ extends to a closed immersion of the projective spaces $\mathbb{P}^n \to \mathbb{P}^{n+1}$ and therefore the frame associated to $F \to \mathbb{A}^{n+1}$ is exactly the same.

Next we consider a second closed immersion $F \to \mathbb{A}^l$. We obtain a diagonal embedding $F \to \mathbb{A}^n \times_{\text{Spec} W} \mathbb{A}^l$. We will see that the frame associated to this embedding is isomorphic to the frame (4.25). Indeed, we take coordinates $Y_1, \ldots, Y_l$ on $\mathbb{A}^l$. We compare the comorphisms of the closed immersions.
We find an epimorphism $W[X_1, \ldots, X_n, Y_1, \ldots, Y_l] \to W[X_1, \ldots, X_n]$, which maps $X_i$ to $X_i$ which makes this diagram commutative. We obtain a diagram

$$
\begin{align*}
W[X_1, \ldots, X_n, Y_1, \ldots, Y_l] & \to B \\
& \\
W[X_1, \ldots, X_n] & \\
\end{align*}
$$

We obtain a diagram

$$
\begin{align*}
F & \longrightarrow \mathbb{A}^n \\
\text{id} \downarrow & \downarrow & \\
F & \longrightarrow \mathbb{A}^n \times \text{Spec } W \mathbb{A}^l,
\end{align*}
$$

where the vertical arrow on the right hand side is the closed immersion define above. But for such a situation we have seen above, that the frames are isomorphic.

It remains to show the functoriality of the dagger spaces. We consider the graph of $F \to F'$. Since the case of a closed immersion is clear we are reduced to the case where we consider a projection:

$$
\begin{align*}
X \times X' & \longrightarrow F \times F' \\
\downarrow & \downarrow & \\
X' & \longrightarrow F'.
\end{align*}
$$

We obtain a morphism of frames if we take embeddings (4.24) for $F$ and $F'$ and then the embedding $\iota : F \times F' \to \mathbb{P}^n \times \mathbb{P}^l$ projecting to $F' \to \mathbb{P}^l$. We have to check that the frame for $X \times X'$ obtained from $\iota$ is isomorphic to the frame associated to $X \times X' \to \mathbb{P}^{n+m}$. For this we may compose $\iota$ with the Segre embedding into $\mathbb{P}^{n+l+m}$. But this frame belongs to the type (4.25):

$$
F \times F' \to \mathbb{A}^{n+l+m} \to \mathbb{P}^{n+l+m}.
$$

Consider a special frame $(X = \text{Spec } A, F = \text{Spec } B)$. If $U = \text{Spec } A \subset X$ then $|U|_{\mathfrak{f}_K \subset} X|_{\mathfrak{f}_K}$ inherits the structure of a dagger space. Let $f \in B$ be a lifting of $\bar{f} \in A$. On the other hand we have on $|U|_{\mathfrak{f}_K}$ the dagger space structure which arises from the special frame $(U, \text{Spec } B_f)$. These two dagger space structures coincide. Indeed, form the commutative diagram

$$
\begin{align*}
U & \longrightarrow \text{Spec } B_f \\
\downarrow & \downarrow & \\
X & \longrightarrow \text{Spec } B \\
\end{align*}
$$

$$
\begin{align*}
\mathbb{A}^n \times \mathbb{A} & \longrightarrow \mathbb{P}^n \times \mathbb{P} \\
\downarrow & \downarrow & \\
\mathbb{A}^n & \longrightarrow \mathbb{P}^n.
\end{align*}
$$
This induces a map of frames in the sense of rigid geometry
\[
\begin{align*}
U & \longrightarrow Y' \longrightarrow \hat{Q}' \\
\downarrow & \quad \downarrow & \downarrow \\
U & \longrightarrow Y \longrightarrow \hat{Q}.
\end{align*}
\]
The last vertical arrow is proper and is an open immersion in a neighborhood of \(U\). We conclude by [2] Thm. 1.3.5 that the strict tubular neighborhoods associated to the two frames are the same. This implies the desired isomorphism of dagger spaces.

We consider the specialization map from the dagger space associated to \((X, F)\). We will denote it by \(\Omega = \Omega_{(X, F)}\) in order to distinguish it from the rigid analytic space \(X[\hat{\mathcal{F}}_X]\). We consider the specialization morphism
\[
\text{sp} : \Omega \rightarrow X.
\]
Let \(\gamma : B \rightarrow W^\dagger(A)\) be a homomorphism which makes \((X, F)\) a \(W^\dagger\)-frame. Then \((U, \text{Spec } B_f)\) inherits the structure of a \(W^\dagger\)-frame. The remark above and [7] Thm. 5.1 shows that we have a sheafified version of the morphism (4.16)
\[
(4.27) \quad \text{sp}_* \Omega_\Omega \rightarrow W^\dagger \Omega_{X/k} \otimes \mathbb{Q}.
\]
Let us assume that \(B/pB \cong A\). In this case \(\Omega\) is an affinoid dagger space. Tate acyclicity for affinoid dagger spaces ([7] Prop. 3.1) implies that the functor \(\text{sp}_*\) is acyclic for coherent sheaves on \(\Omega\). We obtain
\[
R\text{sp}_* \Omega_\Omega \cong \text{sp}_* \Omega_\Omega \rightarrow W^\dagger \Omega_{X/k} \otimes \mathbb{Q}.
\]
The hypercohomology of this arrow is the comparison morphism (4.21).

Let \(X/k\) be a smooth quasiprojective scheme. We use the simplicial methods of [5] to construct a comparison morphism in this case.

We choose a finite affine open covering of \(X = \bigcup_{i \in M} X_i\). We set \(X_i = \text{Spec } A_i\). For a subset \(I \subset M\) we write \(X_I = \bigcap_{i \in I} X_i\). We write \(X_I = \text{Spec } A_I\). We arrange the covering in such a way that for \(i \in I\) we have \(A_I \cong (A_i)_g\) for a suitable element \(g \in A_i\).

We can find for each \(i\) a \(W^\dagger\)-frame \((X_i, F_i = \text{Spec } B_i)\), such that \(B_i/pB_i \cong A_i\). We set
\[
F^I = \prod_{i \in I} F_i,
\]
where the product is taken over \(\text{Spec } W\). The diagonal gives a closed immersion
\[
X_I \rightarrow F^I,
\]
which is naturally endowed with the structure of a \(W^\dagger\)-frame \(\otimes_{i \in I} B_i \rightarrow W^\dagger(A_I)\). We denote the corresponding dagger space by \(\Omega^I\) and regard this as a simplicial dagger space which we denote by \(\Omega^\bullet\). We consider the specialization morphism to the simplicial scheme \(X^\bullet\)
\[
\text{sp} : \Omega^\bullet \rightarrow X^\bullet.
\]
From (4.27) we obtain morphisms
\[
(4.28) \quad \pi_I : \text{sp}_I^* \Omega_{\Omega_I} \rightarrow W^\dagger \Omega_{X_i/k} \otimes \mathbb{Q}.
\]
We regard this as a morphism of simplicial sheaves.

**Proposition 4.29.** For each $I$ the functor $sp_I^*$ is acyclic on coherent sheaves $F$ on $\mathcal{Q}_I$, i.e. $R^i sp_I^* F = 0$ for $i > 0$.

**Proof.** We already remarked this after (4.27) if $I$ consists of one element $i$.

Now let $|I| > 1$. We fix an element $i_0 \in I$. Since $A_I$ is a localization of $A_{i_0}$, we find a localization $B'_{i_0}$ of $B_{i_0}$ which maps surjectively to $A_I$. We set $F'_{i_0} = \text{Spec } B'_{i_0}$. Then $(X_I, F'_{i_0})$ is a $W^I$-frame such that $B'_{i_0}/pB'_{i_0} \cong A_I$. We denote by $\mathcal{Q}_{i_0}'$ the associated dagger space. By the first step of the proof the specialization map

$$sp' : \mathcal{Q}_{i_0}' \rightarrow X_I$$

is acyclic for coherent sheaves. We consider the special frame

$$(4.30) \quad X_I \rightarrow F'_{i_0} \times \prod_{i \neq i_0} F_i.$$

Using [2] Thm. 1.3.5 as above we see that the dagger spaces of the last frame and the frame

$$X_I \rightarrow \prod_i F_i$$

coincide. Therefore it is enough to prove acyclicity for the specialization map associated to (4.30).

We set $F = \prod_{i \neq i_0} F_i$ and we choose a closed immersion $F \rightarrow A^n_W$. We consider the closed immersion of special frames

$$(X_I, F'_{i_0} \times F) \rightarrow (X_I, F'_{i_0} \times A^n_W).$$

This induces a closed immersion of dagger spaces and therefore it is enough to show the assertion of the proposition for the second special frame.

We write the comorphism for the second special frame

$$\gamma : B'_{i_0}[X_1, \ldots, X_n] \rightarrow A_I$$

We choose elements $b_k \in B'_{i_0}$ such that $\gamma(b_k) = \gamma(X_k)$. Let us take $Y_k = X_k - b_k$ as new coordinates. With these new coordinates our special frame looks as follows:

$$(4.31) \quad X_I \cong X_I \times \text{Spec } k \text{ Spec } k \rightarrow F'_{i_0} \times \text{Spec } W A^n_W.$$

Consider the dagger space $\mathcal{Q}_I'$ associated to the last special frame. By (4.31) it is a product of dagger spaces

$$\mathcal{Q}_I' = \mathcal{Q}_{i_0}' \times \mathfrak{D}.$$

Here $\mathfrak{D}$ is the dagger space associated to the special frame $(0 = \text{Spec } k, A^n_W)$ that is the open unit ball in $(A^n_K)^{an}$. The specialization map

$$sp : \mathcal{Q}_{i_0}' \times \mathfrak{D} \rightarrow X_I$$

factors through the projection $\pi$ on the first factor. We see that for each open affine subset of $X_I$ the tube $sp^{-1}(U)$ is a Stein space. We conclude by [7] Thm. 3.2 that coherent sheaves are acyclic for $sp_*$.

$\square$
Let $\epsilon : \{X_I\} \to X$ be the augmentation. By the last proposition we have an isomorphism of simplicial sheaves $sp, \Omega_{\Omega} \to R sp, \Omega_{\Omega}$. Therefore (4.28) gives a morphism
\[
R \epsilon_* R sp_* \Omega_{\Omega} \cong R \epsilon_* W^i \Omega_{X/k} \otimes \mathbb{Q} \cong W^i \Omega_{X/k} \otimes \mathbb{Q}.
\]
The last isomorphism follows because $\epsilon$ is of cohomological descent.

We will verify that the left hand side of (4.32) is a complex on $X$ whose hypercohomology is rigid cohomology. We consider a frame $P : X \to \bar{X} \to \bar{P}$ which gives the rigid cohomology of $X$. If $P' : X \to \bar{X}' \to \bar{P}'$ is a second frame we may form the product as follows: We consider the closure $\bar{X}'''$ of $X$ in $\bar{X}' \times \bar{X}''$. The we obtain a new frame $X \to \bar{X}'' \to \bar{P} \times \bar{P}'$. We denote this frame by $P \times P'$.

By [5] we find a simplicial frame $\{P_I\}$ where $P_I$ is a frame for $X_I$ with an augmentation to $P$. By Proposition 4.26 $(X_I, P_I)$ defines functorially a frame $Q_I$. We obtain a commutative diagram of simplicial schemes
\[
P_I \times Q_I \longrightarrow Q_I \xrightarrow{\bar{\epsilon}} X_I.
\]
Consider the corresponding diagram of dagger spaces. Since each of these dagger space gives the rigid cohomology of $X_I$ we obtain quasi-isomorphisms
\[
R sp, \Omega_{\Omega} \longrightarrow R sp, \Omega_{\Omega_I} \longrightarrow R sp, \Omega_{Q_I}.
\]
Here $\mathcal{R}_I$ denotes the dagger space associated with $P_I \times Q_I$. But this implies that we obtain quasi-isomorphisms of simplicial sheaves too:
\[
R \epsilon_* R sp_* \Omega_{\Omega} \longrightarrow R \epsilon_* R sp_* \Omega_{\Omega_I} \longrightarrow R \epsilon_* R sp_* \Omega_{Q_I}.
\]
If we apply $R \Gamma(R \epsilon_*, ?)$ to the last complex in (4.33) we obtain a quasi-isomorphism with $R \Gamma_{rig}(X)$ by [5] Thm.9.1.1. Therefore we obtain from (4.32):

**Theorem 4.34.** Let $X$ be a smooth scheme over $k$. Then we have a natural quasi-isomorphism
\[
R \Gamma_{rig}(X) \to R \Gamma(X, W^i \Omega_{X/k}) \otimes \mathbb{Q}.
\]

**Proof.** We just proved this in the case where $X$ is quasiprojective. Now we treat the general case. We choose an open embedding $j : X \to \bar{X}$ in some compactification $\bar{X}$. Let $\{V_j\} = \mathcal{V}$ be a Zariski covering of $\bar{X}$ such that $V_j$ admits a closed embedding $V_j \hookrightarrow P_j$ in some formal Spf $W(k)$-scheme smooth around $V_j \cap X$. Let $\{U_i\}_i$ be an affine covering of $X$ such that each $U_i$ admits an overconvergent Witt lift with underlying lift $\bar{X}_i$ of $U_i$.

As before, let $\bar{X}_I$ be the projective closure with respect to some embedding $X_I \hookrightarrow P'_i$ and let $P'_I$ be the formal completion of $\bar{X}_I$ with closed fibre $\bar{X}_I$.

For a finite index set $J$, let $V_j = \bigcap_{j \in J} V_j$, $P_J = \prod_{j \in J} P_j$ and denote by $j_{I,J}$ the embedding $U_I \cap V_j \hookrightarrow V_j$. Let $\pi_J : V_j \to \bar{X}$.

As before we consider for a finite index set $I$ the Dagger space $Q_I$ associated to $|U_I| P'_I$, where $U_I = \bigcap_{i \in I} U_i$ and $P'_I = \prod_{i \in I} P'_i$. For varying $I$ we obtain
compatible isomorphisms
\[(4.35)\quad R\Gamma_{\text{rig}}(U_I, \mathcal{X}) \cong R\Gamma(U_I, Rsp_* \Omega^*_{Q_I})\]

between the rigid cohomology of $U_I$ and the de Rham cohomology of $Q_I$.

Let $\varepsilon : X^{\bullet \bullet} \rightarrow X$ be the canonical augmentation map on the simplicial scheme $X^{\bullet \bullet}$ defined by the open embeddings

$$X^{I,J} = U_I \cap V_J \hookrightarrow X.$$ 

Then we have the following from rigid cohomological descent theory.

**Proposition 4.36** ([5] 10.1.4). The simplicial triple

$$\left( \cosk^X_0(\mathfrak{U}), \cosk^X_0(\mathfrak{V}), \cosk^{\text{Spf} W(k)}_0 P \right)$$

(where $P = \bigsqcup_j P_j$ and $\mathfrak{U} = \left( \bigsqcup_{i,j} (U_i \cap V_j) \right)$

is a universally de Rham descendent hypercovering of the pair $(X, \mathcal{X})$ over $(\text{Spec } k, \text{Spec } k, \text{Spf } W(k))$.

Hence the total (Čech) complex associated to the simplicial complex

$$R\Gamma \left( X, R\varepsilon_* Rsp_* \Omega^*_{|U_I \cap V_J} \right)_{I,J}$$

computes the rigid cohomology of $X/k$. On the other hand, using (4.35), we see that this total complex is isomorphic to

$$R\Gamma \left( X, R\varepsilon_* (sp_* \Omega^*_{|U_I \cap V_J}) \right)_{I,J}.$$ 

We have already proven that there is an isomorphism of simplicial sheaves

$$\left( sp_* \Omega^*_{|U_I \cap V_J} \right)_{I,J} \rightarrow (W^1 \Omega^*_{U_I \cap V_J} \otimes \mathbb{Q})_{I,J}.$$ 

As $W^1 \Omega^*_X$ is a complex of Zariski sheaves we may apply $R\Gamma \circ R\varepsilon_*$ to both sides and finally obtain the desired isomorphism

$$H^*_{\text{rig}}(X/K) \cong H^* \left( X, W^1 \Omega^*_X \otimes \mathbb{Q} \right).$$

This finishes the proof of theorem 4.34.

\[\square\]

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References


Massachusetts Institute of Technology, Dept of Mathematics, Cambridge, MA 02139
E-mail address: davis@math.mit.edu

University of Exeter, Mathematics, Exeter EX4 4QF, Devon, UK
E-mail address: a.langer@exeter.ac.uk

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld
E-mail address: zink@math.uni-bielefeld.de