ON THE p-ADIC UNIFORMIZATION OF QUATERNIONIC SHIMURA CURVES

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Contents

1. Introduction

Let D be a quaternion division algebra over a totally real number field F. We assume that D splits at exactly one infinite place $\chi_0 : F \to \mathbb{R}$. We fix an isomorphism $D \otimes_{F,\chi_0} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$. Let h_D be the homomorphism

$$\mathbb{C}^{\times} \to (D \otimes_{F,\chi_0} \mathbb{R})^{\times} \subset (D \otimes \mathbb{R})^{\times}, \quad z = a + b\mathbf{i} \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{1.1}$$

We regard $D^{\times}$ as an algebraic group over $\mathbb{Q}$. Then $(D^{\times}, h_D)$ is a Shimura datum of a Shimura curve $\mathrm{Sh}(D^{\times}, h_D)$. We prove the p-adic uniformization of these curves under certain conditions, discovered by Cherednik [Ch]. Let us describe our main result.

We fix a prime number p and choose a diagram of field embeddings,

$$\mathbb{C} \leftarrow \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p. \tag{1.2}$$

This gives a bijection

$$\mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(F, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(F, \bar{\mathbb{Q}}_p).$$

Let $\mathfrak{p}_0, \ldots, \mathfrak{p}_s$ be the prime ideals of F over p. We assume that $\mathfrak{p}_0$ is induced by $\chi_0 : F \to \bar{\mathbb{Q}}_p$ and that $D_{\mathfrak{p}_0} = D \otimes_F F_{\mathfrak{p}_0}$ is a division algebra. Let $O_{D_{\mathfrak{p}_0}}$ be the maximal order of $D_{\mathfrak{p}_0}$. We choose an open and compact subgroup $\mathbf{K} \subset (D \otimes_{\mathbb{Q}} \mathbb{A}_f)^{\times}$ as follows. We set $\mathbf{K}_{\mathfrak{p}_0} = O^{\times}_{D_{\mathfrak{p}_0}}$. For $i = 1, \ldots, s$ we choose arbitrarily open and compact subgroups $\mathbf{K}_{\mathfrak{p}_i} \subset D^{\times}_{\mathfrak{p}_i}$. We set

$$\mathbf{K}_p = \prod_{i=0}^{s} \mathbf{K}_{\mathfrak{p}_i} \subset (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}. \tag{1.3}$$

We also choose a sufficiently small open compact subgroup $\mathbf{K}^p \subset (D \otimes_{\mathbb{Q}} \mathbb{A}^p_f)^{\times}$ and set

$$\mathbf{K} = \mathbf{K}_p \mathbf{K}^p.$$

The Shimura field $E(D^{\times}, h_D)$ is $\chi_0(F)$. The diagram (1.2) induces a p-adic place ν of the Shimura field, and χ_0 gives an identification $E(D^{\times}, h_D)_{\nu} \cong F_{\mathfrak{p}_0}$. As an abbreviation we write $E_{\nu} = E(D^{\times}, h_D)_{\nu}$. We denote by $\breve{E}_{\nu}$ the completion of the maximal unramified extension of E_{ν}.

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We will prove (see Corollary 6.8) that the curve $\mathrm{Sh}_{\mathbf{K}}(D^\times, h_D)$ has stable reduction over $\operatorname{Spec} O_{E_\nu}$, i.e., $\mathrm{Sh}_{\mathbf{K}}(D^\times, h_D)$ extends to a stable curve $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)$ over O_{E_ν}, in the sense of Deligne-Mumford [DM]. By [DM, Lem. 1.12], this extension is unique up to unique isomorphism. The action of $(D \otimes \mathbb{A}_f)^\times$ on the tower $\mathrm{Sh}_{\mathbf{K}}(D^\times, h_D)$ for varying $\mathbf{K}$ as above extends to the stable model.

Let $\check{D}$ be a *Cherednik twist* of D. This is a quaternion algebra over F such that

$$\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0} \cong D \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0} \tag{1.4}$$

and such that $\check{D} \otimes_F F_{\mathfrak{p}_0} \cong \mathrm{M}_2(\mathrm{F}_{\mathfrak{p}_0})$ and such that $\check{D}$ is non-split at all infinite places of F. For a more canonical definition of $\check{D}$ see (6.22).

Let $\hat{\Omega}^2_{F_{\mathfrak{p}_0}}$ be the *integral model of the Drinfeld halfplane* for the local field $F_{\mathfrak{p}_0}$, cf. [Dr]. It is a p-adic formal scheme over $\operatorname{Spf} O_{F_{\mathfrak{p}_0}}$ with an action of the group $\check{D}^\times_{\mathfrak{p}_0} = (\check{D} \otimes_F F_{\mathfrak{p}_0})^\times \cong \mathrm{GL}_2(F_{\mathfrak{p}_0})$, cf. [Dr], (5.19). This action factors through an action of $\mathrm{PGL}_2(F_{\mathfrak{p}_0})$. We consider on

$$(\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \times (D \otimes_F F_{\mathfrak{p}_0})^\times / \mathbf{K}_{\mathfrak{p}_0} = (\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \times \mathbb{Z}, \tag{1.5}$$

the action of $\check{D}_{\mathfrak{p}_0}$ which is on the first factor on the right hand side obtained from the action introduced above and which acts on $\mathbb{Z}$ by translation with $\mathrm{ord}_{F_{\mathfrak{p}_0}} \det_{\check{D}_{\mathfrak{p}_0}/F_{\mathfrak{p}_0}}$, cf. Proposition 5.11. We formulate our main result as follows.

Theorem 1.1. *Let $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)^\wedge_{/\operatorname{Spf} O_{\breve{E}_\nu}}$ be the completion of the scheme $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{\breve{E}_\nu}$ along the special fiber. Then there is an isomorphism of formal schemes*

$$\check{D}^\times \backslash \big((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \times (D \otimes \mathbb{A}_f)^\times / \mathbf{K} \big) \xrightarrow{\sim} \widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)^\wedge_{/\operatorname{Spf} O_{\breve{E}_\nu}}. \tag{1.6}$$

The action of $\check{D}^\times$ is given by (1.4) and (1.5). For varying $\mathbf{K}$ this uniformization isomorphism is compatible with the action of Hecke correspondences in $(D \otimes \mathbb{A}_f)^\times$ on both sides.

Let $\Pi \in D_{\mathfrak{p}_0}$ be a prime element in this division algebra over $F_{\mathfrak{p}_0}$. We denote also by Π the image by the canonical embedding $D^\times_{\mathfrak{p}_0} \subset (D \otimes \mathbb{A}_f)^\times$. Let $\tau \in \mathrm{Gal}(\breve{E}_\nu/E_\nu)$ be the Frobenius automorphism and $\tau_c = \operatorname{Spf} \tau^{-1} : \operatorname{Spf} O_{\breve{E}_\nu} \to \operatorname{Spf} O_{\breve{E}_\nu}$. The natural Weil descent datum with respect to $O_{\breve{E}_\nu}/O_{E_\nu}$ on the right hand side of (1.6) induces on the left hand side the Weil descent datum given by the following diagram

$$\begin{array}{ccc} \check{D}^\times \backslash ((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}) & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)^\wedge_{/\operatorname{Spf} O_{\breve{E}_\nu}} \\ \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c \times \Pi^{-1}} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c} \\ \check{D}^\times \backslash ((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}) & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)^\wedge_{/\operatorname{Spf} O_{\breve{E}_\nu}}. \end{array}$$

The left hand side of (1.6) can be written in more concrete terms as follows. We write $\mathbf{K} = \mathbf{K}_{\mathfrak{p}_0} \mathbf{K}^{\mathfrak{p}_0}$ where $\mathbf{K}^{\mathfrak{p}_0} \subset (\check{D} \otimes_F \mathbb{A}^{\mathfrak{p}_0}_{F,f})^\times = (D \otimes_F \mathbb{A}^{\mathfrak{p}_0}_{F,f})^\times$. For $g \in (D \otimes_F \mathbb{A}^{\mathfrak{p}_0}_{F,f})^\times$, let

$$\Gamma_g = \{ d \in \check{D}^\times \cap g \mathbf{K}^{\mathfrak{p}_0} g^{-1} \mid \mathrm{ord}_{F_{\mathfrak{p}_0}} \det d = 0 \}.$$

Let $\bar{\Gamma}_g$ be the image of Γ_g by the natural map $\check{D}^\times \to \check{D}^\times_{\mathfrak{p}_0} \to \mathrm{PGL}_2(F_{\mathfrak{p}_0})$. Then $\bar{\Gamma}_g$ is a discrete cocompact subgroup of $\mathrm{PGL}_2(F_{\mathfrak{p}_0})$, comp. the proof of Proposition 6.6. It acts properly discontinuously on the formal scheme $\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu}$, and the quotients $\mathfrak{X}_{\Gamma_g} := \bar{\Gamma}_g \backslash (\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \chi_0} \operatorname{Spf} O_{\breve{E}_\nu})$ are exactly the connected components of the formal scheme on the LHS of (1.6), for varying g. By [Mum], $\mathfrak{X}_{\Gamma_g}$ is algebraizable, i.e. it is the formal scheme associated to a proper scheme $\mathfrak{X}^{\mathrm{alg}}_{\Gamma_g}$ over $O_{\breve{E}_\nu}$. The general fibers of these schemes for varying g give back the connected components of $\mathrm{Sh}(D^\times, h_D)_{\breve{E}_\nu}$.

We prove Theorem 1.1 using the method which Drinfeld [Dr] used in the case $F = \mathbb{Q}$. The case $F \neq \mathbb{Q}$ becomes more difficult because in this case the Shimura curve is not described by a PEL-moduli problem. In fact, the Shimura curve is then a Shimura variety of abelian type which is not of Hodge type. Also, the weight homomorphism $w\colon \mathbb{G}_m \longrightarrow D^\times_{\mathbb{R}}$ is not defined over $\mathbb{Q}$. The existence of a canonical model is proved by the method of Shimura and Deligne [De, §6], by embedding this Shimura variety into one of PEL type (*méthode des modèles étranges*). We use here a variant of this method to construct integral models over $O_{\breve{E}_\nu}$ of the Shimura curve. A similar approach was used by Carayol [Car]. More precisely, we show that the Shimura curve $\mathrm{Sh}(D^\times, h_D)$ can be embedded as an open and closed subscheme in a Shimura variety which is an unramified twist of a PEL-moduli scheme which has a natural integral model. This PEL-moduli scheme can be so chosen that it has a p-adic uniformization by [RZ, §6]. In this way, we obtain the isomorphism (1.6). Finally we must determine the descent datum to obtain the result over E_ν. Let us explain our strategy in more detail.

Let K/F be a CM-field and assume that each $\mathfrak{p}_i$ is split in K, i.e. $\mathfrak{p}_i O_K = \mathfrak{q}_i \bar{\mathfrak{q}}_i$. By (1.2) we write

$$\begin{aligned}\Phi &:= \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K, \bar{\mathbb{Q}}_p) = \\ &\big(\textstyle\coprod_{i=0}^s \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K_{\mathfrak{q}_i}, \bar{\mathbb{Q}}_p)\big) \coprod \big(\textstyle\coprod_{i=0}^s \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K_{\bar{\mathfrak{q}}_i}, \bar{\mathbb{Q}}_p)\big).\end{aligned} \tag{1.7}$$

We denote by $\varphi_0 \in \Phi$ the extension of χ_0 which on the right hand side of (1.7) lies in the first summand. We define a function $r : \Phi \to \{0, 1, 2\}$ as follows. We set $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$. If the restriction of $\varphi \in \Phi$ to F is not χ_0 we set $r_\varphi = 0$ if φ is in the first s summands on the right hand side and $r_\varphi = 2$ if φ is in the last s summands. If $\chi \neq \chi_0$ the extension φ of χ such that $r_\varphi = 2$ defines an isomorphism $K \otimes_{F,\chi} \mathbb{R} \cong \mathbb{C}$. We define the group homomorphism

$$\begin{array}{cccc} h_K : \mathbb{C}^\times & \to & (K \otimes \mathbb{R})^\times \cong \prod_\chi (K \otimes_{F,\chi} \mathbb{R})^\times = (K \otimes_{F,\chi_0} \mathbb{R})^\times \times \prod_{\chi \neq \chi_0} \mathbb{C}^\times. \\ 1 & \longmapsto & (1, z, z, \ldots, z) \end{array}$$

Let $B = D^{\mathrm{opp}} \otimes_F K$. We denote by $d \mapsto d^\iota$ the main involution of D and by $a \mapsto \bar{a}$ the conjugation of K/F. We denote by $b \mapsto b'$ the involution of the second kind on B/K which is defined by $d \otimes a \mapsto d^\iota \otimes \bar{a}$. Let $V = B$ considered as a B-left module. Multiplication from the right defines a ring homomorphism

$$D \otimes_F K \to \mathrm{End}_B V.$$

In particular the group $D^\times \times K^\times$ acts on V. By (1.1) we obtain a ring homomorphism

$$\mathbb{C} \to D \otimes_{F,\chi_0} \mathbb{R} \to (D \otimes_F K) \otimes_{F,\chi_0} \mathbb{R}$$

and by the isomorphisms $K \otimes_{F,\chi} \mathbb{R} \cong \mathbb{C}$ chosen above for $\chi \neq \chi_0$ we obtain ring homomorphisms

$$\mathbb{C} \to K \otimes_{F,\chi} \mathbb{R} \to (D \otimes_F K) \otimes_{F,\chi} \mathbb{R}.$$

Taking the product of these ring homomorphisms over all $\chi : F \to \mathbb{R}$ we obtain a ring homomorphism

$$\mathbb{C} \to (D \otimes_F K) \otimes_{\mathbb{Q}} \mathbb{R}$$

and therefore a complex structure on the real vector space $V \otimes \mathbb{R}$. Alternatively, this complex structure is given by the group homomorphism

$$h = h_D \times h_K : \mathbb{S} \to \prod_{\chi \in \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(F, \mathbb{C})} \big((D \otimes_{F,\chi} \mathbb{R})^\times \times (K \otimes_{F,\chi} \mathbb{R})^\times\big),$$

where the group on the right hand side acts on $V \otimes \mathbb{R}$ by the action of $D^\times \times K^\times$ on V.

We consider $\mathbb{Q}$-bilinear forms $\psi : V \times V \to \mathbb{Q}$ such that

$$\psi(xb, y) = \psi(x, yb'), \quad \text{for } x, y \in V,\ b \in B.$$

By [De] one can choose ψ in such a way that the complex structure h satisfies the Riemann period relations. We consider $G^\bullet = \{b \in B^{\mathrm{opp}} \mid b'b \in F^\times\}$ as an algebraic group over $\mathbb{Q}$. The right multiplication by elements $d \otimes 1$ and $1 \otimes a$ define elements of $G^\bullet$. This gives a homomorphism of algebraic groups,

$$D^\times \times K^\times \xrightarrow{\kappa} G^\bullet. \tag{1.8}$$

Then $(G^\bullet, h)$ is the Shimura datum for a Shimura variety of PEL-type. By (1.8) we have an embedding $D^\times \to G^\bullet$. The decomposition $B \otimes \mathbb{Q}_p = \prod_{i=0}^s (B_{\mathfrak{q}_i} \times B_{\bar{\mathfrak{q}}_i})$ induces a similiar decomposition of $V \otimes \mathbb{Q}_p$. We choose maximal orders $O_{D_{\mathfrak{p}_i}} \subset D_{\mathfrak{p}_i}$ and hence maximal orders $O_{B_{\mathfrak{q}_i}} \subset B_{\mathfrak{q}_i}$. We assume in the definition (1.3) that $\mathbf{K}_{\mathfrak{p}_i} \subset O_{D_{\mathfrak{p}_i}}^\times$. There is a natural isomorphism $D_{\mathfrak{p}_i}^\times \cong (B_{\mathfrak{q}_i}^{\mathrm{opp}})^\times$. The image $\mathbf{K}_{\mathfrak{p}_i}$ by this isomorphism will be denoted by $\mathbf{K}_{\mathfrak{q}_i}^\bullet$. From these last groups we define a subgroup $\mathbf{K}_p^\bullet \subset G^\bullet(\mathbb{Q}_p)$, cf. (4.6) with $\mathbf{M}_{\mathfrak{p}_i}^\bullet = O_{F_{\mathfrak{p}_i}}^\times$. This subgroup satisfies $\mathbf{K}_p = D^\times(\mathbb{Q}_p) \cap \mathbf{K}_p^\bullet$. Moreover we choose $\mathbf{K}^{\bullet,p}$ such that $\mathbf{K}^p = (D \otimes \mathbb{A}_f^p)^\times \cap \mathbf{K}^{\bullet,p}$.

The form ψ induces an involution $\star$ of the second kind on B,

$$\psi(bx, y) = \psi(b^{\star}x, y), \quad x, y \in V,\ b \in B.$$

We denote by $O_{B_{\bar{\mathfrak{q}}_i}} \subset B_{\bar{\mathfrak{q}}_i}$ the image of $O_{B_{\mathfrak{q}_i}}$ by $\star$. Let $O_{B,(p)}$ be the set of elements of B whose images in $B_{\mathfrak{q}_i}$ and $B_{\bar{\mathfrak{q}}_i}$ lie in the chosen maximal orders. We obtain the lattice $\Lambda_{\mathfrak{q}_i} = O_{B_{\mathfrak{q}_i}} \subset V_{\mathfrak{q}_i}$.

Let $U_p(F) \subset F^{\times}$ be the subgroup of elements which are units in each $F_{\mathfrak{p}_i}$. We define the following functor on the category of O_{E_ν}-schemes S. The upper index t is referring to the fact that $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$, when restricted to the category of E_ν-schemes, is a twisted form of another functor $\mathcal{A}^{\bullet}_{\mathbf{K}^\bullet}$.

Definition 1.2. Let S be an O_{E_ν}-scheme. A point of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(S)$ consists of the following data:

(a) An abelian scheme A over S up to isogeny prime to p with an action $\iota : O_{B,(p)} \to \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

(b) An $U_p(F)$-homogeneous polarization $\bar{\lambda}$ of A which is principal in p.

(c) A class $\bar{\eta}^p$ modulo $\mathbf{K}^{\bullet,p}$ of $B \otimes \mathbb{A}^p_f$-module isomorphisms

$$\eta^p : V \otimes \mathbb{A}^p_f \xrightarrow{\sim} \mathrm{V}^p_f(A),$$

such that

$$\psi(\xi^{(p)}(\lambda)v_1, v_2) = E^{\lambda}(\eta^p(v_1), \eta^p(v_2))$$

for some function $\xi^{(p)}(\lambda) \in (F \otimes \mathbb{A}^p_f)^{\times}(1)$ on $\bar{\lambda}$.

(e) A class $\bar{\eta}_{\mathfrak{q}_i}$ modulo $\mathbf{K}^{\bullet}_{\mathfrak{q}_i}$ of $O_{B_{\mathfrak{q}_i}}$-module isomorphisms for each $i = 1, \ldots s$,

$$\eta_{\mathfrak{q}_i} : \Lambda_{\mathfrak{q}_i} \xrightarrow{\sim} T_{\mathfrak{q}_i}(A).$$

We require that the following Kottwitz condition (KC) holds,

$$\operatorname{char}(T, \iota(b) \mid \operatorname{Lie} A) = \prod_{\varphi : K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}^o_{B/K}(T - b))^{r_\varphi}. \tag{1.9}$$

The general fiber over E_ν of this functor is a Galois form of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h)_{E_\nu}$ but this is irrelevant for this Introduction. We prove that the étale sheafification of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ is representable, cf. Proposition 4.21. We also show that (cf. (4.25))

$$(K \otimes \mathbb{Q}_p)^{\times} = \prod_{i=0}^{s} K^{\times}_{\mathfrak{q}_i} \times \prod_{i=0}^{s} K^{\times}_{\bar{\mathfrak{q}}_i} \tag{1.10}$$

acts by Hecke operators on the functor $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$.

We consider the homomorphism

$$h^{\bullet}_D = h_D \times 1 : \mathbb{C}^{\times} \to (D \otimes \mathbb{R})^{*} \times (K \otimes \mathbb{R})^{*} \to G^{\bullet}_{\mathbb{R}}. \tag{1.11}$$

The Shimura variety $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$ is defined over E_ν. It is a Galois form of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h)$.

We find a model $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$ over O_{E_ν} of this Shimura variety and a commutative diagram

$$\begin{array}{ccc}
\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} \\
\Big\downarrow{\scriptstyle \dot{z} \times \tau_c} & & \Big\downarrow{\scriptstyle \mathrm{id} \times \tau_c} \\
\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu},
\end{array} \tag{1.12}$$

cf. Proposition 4.9 and (4.36). Here E^{nr}_ν is the maximal unramified extension of E_ν, and $\tau \in \operatorname{Gal}(E^{nr}_\nu / E_\nu)$ is the Frobenius automorphism, and $\tau_c = \operatorname{Spec} \tau^{-1}$. We denote by $\pi_{\mathfrak{p}_0} \in F_{\mathfrak{p}_0} = K_{\bar{\mathfrak{q}}_0}$ a prime element and by f_ν the inertia index of $E_\nu/\mathbb{Q}_p$. The element

$$\dot{z} = (1, \ldots, 1) \times (\pi^{-1}_{\mathfrak{p}_0} p^{f_\nu}, p^{f_\nu}, \ldots p^{f_\nu})$$

from the right hand side of (1.10) acts as an Hecke operator.

The horizontal arrow in the diagram (1.12) is the étale sheafification. It follows from [RZ] that the étale sheafification of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ has a p-adic uniformization by the formal scheme $\hat{\Omega}^2_{E_\nu}$, cf. Theorem 6.3. This gives a uniformization of the model $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$. The embedding of

Shimura data $(D^\times, h_D) \subset (G^\bullet, h_D^\bullet)$ and the fact that $\mathbf{K} \subset \mathbf{K}^\bullet$ define a morphism of Shimura varieties $\mathrm{Sh}_{\mathbf{K}}(D^\times, h_D)_{E_\nu} \longrightarrow \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}$. By a theorem of Chevalley, for suitable $\mathbf{K}^{\bullet,p} \in G^\bullet(\mathbb{A}_f^p)$ of the type considered above, this morphism induces an open and closed embedding,

$$\mathrm{Sh}_{\mathbf{K}}(D^\times, h_D)_{E_\nu} \subset \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}. \tag{1.13}$$

The closure of the left hand side in $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)$ gives the stable model $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(D^\times, h_D)$ whose formal scheme inherits a uniformization by $\hat{\Omega}^2_{E_\nu}$, proving the main theorem.

So far, we have only mentioned the Shimura pairs $(G^\bullet, h)$ and $(G^\bullet, h_D^\bullet)$. However, in the body of the paper, also Shimura pairs (G, h) and $(G, h\delta)$ play an important role. Here $G \subset G^\bullet$ is the subgroup where the similitude factor lies in $\mathbb{Q}$, and δ is a central character of G. The Shimura variety for (G, h) is of PEL-type and has the key property that it is a fine moduli scheme for a moduli problem $\mathcal{A}_{\mathbf{K}}$, for small enough level $\mathbf{K}$. Similarly, the Shimura variety for $(G, h\delta)$ is the unramified twist of the fine moduli scheme for a moduli problem $\mathcal{A}^t_{\mathbf{K}}$, which, furthermore, has a natural extension $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ over $\operatorname{Spec} O_{E_\nu}$. The fine moduli scheme for $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ is then used to show that the horizontal arrow in the diagram (1.12) is the étale sheafification.

The lay-out of the paper is as follows. In §2 we explain the linear algebra behind the formation of the Shimura pairs $(G^\bullet, h)$ and (G, h). In §3, we explain the Shimura varieties for (G, h) and $(G, h\delta)$ and the corresponding moduli problems $\mathcal{A}_{\mathbf{K}}$ and $\mathcal{A}^t_{\mathbf{K}}$ and the integral extension $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ of the latter. In §4 we explain the Shimura varieties for $(G^\bullet, h)$ and $(G^\bullet, h_D^\bullet)$ and the corresponding moduli problems $\mathcal{A}^\bullet_{\mathbf{K}}$ and $\mathcal{A}^{\bullet,t}_{\mathbf{K}}$ and the integral extension $\tilde{\mathcal{A}}^t_{\mathbf{K}}$. Furthermore, we establish a relation between the integral extensions $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ and the integral extension $\tilde{\mathcal{A}}^{\bullet,t}_{\mathbf{K}}$ and use this to show that the horizontal arrow in the diagram (1.12) is the étale sheafification. In §5 we explain the Rapoport-Zink spaces relevant to these moduli problems. In §6 we prove the p-adic uniformization of the integral models of the Shimura varieties for the pairs $(G^\bullet, h_D^\bullet)$ and $(D^\times, h_D)$. The last two sections are really appendices. In §7, we clarify our conventions about Galois descent, and in §8 we make precise our sign conventions for Shimura varieties. We formulate a result of Kisin [K] on embeddings of Shimura varieties in the form needed here.

The present paper is an improved version of parts of the preprint [BZ]. The strategy here is the same but some serious gaps in the arguments are repaired. However, not all results of [BZ] are covered.

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2. The Shimura data

In this section, we introduce the linear algebra which leads to the definition of the Shimura pairs $(G^\bullet, h_D^\bullet)$ and $(G^\bullet, h)$ and (G, h).

Let K/F be a CM-field. Let $a \mapsto \bar{a}$ be the conjugation of K/F. We consider a quaternion algebra D over F. Let $d \mapsto d^\iota$ be the main involution of D. We set $B = D^{\mathrm{opp}} \otimes_F K$. We extend the map $d \mapsto d^\iota$ K-linearly to B. Then we obtain the main involution $b \mapsto b^\iota$ of B/K. The conjugation acts via the second factor on $B = D^{\mathrm{opp}} \otimes_F K$. We set $b' = \bar{b}^\iota$. We consider the sesquilinear form

$$\varkappa_0 : B \times B \to K, \quad \varkappa_0(b_1, b_2) = \mathrm{Tr}^o_{B/K} b_2 b_1'.$$

It is K-linear in the second variable and antilinear in the first and it is hermitian

$$\varkappa_0(b_1, b_2) = \overline{\varkappa_0(b_2, b_1)}.$$

Moreover we obtain

$$\varkappa_0(xb, y) = \varkappa_0(x, yb'), \quad \varkappa_0(bx, y) = \varkappa_0(x, b'y), \quad x, y, b \in B. \tag{2.1}$$

We set

$$G^\bullet = \{b \in B^{\mathrm{opp}} \mid b'b \in F^\times\}, \tag{2.2}$$

and consider it as an algebraic group over $\mathbb{Q}$. We write $\tilde{G}^\bullet$ if we consider it as an algebraic group over F, i.e. $\mathrm{Res}_{F/\mathbb{Q}} \tilde{G}^\bullet = G^\bullet$.

We will write $V = B$ considered as a B-left module. The right multiplication by an element of B gives an isomorphism $\mathrm{End}_B V = B^{\mathrm{opp}} = D \otimes_F K$. Therefore we can write

$$G^\bullet = \{g \in GL_B(V) \mid \varkappa_0(gv_1, gv_2) = \mu(g)\varkappa_0(v_1, v_2), \mu(g) \in F^\times\}. \tag{2.3}$$

Lemma 2.1. *There is an exact sequence of algebraic groups over* $\mathbb{Q}$,

$$0 \to F^\times \to D^\times \times K^\times \overset{\kappa}{\to} G^\bullet \to 0.$$

The map κ *maps* (d,k) *to* $d \otimes k$. □

We set $\Phi = \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K,\mathbb{C})$. We assume that there is a unique embedding $\chi_0 : F \to \mathbb{R}$ such that the quaternion algebra $D \otimes_{F,\chi_0} \mathbb{R}$ splits. We consider a generalized CM-type of rank 2 in the sense of [KR2], comp. [KRZ],

$$r : \Phi \to \mathbb{Z}_{\geq 0}, \tag{2.4}$$

such that $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$ for the extensions $\varphi_0, \bar{\varphi}_0 : K \to \mathbb{C}$ of χ_0 and such that $r_\varphi = 0, 2$ for all other $\varphi \in \Phi$.

We will define a complex structure on the $\mathbb{R}$-vector space $V \otimes \mathbb{R} = B \otimes \mathbb{R}$. For this we consider the decomposition

$$B \otimes \mathbb{R} = \bigoplus_{\chi : F \to \mathbb{R}} B \otimes_{F,\chi} \mathbb{R} = \bigoplus_{\chi} \big((D^{\mathrm{opp}} \otimes_{F,\chi} \mathbb{R}) \otimes_{\mathbb{R}} (K \otimes_{F,\chi} \mathbb{R})\big).$$

We define the complex structure on each summand on the right hand side. Let $\chi \neq \chi_0$ and let $\varphi : K \to \mathbb{C}$ be the extension of χ such that $r_\varphi = 2$. Then φ defines an isomorphism $K \otimes_{F,\chi} \mathbb{R} \cong \mathbb{C}$. This induces a complex structure on the summand belonging to χ via the second factor of the tensor product. For χ_0 we have $D \otimes_{F,\chi_0} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$. We endow the $\mathbb{R}$-vector space $D^{\mathrm{opp}} \otimes_{F,\chi_0} \mathbb{R}$ with the complex structure J_{χ_0} given by right multiplication by

$$J_{\chi_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This induces a complex structure on $(D^{\mathrm{opp}} \otimes_{F,\chi_0} \mathbb{R}) \otimes_{\mathbb{R}} (K \otimes_{F,\chi_0} \mathbb{R})$ via the first factor. Together we obtain a complex structure J on $B \otimes_{\mathbb{Q}} \mathbb{R}$ such that

$$\mathrm{Tr}_{\mathbb{C}}(k | (B \otimes_{\mathbb{Q}} \mathbb{R}, J)) = \sum_{\varphi \in \Phi} 2 r_\varphi \varphi(k), \quad k \in K.$$

This complex structure on $V \otimes_{\mathbb{Q}} \mathbb{R}$ commutes with the $B \otimes_{\mathbb{Q}} \mathbb{R}$-module structure and defines therefore a homomorphism $\mathbb{C} \to B^{\mathrm{opp}} \otimes_{\mathbb{Q}} \mathbb{R}$. This homomorphism induces a homomorphism of groups

$$h : \mathbb{S} \to \prod_{\chi \in \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(F,\mathbb{C})} \big((D \otimes_{F,\chi} \mathbb{R})^\times \times (K \otimes_{F,\chi} \mathbb{R})^\times\big). \tag{2.5}$$

Let $z \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$. Then the χ_0-component $h_{\chi_0}(z)$ is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \times 1, \quad z = a + b\mathbf{i},$$

and for $\chi \neq \chi_0$ the component $h_\chi(z)$ is $1 \times 1 \otimes z \in (D \otimes_{F,\chi} \mathbb{R})^\times \times (K \otimes_{K,\varphi} \mathbb{C})^\times$. Here $\varphi \in \Phi$ is the extension of χ with $r_\varphi = 2$. We have used the natural isomorphism $K \otimes_{F,\chi} \mathbb{R} = K \otimes_{K,\varphi} \mathbb{C}$. We can write $h(\mathbf{i}) = J$. The composite with the projection to $G^\bullet_{\mathbb{R}}$ given by Lemma 2.1 is also denoted by h,

$$h \colon \mathbb{S} \longrightarrow G^\bullet_{\mathbb{R}}. \tag{2.6}$$

Lemma 2.2. *There exist elements* $\gamma \in B$ *such that* $\mathfrak{h}(x,y) = \varkappa_0(\gamma x, yJ)$ *is hermitian and positive definite on* $B \otimes \mathbb{R}$. *More precisely, this means that for each* φ *the form*

$$\mathfrak{h}_\varphi : B \otimes_{K,\varphi} \mathbb{C} \times B \otimes_{K,\varphi} \mathbb{C} \to K \otimes_{K,\varphi} \mathbb{C}$$

is hermitian and positive definite.

This follows as in [De]. Note that alternatively we can say that $\mathrm{Tr}_{K/F}\, \mathfrak{h}$ is symmetric and positive definite on $B \otimes \mathbb{R}$. Let

$$G = \{ b \in B^{\mathrm{opp}} \mid b'b \in \mathbb{Q}^\times \} \subset G^\bullet. \tag{2.7}$$

Since $h(z)'h(z) = \bar{z}z \in \mathbb{R}^\times$ for $z \in \mathbb{C}^\times = \mathcal{S}(\mathbb{R})$, the morphism h factors through $G_{\mathbb{R}}$. We define

$$\varkappa : V \times V \to K, \qquad \varkappa(x,y) = \varkappa_0(\gamma x, y), \quad x, y \in V = B.$$

The first equation of (2.1) continues to hold for $\varkappa$. We have $\mathfrak{h}(x,y) = \varkappa(x, yJ)$. We note that $\varkappa$ is an antihermitian form:

$$\overline{\varkappa(y,x)} = -\overline{\varkappa(y, xJ^2)} = -\overline{\mathfrak{h}(y, xJ)} = -\mathfrak{h}(xJ, y) = -\varkappa(xJ, yJ) = -\varkappa(x,y).$$

It is easily seen that $\gamma' = -\gamma$ is equivalent with the property that $\varkappa$ is antihermitian or that $\mathfrak{h}$ is hermitian. Equivalently one could use the alternating form

$$\psi : V \times V \to \mathbb{Q}, \quad \psi(x,y) = \mathrm{Tr}_{K/\mathbb{Q}}\, \varkappa(x,y), \quad x, y \in B, \tag{2.8}$$

which satisfies

$$\psi(kx, y) = \psi(x, \bar{k}y), \quad k \in K.$$

Then $\psi(x, yJ)$ is symmetric and positive definite. We define an involution $b \mapsto b^\star$ on B by

$$\varkappa(bx, y) = \varkappa(x, b^\star y). \tag{2.9}$$

Because the same equation holds for $\mathfrak{h}$, the involution $b \mapsto b^\star$ is positive. From the definition we obtain $b = \gamma^{-1} b' \gamma$. We obtain

$$\psi(bx, y) = \psi(x, b^\star y), \quad \psi(xb, y) = \psi(x, yb'), \quad x, y, b \in B. \tag{2.10}$$

We can also write

$$G = \{g \in \mathrm{End}_B(V) \mid \psi(gx, gy) = \mu(g)\psi(x,y), \text{ for } \mu(g) \in \mathbb{Q}^\times\}. \tag{2.11}$$

We also obtain G if we replace on the right hand side ψ by $\varkappa$.

The action of $g = (d,k) \in D^\times \times K^\times$ on B is by definition

$$(d,k)(u \otimes a) = ud \otimes ak, \quad u \otimes a \in D^{\mathrm{opp}} \otimes_F K = B.$$

The product ud is taken in D^{opp}.

The homomorphism induced by (2.5)

$$h : \mathbb{S} \to G_{\mathbb{R}} \tag{2.12}$$

gives a Shimura datum in the sense of [De], except that we denote by h what is h^{-1} in Deligne's normalization. The Hodge structure on V is therefore in this paper of type $(1,0), (0,1)$.

We fix a prime number p and we choose a diagram

$$\mathbb{C} \leftarrow \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p. \tag{2.13}$$

By this diagram we obtain $\Phi = \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K, \bar{\mathbb{Q}}_p)$. We assume that all prime ideals of O_F containing pO_F are split in K/F. We denote these prime ideals of O_F by

$$\mathfrak{p}_0, \dots, \mathfrak{p}_s. \tag{2.14}$$

Let $\mathfrak{q}_i, \bar{\mathfrak{q}}_i$ the two prime ideals of O_K over $\mathfrak{p}_i$. We obtain

$$\mathfrak{p}_i O_K = \mathfrak{q}_i \bar{\mathfrak{q}}_i.$$

We obtain a decomposition

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(K, \bar{\mathbb{Q}}_p) = \\ &\quad (\textstyle\coprod_{i=0}^s \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{Alg}}(K_{\mathfrak{q}_i}, \bar{\mathbb{Q}}_p)) \ \amalg \ (\textstyle\coprod_{i=0}^s \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{Alg}}(K_{\bar{\mathfrak{q}}_i}, \bar{\mathbb{Q}}_p)). \end{aligned} \tag{2.15}$$

We denote the components of this disjoint sum by $\Phi_{\mathfrak{q}_i}$, resp. $\Phi_{\bar{\mathfrak{q}}_i}$. We assume that $\varphi_0 \in \Phi_{\mathfrak{q}_0}$ and $\bar{\varphi}_0 \in \Phi_{\bar{\mathfrak{q}}_0}$. For all other φ we require that

$$\begin{aligned} r_\varphi = 0 \quad &\text{if } \varphi \in \Phi_{\mathfrak{q}_i} \text{ for some } i = 0, \dots, s, \text{ and } \varphi \neq \varphi_0 \\ r_\varphi = 2 \quad &\text{if } \varphi \in \Phi_{\bar{\mathfrak{q}}_i} \text{ for some } i = 0, \dots, s, \text{ and } \varphi \neq \bar{\varphi}_0. \end{aligned} \tag{2.16}$$

Let $E = E(G,h)$ be the reflex field, i.e.,

$$\mathrm{Gal}(\bar{\mathbb{Q}}/E) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid r_{\sigma\varphi} = r_\varphi, \text{ for all } \varphi \in \Phi\}. \tag{2.17}$$

The embedding $E \to \bar{\mathbb{Q}}_p$ in the sense of diagram (2.13) defines a place $E_\nu \subset \bar{\mathbb{Q}}_p$. We call this the *local Shimura field.* If $\varphi \neq \varphi_0, \bar{\varphi}_0$, the number r_φ depends only on the place $\mathfrak{q}_i$ of K which is induced by $\varphi : K \to \bar{\mathbb{Q}}_p$. We conclude that

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_\nu) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\sigma\varphi_0} = r_{\varphi_0}\}. \tag{2.18}$$

The condition (2.18) on σ signifies that σ fixes the embedding $F_{\mathfrak{p}_0} \to \bar{\mathbb{Q}}_p$ induced by φ_0. We obtain that

$$E_\nu = \varphi_0(F_{\mathfrak{p}_0}). \tag{2.19}$$

We remark that E_ν coincides with the localization of the Shimura field $\chi_0(F)$ of the Shimura curve we have chosen, cf. the beginning of this section.

3. The moduli problem for $\mathrm{Sh}(G,h)$ and a reduction modulo p

We consider the alternating $\mathbb{Q}$-bilinear form ψ on the B-module V (2.8). It satisfies

$$\psi(bv_1, v_2) = \psi(v_1, b^\star v_2) \quad v_1, v_2 \in V.$$

We state the moduli problem associated to the B-module V and the alternating form ψ, cf. [De, 4.10]. Recall (G,h) from (2.11), (2.12).

Let $\mathbf{K} \subset G(\mathbb{A}_f)$ be an open compact subgroup. The Shimura variety $\mathrm{Sh}(G,h)_{\mathbf{K}}$ is the coarse moduli scheme of the following functor $\mathcal{A}_{\mathbf{K}}$ on the category of schemes over $E = E(G,h)$. If $\mathbf{K}$ satisfies the condition (3.6) below, the functor $\mathcal{A}_{\mathbf{K}}$ is representable.

Definition 3.1. Let S be a scheme over E. A point of $\mathcal{A}_{\mathbf{K}}(S)$ is given by the following data:

(a) An abelian scheme A over S up to isogeny with an action $\iota : B \to \mathrm{End}^o A$.

(b) A $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}$ of A which induces on B the involution $b \mapsto b^\star$.

(c) A class $\bar{\eta}$ modulo $\mathbf{K}$ of $B \otimes \mathbb{A}_f$-module isomorphisms

$$\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} V_f(A)$$

such that for each $\lambda \in \bar{\lambda}$ there is locally for the Zariski topology on S a constant $\xi \in \mathbb{A}_f^\times(1)$ with

$$\xi\psi(v_1, , v_1) = E^\lambda(\eta(v_1), \eta(v_2)). \tag{3.1}$$

We require that the following condition (KC) holds,

$$\mathrm{char}(T, \iota(b) \mid \mathrm{Lie}\, A) = \prod_{\varphi : K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}^o_{B/K}(T-b))^{r_\varphi}. \tag{3.2}$$

A more precise formulation of the datum (c) is as follows. We assume that S is connected and we choose a geometric point $\bar{s}$ of S. Then we may regard $V_f(A)$ resp. $V \otimes \mathbb{A}_f$ as continuous representation of the fundamental group $\pi_1(\bar{s}, S)$. We denote by $\mathbb{A}_f(1)$ the group $\mathbb{A}_f$ endowed with the action of $\pi_1(\bar{s}, S)$ via the cyclotomic character

$$\varsigma : \pi_1(\bar{s}, S) \to \hat{\mathbb{Z}}^\times \subset \mathbb{A}_f^\times.$$

Then $\bar{\eta}$ is determined by a $B \otimes \mathbb{A}_f$-linear symplectic similitude η, i.e. (3.1) holds with $\xi \in \mathbb{A}_f$. The class $\bar{\eta}$ must be invariant by the action of $\pi_1(\bar{s}, S)$, i.e., for each $\gamma \in \pi_1(\bar{s}, S)$ there is $k(\gamma) \in \mathbf{K}$ such that

$$\gamma\eta(v) = \eta(k(\gamma)v), \quad v \in V \otimes \mathbb{A}_f. \tag{3.3}$$

Since the polarization λ is defined over S the form E^λ satisfies

$$E^\lambda(\gamma\eta(v_1), \gamma\eta(v_2)) = \gamma(E^\lambda(v_1, v_2)) = \varsigma(\gamma)E^\lambda(v_1, v_2).$$

When we apply the symplectic similitude η, this translates into

$$\varsigma(\gamma) = \mu(k(\gamma)), \tag{3.4}$$

where μ is the multiplicator (2.11). The datum $\bar{\eta}$ of (c) for a connected scheme S is now equivalently a class modulo $\mathbf{K}$ of symplectic similitudes η of $B \otimes \mathbb{A}_f$-modules $V \otimes \mathbb{A}_f \xrightarrow{\sim} V_f(A_{\bar{s}})$ such that (3.3) and (3.4) hold.

An alternative way to describe the functor $\mathcal{A}_{\mathbf{K}}$ is as follows, cf. [De, 4.12]. We fix a $\mathbb{Z}$-lattice $\Gamma \subset V$ such that $\psi(\Gamma \times \Gamma) \subset \mathbb{Z}$. Let $m > 0$ an integer and assume that $\mathbf{K} = \mathbf{K}_m$ is the subgroup of all $g \in G(\mathbb{A}_f)$, such that $g\hat{\Gamma} = \hat{\Gamma}$ and $g \equiv \mathrm{id}_{\hat{\Gamma}}$ modulo $m\hat{\Gamma}$. Let $O_B \subset B$ the order of all elements b such that $b\Gamma \subset \Gamma$. Then for a connected scheme S over E a point of $\mathcal{A}_{\mathbf{K}_m}(S)$ consists of

(a) An abelian scheme A_0 over S with an action $\iota : O_B \to \mathrm{End}\, A_0$.

(b) A polarization λ of A_0 which induces on B the involution $b \mapsto b^\star$.

(c) An isomorphism of étale sheaves on S

$$\eta_m : \Gamma/m\Gamma \to A_0[m]$$

such that η_m lifts to an isomorphism of O_B-modules

$$\eta : \Gamma \otimes \hat{\mathbb{Z}} \to \hat{T}(A_{0,\bar{s}})$$

and such that there is $\xi \in \hat{\mathbb{Z}}^\times(1)$ with

$$\xi\psi(v_1, , v_1) = E^\lambda(\eta(v_1), \eta(v_2)), \quad v_1, v_2 \in \Gamma.$$

We require that the following condition (KC) holds,

$$\mathrm{char}(T, \iota(b) \mid \mathrm{Lie}\, A_0) = \prod_{\varphi: K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}^o_{B/K}(T - b))^{r_\varphi}, \quad b \in O_B. \tag{3.5}$$

If we start with a point $(A, \iota, \bar{\lambda}, \bar{\eta})$ of the first description of $\mathcal{A}_{\mathbf{K}}(S)$, we construct a point of the second description as follows. We choose $\eta \in \bar{\eta}$. Then there is an abelian variety $A_0 \in A$ such that

$$\eta : \Gamma \otimes \hat{\mathbb{Z}} \xrightarrow{\sim} \hat{T}(A_0).$$

Then A_0 is independent of the choice of η. There exists a unique $\lambda \in \bar{\lambda}$ such that the equation (3.1) holds with $\xi \in \hat{\mathbb{Z}}^\times(1)$. Modulo m we obtain an isomorphism

$$\eta_m : \Gamma/m\Gamma \xrightarrow{\sim} \hat{T}(A_0)/m\hat{T}(A_0) \cong A_0[m],$$

which is also independent of the choice of η. Conversely, to produce from a point of the second decription a point of the first description is even more obvious and we omit it.

It follows from these considerations that the functor $\mathcal{A}_{\mathbf{K}}$ is representable if $\mathbf{K}$ satisfies the following condition:

$$\begin{gathered} \textit{There is a } \mathbb{Z}\textit{-lattice } \Gamma \subset V \textit{ and an integer } m \geq 3 \textit{ such that} \\ \mathbf{K} \subset \{g \in G(\mathbb{A}_f) \mid g(\Gamma \otimes \hat{\mathbb{Z}}) \subset \Gamma \otimes \hat{\mathbb{Z}},\, g \equiv \mathrm{id} \mod m(\Gamma \otimes \hat{\mathbb{Z}})\}. \end{gathered} \tag{3.6}$$

As above (2.14) we will assume that all prime ideals of O_F over p are split in K/F. Let $K \to F_{\mathfrak{p}_i}$ be the embedding over F which induces the prime ideal $\mathfrak{q}_i$ of K. It we compose the embedding with the conjugation on K, the induced prime ideal is $\bar{\mathfrak{q}}_i$. We write

$$\begin{array}{rcl} K \otimes_F F_{\mathfrak{p}_i} & \xrightarrow{\sim} & F_{\mathfrak{p}_i} \times F_{\mathfrak{p}_i} = K_{\mathfrak{q}_i} \times K_{\bar{\mathfrak{q}}_i} \\ x \otimes f & \longmapsto & (xf, \bar{x}f) \end{array}$$

We will from now on always assume that the function r_φ satifies (2.16). We consider the moduli problem $\mathcal{A}_{\mathbf{K}}$ over the local reflex field E_ν (2.18). We will extend it to a moduli problem over O_{E_ν}. For this, we need to impose some restrictions on $\mathbf{K}$.

We set

$$V_{\mathfrak{q}_i} = V \otimes_K K_{\mathfrak{q}_i}, \quad V_{\bar{\mathfrak{q}}_i} = V \otimes_K K_{\bar{\mathfrak{q}}_i}, \quad V_{\mathfrak{p}_i} = V \otimes_F F_{\mathfrak{p}_i} = V_{\mathfrak{q}_i} \oplus V_{\bar{\mathfrak{q}}_i}.$$

We use the decompositions

$$B \otimes \mathbb{Q}_p = \prod_{i=0}^{s} (B_{\mathfrak{q}_i} \times B_{\bar{\mathfrak{q}}_i}), \quad V \otimes \mathbb{Q}_p = \bigoplus_{i=0}^{s} V_{\mathfrak{p}_i} = \bigoplus_{i=0}^{s} (V_{\mathfrak{q}_i} \oplus V_{\bar{\mathfrak{q}}_i}), \tag{3.7}$$

cf. (2.15). All $V_{\mathfrak{p}_i}$ in the last decomposition are orthogonal with respect to $\psi_p : V \otimes \mathbb{Q}_p \times V \otimes \mathbb{Q}_p \to \mathbb{Q}_p$.

An element $g \in G(\mathbb{Q}_p)$ has the form $g = (\ldots, g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i}, \ldots)$, where $g_{\mathfrak{q}_i} \in \mathrm{End}_{B_{\mathfrak{q}_i}} V_{\mathfrak{q}_i}$ and $g_{\bar{\mathfrak{q}}_i} \in \mathrm{End}_{B_{\bar{\mathfrak{q}}_i}} V_{\bar{\mathfrak{q}}_i}$. We define $g'_{\mathfrak{q}_i} \in \mathrm{End}_{B_{\bar{\mathfrak{q}}_i}} V_{\bar{\mathfrak{q}}_i}$ by

$$\psi_p(g_{\mathfrak{q}_i} v, w) = \psi_p(v, g'_{\mathfrak{q}_i} w), \quad v \in V_{\mathfrak{q}_i},\ w \in V_{\bar{\mathfrak{q}}_i}. \tag{3.8}$$

We see that $g \in G(\mathbb{Q}_p)$ if and only if

$$g'_{\mathfrak{q}_i} g_{\bar{\mathfrak{q}}_i} \in \mathbb{Q}_p^\times \tag{3.9}$$

and if this value is independent of i. We set

$$G_{\mathfrak{q}_i} = \mathrm{Aut}_{B_{\mathfrak{q}_i}} V_{\mathfrak{q}_i} \tag{3.10}$$

By (3.9) we obtain a canonical isomorphism

$$G(\mathbb{Q}_p) \cong G_{\mathfrak{q}_0} \times \ldots \times G_{\mathfrak{q}_s} \times \mathbb{Q}_p^\times. \tag{3.11}$$

The multiplier homomorphism $\mu : G(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ corresponds on the right hand side to the projection on the factor $\mathbb{Q}_p^\times$.

We are only interested in the case where

$$D_{\mathfrak{p}_0}^{\text{opp}} \cong B_{\mathfrak{q}_0} \cong B_{\bar{\mathfrak{q}}_0} \quad \textit{is a quaternion division algebra over } F_{\mathfrak{p}_0}. \tag{3.12}$$

For each prime $\mathfrak{q}_i$, $i = 0, 1, \ldots, s$ we choose a maximal order $O_{B_{\mathfrak{q}_i}} \subset B_{\mathfrak{q}_i}$. The image of $O_{B_{\mathfrak{q}_i}}$ by the involution $\star : B_{\mathfrak{q}_i} \to B_{\bar{\mathfrak{q}}_i}$ will be denoted by $O_{B_{\bar{\mathfrak{q}}_i}}$.

We set $\Lambda_{\mathfrak{q}_i} = O_{B_{\mathfrak{q}_i}} \subset V_{\mathfrak{q}_i}$. Moreover we set

$$\Lambda_{\bar{\mathfrak{q}}_i} = \{u \in V_{\bar{\mathfrak{q}}_i} \mid \psi_{\mathfrak{p}_i}(x, u) \in \mathbb{Z}_p, \text{ for all } x \in \Lambda_{\mathfrak{q}_i}\},$$

where $\psi_{\mathfrak{p}_i}$ is the restriction of ψ_p to $V_{\mathfrak{p}_i}$. Then $\Lambda_{\bar{\mathfrak{q}}_i}$ is an $O_{B_{\bar{\mathfrak{q}}_i}}$-module and the pairings

$$\psi_{\mathfrak{p}_i} : \Lambda_{\mathfrak{q}_i} \times \Lambda_{\bar{\mathfrak{q}}_i} \to \mathbb{Z}_p \tag{3.13}$$

are perfect. We write $\Lambda_{\mathfrak{p}_i} = \Lambda_{\mathfrak{q}_i} \oplus \Lambda_{\bar{\mathfrak{q}}_i}$ and $\Lambda_p = \oplus_{i=0}^s \Lambda_{\mathfrak{p}_i}$.

We choose an open subgroup $\mathbf{M} \subset \mathbb{Z}_p^\times$. We set $\mathbf{K}_{\mathfrak{q}_0} = \operatorname{Aut}_{O_{B_{\mathfrak{q}_0}}} \Lambda_{\mathfrak{q}_0}$. For $i > 0$ we choose arbitrarily open and compact subgroups $\mathbf{K}_{\mathfrak{q}_i} \subset \operatorname{Aut}_{O_{B_{\mathfrak{q}_i}}} \Lambda_{\mathfrak{q}_i}$. We set

$$G_{\mathfrak{p}_i} = \{g \in \operatorname{Aut}_{B_{\mathfrak{p}_i}} V_{\mathfrak{p}_i} \mid \psi_{\mathfrak{p}_i}(gv, gw) = \mu_{\mathfrak{p}_i}(g)\psi_{\mathfrak{p}_i}(v, w) \text{ for } \mu_{\mathfrak{p}_i}(g) \in \mathbb{Q}_p^\times\}. \tag{3.14}$$

We define $\mathbf{K}_{\mathfrak{p}_i} \subset G_{\mathfrak{p}_i}$ as the group of all pairs $g = (c_1, c_2)$ of automorphisms

$$c_1 \in \mathbf{K}_{\mathfrak{q}_i}, \quad c_2 \in \operatorname{Aut}_{B_{\bar{\mathfrak{q}}_i}} V_{\bar{\mathfrak{q}}_i}$$

such that for some $m \in \mathbf{M}$

$$\psi(c_1 v, c_2 w) = m\psi(v, w), \quad \text{for all } v \in V_{\mathfrak{q}_i}, w \in V_{\bar{\mathfrak{q}}_i}.$$

Since $c_1(\Lambda_{\mathfrak{q}_i}) \subset \Lambda_{\mathfrak{q}_i}$ it follows from (3.8) that $c_1'(\Lambda_{\bar{\mathfrak{q}}_i}) \subset \Lambda_{\bar{\mathfrak{q}}_i}$. Since $c_1' c_2 = m$, this implies that $c_2 \in \operatorname{Aut}_{O_{B_{\bar{\mathfrak{q}}_i}}} \Lambda_{\bar{\mathfrak{q}}_i}$.

We obtain an isomorphism

$$\begin{array}{ccc} \mathbf{K}_{\mathfrak{p}_i} & \cong & \mathbf{K}_{\mathfrak{q}_i} \times \mathbf{M} \\ (c_1, c_2) & \longmapsto & c_1 \times m. \end{array}$$

We define the subgroup $\mathbf{K}_p \subset G(\mathbb{Q}_p)$ as

$$\begin{aligned} \mathbf{K}_p = \quad & \{g = (g_{\mathfrak{p}_i}) \in \textstyle\prod_i \mathbf{K}_{\mathfrak{p}_i} \mid \mu(g_{\mathfrak{p}_0}) = \ldots = \mu(g_{\mathfrak{p}_s}) \in \mathbf{M}\} \\ \cong \quad & \mathbf{K}_{\mathfrak{q}_0} \times \ldots \times \mathbf{K}_{\mathfrak{q}_s} \times \mathbf{M}. \end{aligned} \tag{3.15}$$

The last equation follows from (3.11). We choose an arbitrary open compact subgroup $\mathbf{K}^p \subset G(\mathbb{A}_f^p)$ and define

$$\mathbf{K} = \mathbf{K}_p \mathbf{K}^p \subset G(\mathbb{A}_f). \tag{3.16}$$

This concludes the description of the class of open compact subgroups $\mathbf{K}$ for which we will extend $\mathcal{A}_{\mathbf{K}}$ to a moduli problem over $\operatorname{Spec} O_{E_\nu}$. For these $\mathbf{K}$ we may reformulate the Definition 3.1 of the functor $\mathcal{A}_{\mathbf{K}}$. The datum $\bar{\eta}$ is then the product of two classes $\bar{\eta}^p$ modulo $\mathbf{K}^p$, resp. $\bar{\eta}_p$ modulo $\mathbf{K}_p$, of isomorphisms

$$\eta^p : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} V_f^p(A), \quad \text{resp.} \quad \eta_p : V \otimes \mathbb{Q}_p \xrightarrow{\sim} V_p(A),$$

which respect the bilinear forms on both sides up to a constant in $(\mathbb{A}_f^p)^\times(1)$, resp. $\mathbb{Q}_p^\times(1)$. In particular there is for each $\lambda \in \bar{\lambda}$ locally on S a constant $\xi_p(\lambda) \in \mathbb{Q}_p^\times(1)$ such that for the Riemann form E^λ

$$E^\lambda(\eta_p(v), \eta_p(w)) = \xi_p(\lambda)\psi(v, w), \quad v, w \in V \otimes_{\mathbb{Q}} \mathbb{Q}_p. \tag{3.17}$$

If we change η_p in its class by an element $g \in \mathbf{K}_p$ we find

$$E^\lambda(\eta_p(gv), \eta_p(gw)) = \xi_p(\lambda)\psi(gv, gw) = \xi_p(\lambda)\mu(g)\psi(v, w).$$

Since $\mu(g) \in \mathbf{M}$, the class of $\xi_p(\lambda) \in \mathbb{Q}_p^\times(1)/\mathbf{M}$ is well-defined by the class $\bar{\eta}_p$. If we change λ into $u\lambda$ for $u \in \mathbb{Q}^\times$, we obtain

$$\xi_p(u\lambda) = u\xi_p(\lambda).$$

By (3.7) η_p decomposes into isomorphisms

$$\eta_{\mathfrak{q}_i} : V \otimes_K K_{\mathfrak{q}_i} \xrightarrow{\sim} V_{\mathfrak{q}_i}(A), \quad \eta_{\bar{\mathfrak{q}}_i} : V \otimes_K K_{\bar{\mathfrak{q}}_i} \xrightarrow{\sim} V_{\bar{\mathfrak{q}}_i}(A), \quad \text{for } i = 0, \dots, s.$$

The equation (3.17) becomes equivalent to the equations for $i = 0, \dots, s$,

$$E^{\lambda}(\eta_{\mathfrak{q}_i}(v_i), \eta_{\bar{\mathfrak{q}}_i}(w_i)) = \xi_p(\lambda)\psi(v_i, w_i), \quad v_i \in V \otimes_K K_{\mathfrak{q}_i},\ w_i \in V \otimes_K K_{\bar{\mathfrak{q}}_i}. \tag{3.18}$$

From these equations it is clear that the set of data $\eta_{\mathfrak{q}_i}, \eta_{\bar{\mathfrak{q}}_i}$ is determined by $\eta_{\mathfrak{q}_i}, \xi_p(\lambda)$.

We obtain the following reformulation of Definition 3.1.

Definition 3.2. (alternative of Definition 3.1 for $\mathcal{A}_{\mathbf{K}}$) Let $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p \subset G(\mathbb{A}_f)$, where $\mathbf{K}_p$ is defined as in (3.15). Then we can replace the datum (c) of Definition 3.1 by the following data

(c^p) A class $\bar{\eta}^p$ modulo $\mathbf{K}^p$ of $B \otimes \mathbb{A}_f^p$-module isomorphisms

$$\eta^p : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathrm{V}_f^p(A)$$

such that for each $\lambda \in \bar{\lambda}$ there is a constant $\xi^{(p)}(\lambda) \in \mathbb{A}_f^p(1)$ with

$$\xi^{(p)}(\lambda)\psi(v_1, v_2) = E^{\lambda}(\eta^p(v_1), \eta^p(v_2)).$$

(c_p) For each $i = 0, \dots s$ a class $\bar{\eta}_{\mathfrak{q}_i}$ modulo $\mathbf{K}^{\bullet}_{\mathfrak{q}_i}$ of $B_{\mathfrak{q}_i}$-module isomorphisms

$$\eta_{\mathfrak{q}_i} : V \otimes_K K_{\mathfrak{q}_i} \xrightarrow{\sim} V_{\mathfrak{q}_i}(A).$$

(ξ_p) A function

$$\xi_p : \bar{\lambda} \to \mathbb{Q}_p^{\times}(1)/\mathbf{M}$$

such that $\xi_p(u\lambda) = u\xi_p(\lambda)$ for each $u \in \mathbb{Q}^{\times}$.

Let $g \in G(\mathbb{Q}_p) \subset G(\mathbb{A}_f)$ and consider the Hecke operator

$$g : \mathcal{A}_{\mathbf{K}} \to \mathcal{A}_{g^{-1}\mathbf{K}g} \tag{3.19}$$

which maps a point $(A, \iota, \bar{\lambda}, \bar{\eta})$ to $(A, \iota, \bar{\lambda}, \overline{\eta g})$. This makes sense, since with $\mathbf{K}$ also $g^{-1}\mathbf{K}g$ satisfies the conditions imposed in (3.16). According to the isomorphism (3.11) we represent g in the form $(\dots, g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i}, \dots, \mu(g))$. In terms of the Definition 3.2, the point is represented in the form $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \xi_p)$. The Hecke operator (3.19) maps it to the point $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\overline{\eta_{\mathfrak{q}_i} g_{\mathfrak{q}_i}})_i, \mu(g)\xi_p)$.

Let $u \in \mathbb{Q}_p^{\times}$. The action which maps $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \xi_p)$ to $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, u\xi_p)$ is denoted by

$$u_{|\xi} : \mathcal{A}_{\mathbf{K}} \to \mathcal{A}_{\mathbf{K}}. \tag{3.20}$$

The element

$$s_u := (1, \dots 1) \times (u, \dots, u) \in (K \otimes \mathbb{Q}_p)^{\times} = (\prod_{i=0}^{s} K_{\mathfrak{q}_i}^{\times}) \times (\prod_{i=0}^{s} K_{\bar{\mathfrak{q}}_i}^{\times})$$

lies in $G(\mathbb{Q}_p)$. The action of the Hecke operator s_u coincides with the action of $u_{|\xi}$. The notation $u_{|\xi}$ is a reminder that the action of s_u only changes the last entry in a tuple $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \xi_p)$.

We set

$$O_{B,p} = \prod_{i=0}^{s} (O_{B_{\mathfrak{q}_i}} \times O_{B_{\bar{\mathfrak{q}}_i}}) \subset B \otimes \mathbb{Q}_p. \tag{3.21}$$

Let $O_{B,(p)} \subset B$ the subring of elements which lie in $O_{B,p}$. This subring is invariant under the involution $\star$, cf. (2.9). We will consider abelian schemes A with an action

$$O_{K,(p)} = O_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Then $O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ acts on the p-divisible group X of A. Therefore we obtain a decomposition

$$X = (\prod_{i=0}^{s} X_{\mathfrak{q}_i}) \times (\prod_{i=0}^{s} X_{\bar{\mathfrak{q}}_i}).$$

We will write $X_{\mathfrak{p}_i} = X_{\mathfrak{q}_i} \times X_{\bar{\mathfrak{q}}_i}$. This continues to make sense if A is an abelian scheme up to isogeny of order prime to p. We set

$$U_p(\mathbb{Q}) = \{d \in \mathbb{Q}^{\times} \mid \operatorname{ord}_p d = 0\} = \mathbb{Z}_{(p)}^{\times}.$$

Definition 3.3. Let $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p \subset G(\mathbb{A}_f)$, where $\mathbf{K}_p$ is defined as in (3.15). We define a variant $\mathcal{A}^{bis}_{\mathbf{K}}$ of the functor $\mathcal{A}_{\mathbf{K}}$. A point of $\mathcal{A}^{bis}_{\mathbf{K}}$ with values in an E-scheme S consists of:

(a) An abelian scheme A over S up to isogeny of degree prime to p with an action $\iota : O_{B,(p)} \rightarrow \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$,

(b) A $U_p(\mathbb{Q})$-homogeneous polarization $\bar{\lambda}$ of A which induces on B the involution $b \mapsto b^{\star}$ and which is principal in p, i.e. of degree prime to p.

(c) A class $\bar{\eta}^p$ modulo $\mathbf{K}^p$ of $B \otimes \mathbb{A}^p_f$-module isomorphisms

$$\eta^p : V \otimes \mathbb{A}^p_f \xrightarrow{\sim} \mathrm{V}^p_f(A).$$

such that for each $\lambda \in \bar{\lambda}$ there is a constant $\xi^{(p)}(\lambda) \in \mathbb{A}^p_f(1)$ with

$$\xi^{(p)}(\lambda)\psi(v_1, v_2) = E^{\lambda}(\eta^p(v_1), \eta^p(v_2)).$$

(d) For each polarization $\lambda \in \bar{\lambda}$ a section $\xi_p(\lambda) \in \mathbb{Z}^{\times}_p(1)/\mathbf{M}$ such that $\xi_p(u\lambda) = u\xi_p(\lambda)$ for each $u \in U_p(\mathbb{Q})$.

(e) A class $\bar{\eta}_{\mathfrak{q}_j}$ modulo $\mathbf{K}_{\mathfrak{q}_i}$ of $O_{B_{\mathfrak{q}_j}}$-module isomorphisms for each $j = 1, \dots s$,

$$\eta_{\mathfrak{q}_j} : \Lambda_{\mathfrak{q}_j} \xrightarrow{\sim} T_{\mathfrak{q}_j}(A).$$

We require that the condition (KC) holds

$$\operatorname{char}(T, \iota(b) \mid \operatorname{Lie} A) = \prod_{\varphi : K \rightarrow \bar{\mathbb{Q}}} \varphi(\operatorname{Nm}^o_{B/K}(T - b))^{r_{\varphi}}. \tag{3.22}$$

Here and in the sequel, the index i will run through $0, \dots, s$ and the index j through $1, \cdots, s$.

Proposition 3.4. *The functors $\mathcal{A}_{\mathbf{K}}$ and $\mathcal{A}^{bis}_{\mathbf{K}}$ on the category of E-schemes are isomorphic.*

Proof. We begin with a point $(A, \iota, \bar{\lambda}, \bar{\eta})$ of $\mathcal{A}_{\mathbf{K}}(S)$ and construct a point of $\mathcal{A}^{bis}_{\mathbf{K}}(S)$. We choose an element $\eta \in \bar{\eta}$ and consider the component $\eta_p : V \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{V}_p(A)$. By the choice of $\mathbf{K}_p$, the image of Λ_p by this morphism is independent of the choice of η_p. Therefore we find an abelian variety $A_0 \in A$ up to isogeny prime to p such that η_p induces an isomorphism

$$\eta_p : \Lambda_p \xrightarrow{\sim} T_p(A_0).$$

We choose a polarization $\lambda_0 \in \bar{\lambda}$. Then we obtain

$$E^{\lambda_0}_p(\eta_p(v_1), \eta_p(v_2)) = \xi\psi_p(v_1, v_2), \quad v_1, v_2 \in \Lambda_p, \ \xi \in \mathbb{Q}^{\times}_p(1).$$

After multiplying λ_0 by a power of p we may assume that $\xi \in \mathbb{Z}^{\times}_p(1)$. This determines an $U_p(\mathbb{Q})$-homogeneous polarization $\bar{\lambda}_0$ on A_0. We remark that the class of ξ in $\mathbb{Z}^{\times}_p(1)/\mathbf{M}$ is independent of the choice of η. Finally η_p induces $O_{B_{\mathfrak{q}_i}}$-module isomorphisms $\eta_{\mathfrak{q}_i} : \Lambda_{\mathfrak{q}_i} \xrightarrow{\sim} T_{\mathfrak{q}_i}(A_0)$. We have obtained a point $(A_0, \iota, \bar{\lambda}, \bar{\eta}^p, (\eta_{\mathfrak{q}_j})_j, \xi_p)$ of $\mathcal{A}^{bis}_{\mathbf{K}}$.

Conversely assume that $(A_0, \iota, \bar{\lambda}, \bar{\eta}^p, (\eta_{\mathfrak{q}_j})_j, \xi_p) \in \mathcal{A}^{bis}_{\mathbf{K}}$ is given. Let $\lambda \in \bar{\lambda}_0$. The Riemann form E^{λ} on $T_{\mathfrak{p}_0}(A_0)$ is by assumption perfect. Therefore we find for any given $\xi' \in \mathbb{Z}^{\times}_p(1)$ an isomorphism of $O_{B_{\mathfrak{p}_0}}$-modules

$$\eta_{\mathfrak{p}_0} : \Lambda_{\mathfrak{p}_0} \xrightarrow{\sim} T_{\mathfrak{p}_0}(A_0)$$

such that

$$\xi'\psi(v_1, v_2) = E^{\lambda}(\eta_{\mathfrak{p}_0}(v_1), \eta_{\mathfrak{p}_0}(v_2)), \quad v_1, v_2 \in \Lambda_{\mathfrak{p}_0}.$$

We choose $\eta_{\mathfrak{p}_0}$ such that $\xi' = \xi_p(\lambda)$. For $i > 0$ the isomorphism $\eta_{\mathfrak{q}_i}$ induces by duality $\operatorname{Hom}(?, \mathbb{Z}_p)$ an isomorphism $T_{\bar{\mathfrak{q}}_i}(A_0) \xrightarrow{\sim} \Lambda_{\bar{\mathfrak{q}}_i}(1)$. We multiply the inverse map with $\xi_p(\lambda)$ and obtain

$$\eta_{\bar{\mathfrak{q}}_i} : \Lambda_{\bar{\mathfrak{q}}_i} \xrightarrow{\sim} T_{\bar{\mathfrak{q}}_i}(A_0).$$

We obtain an isomorphism

$$\eta_{\mathfrak{p}_i} = \eta_{\mathfrak{q}_i} \oplus \eta_{\bar{\mathfrak{q}}_i} : \Lambda_{\mathfrak{p}_i} \xrightarrow{\sim} T_{\mathfrak{p}_i}(A_0)$$

which respects the bilinear forms on both sides up to the factor $\xi_p(\lambda)$. Then $\eta_p = \oplus^s_{i=0}\eta_{\mathfrak{p}_i}$ defines an isomorphism $\eta_p : \Lambda_p \rightarrow T_p(A_0)$ such that

$$\xi_p(\lambda)\psi(v_1, v_2) = E^{\lambda}(\eta_p(v_1), \eta_p(v_2)), \quad v_1, v_2 \in \Lambda_p.$$

Therefore we have constructed a point of $\mathcal{A}_{\mathbf{K}}(S)$. The two procedures are inverses of each other, proving the proposition. □

From now on we will always assume that $\mathbf{K}$ satisfies the assumptions made in Definition 3.3. If we write $\mathcal{A}_{\mathbf{K}}$, we understand that this functor is given in the form of Definition 3.3. To extend the functor $\mathcal{A}_{\mathbf{K}}(S)$ to an arbitrary O_{E_ν}-scheme S, the main obstacle is the datum (d), since we do not have the $(\mathbb{Z}/p\mathbb{Z})^\times$-étale torsor of primitive pth roots of unity. Therefore we define a new functor $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ by replacing the data $\xi_p(\lambda) \in \mathbb{Z}_p^\times(1)/\mathbf{M}$ by sections in the constant sheaf $\xi_p(\lambda) \in \mathbb{Z}_p^\times/\mathbf{M}$. Here the upper index t in $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ indicates that this functor, when restricted to the category of E_ν-schemes, is a twisted version of $\mathcal{A}_{\mathbf{K}}$.

Definition 3.5. Let $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p \subset G(\mathbb{A}_f)$, where $\mathbf{K}_p$ is defined as in (3.15). We define a functor $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ on the category of O_{E_ν}-schemes S. An S-valued point consists of the data $(a), (b), (c), (e)$ as in Definition 3.3. But we replace (d) by the following datum

(d^t) A section $\xi_p(\lambda) \in \mathbb{Z}_p^\times/\mathbf{M}$ for each polarization $\lambda \in \bar{\lambda}$ such that

$$\xi_p(u\lambda) = u\xi_p(\lambda), \quad u \in U_p(\mathbb{Q}).$$

We continue to impose the condition (KC).

The data (a)–(c) continue to make sense over a O_{E_ν}-scheme S. Since an isogeny of degree prime to p induces an isomorphism on tangent spaces, the condition (KC) also makes sense. Since $r_\varphi = 0$ for each $\varphi : K_{\mathfrak{q}_j} \to \bar{\mathbb{Q}}_p$ for $j = 1, \ldots, s$, the p-divisible groups $X_{\mathfrak{q}_j}$ are étale. We note that $X_{\mathfrak{q}_j}$ is a p-divisible group of height $4[K_{\mathfrak{q}_j} : \mathbb{Q}_p]$ and this implies that $T_{\mathfrak{q}_j}(A)$ is a free $O_{B_{\mathfrak{q}_j}}$-module of rank 1. Therefore the datum (e) also continues to make sense.

The functor $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ is representable if the group $\mathbf{K}$ satisfies the condition (3.6) for some integer $m \geq 3$ which is prime to p. A standard argument shows that $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ is proper over $\operatorname{Spec} O_{E_\nu}$, cf. [Dr, Prop. 4.1]. If we have another open and compact subgroup $\tilde{\mathbf{K}} \subset \mathbf{K}$ as in Definition 3.5, we obtain an étale covering

$$\tilde{\mathcal{A}}^t_{\tilde{\mathbf{K}}} \to \tilde{\mathcal{A}}^t_{\mathbf{K}}.$$

The general fibre $\tilde{\mathcal{A}}^t_{\mathbf{K},E_\nu}$ of $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ is a Galois twist of $\mathcal{A}_{\mathbf{K},E_\nu} = \mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} E_\nu$. Let us explain this. The definition of $\tilde{\mathcal{A}}^t_{\mathbf{K}}(S)$ makes sense for any E-scheme S. We denote this functor on the category of E-schemes by $\mathcal{A}^t_{\mathbf{K}}$. (The fact, used above, that the p-divisible groups $X_{\mathfrak{q}_j}$ are étale is automatic in characteristic 0, even for $X_{\mathfrak{q}_0}$.) We may represent a point of $\mathcal{A}^t_{\mathbf{K}}(S)$ in the same way as in Definition 3.2 by

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \xi_p), \tag{3.23}$$

except that now ξ_p is a function $\xi_p : \bar{\lambda} \to \mathbb{Q}_p^\times/\mathbf{M}$. There is the canonical isomorphism

$$\mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} E_\nu \cong \tilde{\mathcal{A}}^t_{\mathbf{K}} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} E_\nu. \tag{3.24}$$

We will identify $\mathcal{A}^t_{\mathbf{K}}$ with a Shimura variety of the form $\operatorname{Sh}(G, h \cdot c)$ for some $c : \mathbb{S} \to G_{\mathbb{R}}$ which factors through the center of G. We consider the cyclotomic character

$$\varsigma_{p^\infty} : \operatorname{Gal}(\bar{\mathbb{Q}}/E) \to \mathbb{Z}_p^\times/\mathbf{M}.$$

Let $L \subset \bar{\mathbb{Q}}$ be the subfield fixed by the kernel of this homomorphism. Let $\zeta_{p^\infty} \in \mathbb{C}$ be a compatible system of primitive p^n-th roots of unity. We obtain a natural isomorphism

$$\begin{array}{ccc} \mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L & \xrightarrow{\sim} & \mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L, \\ \xi_p & \longmapsto & \zeta_{p^\infty}\xi_p \end{array} \tag{3.25}$$

i.e., in Definition 3.3 we have to change only (d) to pass from one functor to the other.

Proposition 3.6. *Let $\tau \in \operatorname{Gal}(L/E)$ be an automorphism. The morphism (3.25) fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L & \longrightarrow & \mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L \\ \Big\downarrow {\scriptstyle \operatorname{id} \times \tau_c} & & \Big\downarrow {\scriptstyle \varsigma_{p^\infty}(\tau^{-1})_{|\xi} \times \tau_c} \\ \mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L & \longrightarrow & \mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L. \end{array}$$

(see (3.20) for the notation $\varsigma_{p^\infty}(\tau^{-1})_{|\xi}$).

Proof. Let $\pi\colon S \to \operatorname{Spec} L$ be an L-scheme. Denote by $S_{[\tau_c]}$ the L-scheme obtained when the structure morphism is changed to $\tau_c \circ \pi$. Our assertion says that the following diagram is commutative

$$\begin{array}{ccc} (\mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L)(S) & \longrightarrow & (\mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L)(S) \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle \varsigma_{p^\infty}(\tau^{-1})_{|\xi}\circ can} \\ (\mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L)(S_{[\tau_c]}) & \longrightarrow & (\mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} L)(S_{[\tau_c]}). \end{array} \tag{3.26}$$

The morphism *can* on the left hand side is defined by the identification $\mathcal{A}^t_{\mathbf{K}}(S) = \mathcal{A}^t_{\mathbf{K}}(S_{[\tau_c]})$ which exists because the functor is defined over E. In the same way there is *can* on the right hand side. To show the commutativity of (3.26), we start with a point $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \xi_p)$ of the upper left corner. The image Θ of this point by the left *can* in $\mathcal{A}^t_{\mathbf{K}}(S_{[\tau_c]})$ is given by the same tuple. By the upper horizontal map the point is mapped to

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \pi^*(\zeta_{p^\infty})\xi_p) \tag{3.27}$$

where $\pi^* : L \to \Gamma(S, \mathcal{O}_S)$ is the comorphism of the structure map. The image of (3.27) by the right *can*-morphism is represented by the same tuple but we write the last item as

$$\pi^*(\tau^{-1}(\tau(\zeta_{p^\infty})))\xi_p = \varsigma_{p^\infty}(\tau)\pi^*(\tau^{-1}(\zeta_{p^\infty}))\xi_p.$$

On the other hand the image of Θ by the lower horizontal bijection is the tuple

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, \pi^*(\tau^{-1}(\zeta_{p^\infty}))\xi_p).$$

This shows the commutativity of (3.26). □

Let $\Xi \subset \Phi$ be the CM-type of K defined by

$$\Xi = \coprod_{i=0}^{s} \operatorname{Hom}_{\mathbb{Q}_p-\mathrm{Alg}}(K_{\bar{\mathfrak{q}}_i}, \bar{\mathbb{Q}}_p).$$

As defined by diagram (2.13), we can write

$$\Xi = \{\bar{\varphi}_0\} \cup \{\varphi \in \Phi \mid r_\varphi = 2\}. \tag{3.28}$$

We obtain an isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \prod\nolimits_{\Xi} \mathbb{C}, \quad a \otimes \rho \longmapsto (\varphi(a)\rho)_{\varphi\in\Xi}. \tag{3.29}$$

This puts a complex structure on the left hand side and hence defines a homomorphism $\mathbb{C}^\times \to (K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ which we view as a morphism of algebraic groups $\delta : \mathbb{S} \to (\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_{m,K})_{\mathbb{R}}$. We consider the algebraic torus over $\mathbb{Q}$ given by

$$T(\mathbb{Q}) = \{t \in K^\times \mid t\bar{t} \in \mathbb{Q}^\times\} \subset G(\mathbb{Q}).$$

This is a central subtorus of G. Clearly δ factors through

$$\delta : \mathbb{S} \to T_{\mathbb{R}} \subset G_{\mathbb{R}}. \tag{3.30}$$

Let $\operatorname{Sh}(G, h\delta^{-1})$ be the canonical model over $E(G, h\delta^{-1})$ of the Shimura variety attached to the Shimura datum $(G, h\delta^{-1})$. Note that $E(G, h\delta^{-1}) \subset E_\nu$. Indeed, if we restrict $\delta_{\mathbb{C}} : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to (K \otimes_{\mathbb{Q}} \mathbb{C})^\times$ to the first factor, we obtain

$$\mu_\delta : \mathbb{G}_{m,\mathbb{C}} \to (K \otimes_{\mathbb{Q}} \mathbb{C})^\times \cong \prod_{\Phi} \mathbb{C}^\times, \quad z \longmapsto (1, \ldots 1, z, \ldots, z). \tag{3.31}$$

Here z appears exactly at the places $\Xi \subset \Phi$. By our choice of the diagram (2.13) μ_δ is defined over $\mathbb{Q}_p$ since it comes from the canonical embedding

$$\mathbb{Q}_p^\times \to \prod_{i=0}^{s} K_{\bar{\mathfrak{q}}_i} \subset (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times.$$

Therefore $E(G, \delta^{-1}) \subset \mathbb{Q}_p$, which shows our assertion.

Proposition 3.7. *Let* $\mathbf{K} \subset G(\mathbb{A}_f)$ *be as in Definition 3.5. We denote by* $\mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1})_{E_\nu}$ *the scheme obtained by base change via* $E(G, h\delta^{-1}) \subset E_\nu$ *from the canonical model. We set* $\mathcal{A}^t_{\mathbf{K},E_\nu} = \tilde{\mathcal{A}}^t_{\mathbf{K}} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} E_\nu = \mathcal{A}^t_{\mathbf{K}} \times_{\operatorname{Spec} E} \operatorname{Spec} E_\nu$ *(cf. (3.24)). There exists an isomorphism over the maximal unramified extension* E^{nr}_ν

$$\mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1})_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu \xrightarrow{\sim} \mathcal{A}^t_{\mathbf{K},E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu, \tag{3.32}$$

which for varying $\mathbf{K}$ *is compatible with the Hecke operators in* $G(\mathbb{A}^p_f)$.

Let $\tau \in \mathrm{Gal}(E^{nr}_\nu/E_\nu)$ *be the Frobenius automorphism. Let* f_ν *be the inertia index of* $E_\nu/\mathbb{Q}_p$. *Then the following diagram is commutative,*

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1})_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu & \longrightarrow & \mathcal{A}^t_{\mathbf{K},E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\tau_c} & & \Big\downarrow{\scriptstyle p^{f_\nu}{}_{|\xi}\times\tau_c} \\ \mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1})_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu & \longrightarrow & \mathcal{A}^t_{\mathbf{K},E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{nr}_\nu . \end{array}$$

(See (3.20) for the notation $p^{f_\nu}{}_{|\xi}$).

Proof. Let $e \in E^\times_\nu$ and let $\sigma \in \mathrm{Gal}(E^{ab}_\nu/E_\nu)$ be the automorphism that corresponds to it by local class field theory. We set $\varsigma_{p^\infty}(e) = \varsigma_{p^\infty}(\sigma)$. Then we obtain from local class field theory

$$\varsigma_{p^\infty}(e) = (\mathrm{Nm}_{E_\nu/\mathbb{Q}_p} e)^{-1} p^{f_\nu \operatorname{ord} e}, \tag{3.33}$$

where $\operatorname{ord} = \operatorname{ord}_\nu \colon E^\times_\nu \to \mathbb{Z}$ maps a uniformizer to 1. Indeed, this formula makes sense for an arbitrary p-adic local field E_ν. In the case $E_\nu = \mathbb{Q}_p$ (3.33) follows from [CF] Chapt. VI, Thm. 3.2. In the general case the action of σ on μ_{p^∞} depends only on the restriction of σ to $\mathbb{Q}^{\mathrm{ab}}$. But this restriction corresponds to $\mathrm{Nm}_{E_\nu/\mathbb{Q}_p} e$ by the reciprocity law of the local field $\mathbb{Q}_p$ by the last diagram of [CF] Chapt. VI, §2.4. This shows (3.33).

It is enough to show that the two squares in the following diagram are commutative.

$$\begin{array}{ccccc} \mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1}) \times \operatorname{Spec} E^{ab}_\nu & \longrightarrow & \mathcal{A}_{\mathbf{K}} \times \operatorname{Spec} E^{ab}_\nu & \longleftarrow & \mathcal{A}^t_{\mathbf{K},E_\nu} \times \operatorname{Spec} E^{ab}_\nu \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\sigma_c} & & \Big\downarrow{\scriptstyle (\varsigma_{p^\infty}(e^{-1})p^{f_\nu \operatorname{ord} e})_{|\xi}\times\sigma_c} & & \Big\downarrow{\scriptstyle (p^{f_\nu \operatorname{ord} e})_{|\xi}\times\sigma_c} \\ \mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1}) \times \operatorname{Spec} E^{ab}_\nu & \longrightarrow & \mathcal{A}_{\mathbf{K}} \times \operatorname{Spec} E^{ab}_\nu & \longleftarrow & \mathcal{A}^t_{\mathbf{K},E_\nu} \times \operatorname{Spec} E^{ab}_\nu \end{array}$$

The commutativity of the diagram on the right hand side follows from Proposition 3.6. Since $\mathcal{A}_{\mathbf{K}} \cong \mathrm{Sh}_{\mathbf{K}}(G, h)$, the square on the left hand side becomes commutative if we replace the vertical arrow in the middle by

$$r^{\mathrm{cft}}_\nu(T, \delta^{-1})(\sigma) \times \sigma_c \colon \mathrm{Sh}_{\mathbf{K}}(G, h)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{ab}_\nu \to \mathrm{Sh}_{\mathbf{K}}(G, h)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E^{ab}_\nu.$$

This is a consequence of Corollary 8.5. It remains to compute the class field theory version $r^{\mathrm{cft}}_\nu(T, \delta^{-1})$ of the reciprocity law.

The morphism $\mu_{\delta^{-1}}$ is the inverse of (3.31). Therefore we find the local reciprocity law

$$\begin{array}{rcl} r_\nu(T, \delta^{-1}) : E^\times_\nu & \to & \prod_{i=0}^s K^\times_{\mathfrak{q}_i} \times \prod_{i=0}^s K^\times_{\bar{\mathfrak{q}}_i} \\ e & \longmapsto & (1, \ldots, 1) \times (\mathrm{Nm}_{E_\nu/\mathbb{Q}_p} e, \ldots, \mathrm{Nm}_{E_\nu/\mathbb{Q}_p} e). \end{array} \tag{3.34}$$

We write $\mathrm{Nm}_{E_\nu/\mathbb{Q}_p} e = \varsigma_{p^\infty}(e^{-1}) p^{f \operatorname{ord} e}$, cf. (3.33). Under the isomorphism (3.11), the element on the right hand side of (3.34) corresponds to $(1, \ldots, 1, \varsigma_{p^\infty}(e^{-1})p^{f \operatorname{ord} e}) \in G_{\mathfrak{q}_0} \times \ldots \times G_{\mathfrak{q}_s} \times \mathbb{Q}^\times_p$. By the description of the Hecke operators (3.19) we obtain the proposition. ☐

We will next show how the action of $p^{f_\nu}_{|\xi}$ on $\mathcal{A}^t_{\mathbf{K},E_\nu}$ extends naturally to the model $\widetilde{\mathcal{A}}^t_{\mathbf{K}}$ over O_{E_ν}. It is enough to define $p_{|\xi}$. Let

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, \xi_p) \in \widetilde{\mathcal{A}}^t_{\mathbf{K}}(S) \tag{3.35}$$

be a point as in Definition 3.5. Let $\bar{\lambda}_{\mathbb{Q}}$ be the $\mathbb{Q}$-homogeneous polarization which contains $\bar{\lambda}$. We extend ξ_p to a map $\xi_p : \bar{\lambda}_{\mathbb{Q}} \to \mathbb{Q}^\times_p/\mathbf{M}$ such that $\xi_p(u\lambda) = u\xi_p(\lambda)$ for $u \in \mathbb{Q}^\times$. Let

$X = \prod_{i=0}^{s}(X_{\mathfrak{q}_i} \times X_{\bar{\mathfrak{q}}_i})$ be the p-divisible group of A. We consider the isogeny

$$a : \prod_{i=0}^{s}(X_{\mathfrak{q}_i} \times X_{\bar{\mathfrak{q}}_i}) \to \prod_{i=0}^{s}(X_{\mathfrak{q}_i} \times X_{\bar{\mathfrak{q}}_i}) \tag{3.36}$$

which is the identity on the factors $X_{\mathfrak{q}_i}$ and multiplication by p on the factors $X_{\bar{\mathfrak{q}}_i}$. Let us fix a polarization $\lambda \in \bar{\lambda}$. Since λ is principal in p it is given on X by an isomorphism of $X_{\bar{\mathfrak{q}}_i}$ with the dual of $X_{\mathfrak{q}_i}$ for each $i = 0, \ldots, s$. The inverse image of λ on X by (3.36) is $p\lambda$. There is an isogeny of abelian varieties with an $O_{B,(p)}$-action

$$\alpha : A' \to A,$$

of order a power of p such that the induced homomorphism of p-divisible groups is isomorphic to (3.36). A polarization λ of A induces a polarization $\lambda' = \alpha^*(\lambda) := \hat{\alpha}\lambda\alpha$ on A'. This defines a $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}'_{\mathbb{Q}}$ of A' and a bijection $\bar{\lambda}_{\mathbb{Q}} \to \bar{\lambda}'_{\mathbb{Q}}$. By this bijection ξ_p induces

$$\xi'_p : \bar{\lambda}'_{\mathbb{Q}} \to \mathbb{Q}_p^{\times}/\mathbf{M}.$$

We obtain $\xi'_p(\lambda') = \xi_p(\lambda) \in \mathbb{Z}_p^{\times}$. But the polarization $\lambda_1 = (1/p)\lambda'$ is principal in p, as we see by looking at the p-divisible groups (3.36). Then

$$(A', \iota', \bar{\lambda}_1, \bar{\eta}'^p, (\bar{\eta}'_{\mathfrak{q}_j})_j, p\xi'_p) \tag{3.37}$$

is a point of $\tilde{\mathcal{A}}^t_{\mathbf{K}}(S)$. Here η'^p, resp. $\eta'_{\mathfrak{q}_j}$, denotes the composite

$$\eta'^p : V \otimes \mathbb{A}_f^p \xrightarrow{\eta^p} \mathrm{V}_f^p(A) \xrightarrow{\alpha^{-1}} \mathrm{V}_f^p(A'),$$

resp.

$$\eta'_{\mathfrak{q}_j} : \Lambda_{\mathfrak{q}_j} \xrightarrow{\eta_{\mathfrak{q}_j}} T_{\mathfrak{q}_j}(A) \xrightarrow{\alpha^{-1}} T_{\mathfrak{q}_j}(A').$$

Note that the last arrow is an isomorphism by definition of α. The map which assigns (3.37) to (3.35) defines the desired extension of the operator $p_{|\xi}$ to $\tilde{\mathcal{A}}^t_{\mathbf{K}}$.

Corollary 3.8. *The Shimura variety* $\mathrm{Sh}_{\mathbf{K}}(G, h\delta^{-1})_{E_\nu}$ *has a unique model* $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(G, h\delta^{-1})$ *over* O_{E_ν} *with the following properties. There is an isomorphism*

$$\widetilde{\mathrm{Sh}}_{\mathbf{K}}(G, h\delta^{-1}) \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} \xrightarrow{\sim} \tilde{\mathcal{A}}^t_{\mathbf{K}} \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} \tag{3.38}$$

which is compatible with the Hecke operators $G(\mathbb{A}_f^p)$. *Let* $\tau \in \mathrm{Gal}(E_\nu^{nr}/E_\nu)$ *be the Frobenius automorphism. Let* f_ν *be the inertia index of* $E_\nu/\mathbb{Q}_p$. *Then the following diagram is commutative.*

$$\begin{array}{ccc} \widetilde{\mathrm{Sh}}_{\mathbf{K}}(G, h\delta^{-1}) \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} & \longrightarrow & \tilde{\mathcal{A}}^t_{\mathbf{K}} \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} \\ \downarrow{\scriptstyle \mathrm{id}\times\tau_c} & & \downarrow{\scriptstyle (p^{f_\nu})_{|\xi}\times\tau_c} \\ \widetilde{\mathrm{Sh}}_{\mathbf{K}}(G, h\delta^{-1}) \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} & \longrightarrow & \tilde{\mathcal{A}}^t_{\mathbf{K}} \times_{\mathrm{Spec}\, O_{E_\nu}} \mathrm{Spec}\, O_{E_\nu^{nr}} \end{array}$$

□

The homomorphism $h\delta^{-1}$ may be written as follows. It factors as

$$\mathbb{S} \longrightarrow \prod_{\chi \in \mathrm{Hom}_{\mathbb{Q}-\mathrm{Alg}}(F,\mathbb{C})} (D \otimes_{F,\chi} \mathbb{R})^{\times} \times (K \otimes_{F,\chi} \mathbb{R})^{\times} \xrightarrow{\kappa} G^{\bullet}(\mathbb{R}),$$

The image of $z = a + b\mathbf{i} \in \mathbb{C} = \mathbb{S}(\mathbb{R})$ by the first arrow is given by

$$\left(\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \times z^{-1}\right), 1, \ldots, 1\right),$$

where the first entry in this vector is at the place χ_0.

4. The moduli problem for $\mathrm{Sh}(G^\bullet, h)$ and a reduction modulo p

We recall the groups G of (2.11) and $G^\bullet$ of (2.3). We consider the map of Shimura data $(G,h) \to (G^\bullet, h)$, cf. (2.12) and (2.6). The Shimura fields coincide, i.e. $E = E(G,h) = E(G^\bullet, h)$. We consider a pair of open compact subgroups $\mathbf{K} \subset G(\mathbb{A}_f)$ and $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ such that $\mathbf{K} \subset \mathbf{K}^\bullet$ and such that the induced map $\mathrm{Sh}(G,h)_{\mathbf{K}} \to \mathrm{Sh}(G^\bullet, h)_{\mathbf{K}^\bullet}$ is a closed immersion, cf. [De, Prop. 1.15.].

The Shimura variety $\mathrm{Sh}(G^\bullet, h)_{\mathbf{K}^\bullet}$ is the coarse moduli scheme associated to the following functor.

Definition 4.1. Let $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ be an open compact subgroup. We define a functor $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}$ on the category of E-schemes. A point of $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}(S)$ is given by the following data:

(a) An abelian scheme A over S up to isogeny with an action $\iota : B \to \mathrm{End}^o A$.

(b) A F-homogeneous polarization $\bar{\lambda}$ of A which induces on B the involution $b \mapsto b^\star$.

(c) A class $\bar{\eta}$ modulo $\mathbf{K}^\bullet$ of $B \otimes \mathbb{A}_f$-module isomorphisms

$$\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} \mathrm{V}_f(A)$$

such that for each $\lambda \in \bar{\lambda}$ there is locally for the Zariski-topology on S a constant $\xi \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f)(1)$ with

$$\psi(\xi v_1, v_2) = E^\lambda(\eta(v_1), \eta(v_2)).$$

We require that the following condition (KC) holds

$$\mathrm{char}(T, \iota(b) \mid \mathrm{Lie}\, A) = \prod_{\varphi: K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}^o_{B/K}(T-b))^{r_\varphi}.$$

We will reformulate this moduli problem in a way that makes sense over O_{E_ν}. As in the previous section, we need additional requirements for the group $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$, similar to those discussed before (3.16). We assume that $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p}$ where $\mathbf{K}^\bullet_p \subset G(\mathbb{Q}_p)$. The decomposition $V \otimes \mathbb{Q}_p = \oplus_{i=0}^s V_{\mathfrak{p}_i}$ is orthogonal with respect to ψ. We obtain that

$$G^\bullet(\mathbb{Q}_p) = \prod_{i=0}^s G^\bullet_{\mathfrak{p}_i},$$

where

$$G^\bullet_{\mathfrak{p}_i} = \{g \in \mathrm{Aut}_{B_{\mathfrak{p}_i}} V_{\mathfrak{p}_i} \mid \psi_{\mathfrak{p}_i}(gv, gw) = \psi_{\mathfrak{p}_i}(\mu_{\mathfrak{p}_i}(g)v, w),\ \mu_{\mathfrak{p}_i}(g) \in F^\times_{\mathfrak{p}_i}\}. \tag{4.1}$$

This defines the homomorphism $\mu_{\mathfrak{p}_i} : G^\bullet_{\mathfrak{p}_i} \to F^\times_{\mathfrak{p}_i}$. According to the decomposition $V \otimes_F F_{\mathfrak{p}_i} = V_{\mathfrak{q}_i} \oplus V_{\bar{\mathfrak{q}}_i}$ we can write $g = (g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i})$. Then we obtain

$$G^\bullet_{\mathfrak{p}_i} = \{g = (g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i}),\ g_{\mathfrak{q}_i} \in \mathrm{Aut}_{B_{\mathfrak{q}_i}} V_{\mathfrak{q}_i},\ g_{\bar{\mathfrak{q}}_i} \in \mathrm{Aut}_{B_{\bar{\mathfrak{q}}_i}} V_{\bar{\mathfrak{q}}_i} \mid g'_{\mathfrak{q}_i} g_{\bar{\mathfrak{q}}_i} \in F^\times_{\mathfrak{p}_i}\}, \tag{4.2}$$

cf. (3.8). We note that $\mu_{\mathfrak{p}_i}(g) = g'_{\mathfrak{q}_i} g_{\bar{\mathfrak{q}}_i}$. By this equation we obtain an isomorphism of groups

$$\begin{array}{ccc} G^\bullet_{\mathfrak{p}_i} & \cong & G^\bullet_{\mathfrak{q}_i} \times F^\times_{\mathfrak{p}_i}, \\ g & \longmapsto & g_{\mathfrak{q}_i} \times \mu_{\mathfrak{p}_i}(g) \end{array} \qquad G^\bullet_{\mathfrak{q}_i} = \mathrm{Aut}_{B_{\mathfrak{q}_i}} V_{\mathfrak{q}_i}, \tag{4.3}$$

cf. (3.10). Altogether we obtain an isomorphism

$$G^\bullet(\mathbb{Q}_p) = \prod_{i=0}^s G^\bullet_{\mathfrak{q}_i} \times \prod_{i=0}^s F^\times_{\mathfrak{p}_i}. \tag{4.4}$$

We use the notations $\Lambda_{\mathfrak{q}_i}$, $\Lambda_{\bar{\mathfrak{q}}_i}$, $\Lambda_{\mathfrak{p}_i}$, $O_{B_{\mathfrak{q}_i}}$, $O_{B_{\bar{\mathfrak{q}}_i}}$, and $O_{B_{\mathfrak{p}_i}} = O_{B_{\mathfrak{q}_i}} \oplus O_{B_{\bar{\mathfrak{q}}_i}}$ as before (3.13). For each prime $\mathfrak{p}_i$, $i = 0, \ldots, s$, we choose an open subgroup $\mathbf{M}^\bullet_{\mathfrak{p}_i} \subset O^\times_{F_{\mathfrak{p}_i}}$. We set

$$\mathbf{M}^\bullet = \prod_{i=0}^s \mathbf{M}^\bullet_{\mathfrak{p}_i} \subset \prod_{i=0}^s O^\times_{F_{\mathfrak{p}_i}} = (O_F \otimes \mathbb{Z}_p)^\times. \tag{4.5}$$

As in the previous section (see right after (3.13)), we set $\mathbf{K}_{\mathfrak{q}_0} = \mathrm{Aut}_{O_{B_{\mathfrak{q}_0}}} \Lambda_{\mathfrak{q}_0}$, and choose for $j > 0$ arbitrarily open compact subgroups $\mathbf{K}_{\mathfrak{q}_j} \subset \mathrm{Aut}_{O_{B_{\mathfrak{q}_j}}} \Lambda_{\mathfrak{q}_j}$. We define, for $i = 0, \ldots, s$,

$$\mathbf{K}_{\mathfrak{p}_i} = \mathbf{K}^\bullet_{\mathfrak{q}_i} \times \mathbf{M}^\bullet_{\mathfrak{p}_i} \subset G^\bullet_{\mathfrak{q}_i} \times F^\times_{\mathfrak{p}_i} \cong G_{\mathfrak{p}_i},$$

cf. (4.3). We obtain

$$\begin{aligned}\mathbf{K}^\bullet_{\mathfrak{p}_i} = \quad & \{g = (g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i}),\ g_{\mathfrak{q}_i} \in \mathbf{K}^\bullet_{\mathfrak{q}_i},\ g_{\bar{\mathfrak{q}}_i} \in \mathrm{Aut}_{O_{B_{\bar{\mathfrak{q}}_i}}} \Lambda_{\bar{\mathfrak{q}}_i} \mid \\ & \psi_{\mathfrak{p}_i}(g_{\mathfrak{q}_i}v_1, g_{\bar{\mathfrak{q}}_i}v_2) = \psi_{\mathfrak{p}_i}(mv_1, v_2),\ v_1 \in \Lambda_{\mathfrak{q}_i},\ v_2 \in \Lambda_{\bar{\mathfrak{q}}_i},\ m \in \mathbf{M}^\bullet_{\mathfrak{p}_i}\},\end{aligned}$$

and in particular

$$\mathbf{K}^\bullet_{\mathfrak{p}_0} = \{g \in \mathrm{Aut}_{O_{B_{\mathfrak{p}_0}}} \Lambda_{\mathfrak{p}_0} \mid \psi_{\mathfrak{p}_0}(gv_1, gv_2) = \psi_{\mathfrak{p}_0}(\mu(g)v_1, v_2),\ \mu(g) \in \mathbf{M}^\bullet_{\mathfrak{p}_0}\}.$$

Finally we set

$$\mathbf{K}^\bullet_p = \prod_{i=0}^{s} \mathbf{K}^\bullet_{\mathfrak{p}_i} = \prod_{i=0}^{s} (\mathbf{K}_{\mathfrak{q}_i} \times \mathbf{M}^\bullet_{\mathfrak{p}_i}). \tag{4.6}$$

We choose $\mathbf{K}^{\bullet,p}$ arbitrarily and set

$$\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p}. \tag{4.7}$$

With these assumptions on $\mathbf{K}^\bullet$ we can rewrite the definition of the functor $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}$ as follows.

Definition 4.2. (alternative of Definition 4.1) Let $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}^\bullet_p$ as in (4.6). Let S be an E-scheme. A point of $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}(S)$ consists of the following data:

(a) An abelian scheme A over S up to isogeny with an action $\iota : B \to \mathrm{End}^o A$.

(b) A F-homogeneous polarization $\bar{\lambda}$ of A which induces on B the involution $b \mapsto b^\star$.

(c) A class $\bar{\eta}^p$ modulo $\mathbf{K}^{\bullet,p}$ of $B \otimes \mathbb{A}^p_f$-module isomorphisms

$$\eta^p : V \otimes \mathbb{A}^p_f \xrightarrow{\sim} \mathrm{V}^p_f(A),$$

such that for each $\lambda \in \bar{\lambda}$ there is locally for the Zariski topology a constant $\xi^{(p)}(\lambda) \in (F \otimes \mathbb{A}^p_f)(1)$ with

$$\psi(\xi^{(p)}(\lambda)v_1, v_2) = E^\lambda(\eta^p(v_1), \eta^p(v_2)).$$

(d) For each polarization $\lambda \in \bar{\lambda}$ and for each prime $\mathfrak{p}|p$ of O_F a section $\xi_\mathfrak{p}(\lambda) \in F^\times_\mathfrak{p}(1)/\mathbf{M}^\bullet_\mathfrak{p}$ such that $\xi_\mathfrak{p}(\lambda u) = u\xi_\mathfrak{p}(\lambda)$ for each $u \in F^\times$.

(e) For each $i = 0, 1, \ldots, s$ a class $\bar{\eta}_{\mathfrak{q}_i}$ modulo $\mathbf{K}^\bullet_{\mathfrak{q}_i}$ of $B_{\mathfrak{q}_i}$-module isomorphisms

$$\eta_{\mathfrak{q}_i} : V_{\mathfrak{q}_i} \xrightarrow{\sim} \mathrm{V}_{\mathfrak{q}_i}(A).$$

We require that the following condition (KC) holds,

$$\mathrm{char}(T, \iota(b) \mid \mathrm{Lie}\, A) = \prod_{\varphi : K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}^o_{B/K}(T - b))^{r_\varphi}.$$

We write a point of this functor in the form

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\xi_\mathfrak{p})_\mathfrak{p}) \tag{4.8}$$

or alternatively $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\xi_{\mathfrak{p}_i})_i)$, $i = 0, \ldots, s$.

To make the relationship of the last two Definitions 4.1 and 4.2 explicit, we consider an S-valued point $(A, \iota, \bar{\lambda}, \bar{\eta})$ of Definition 4.1. We fix $\eta \in \bar{\eta}$. Then η_p is an isomorphism

$$\eta_p = \oplus_{i=0}^s \eta_{\mathfrak{p}_i} : \bigoplus_{i=0}^{s} V_{\mathfrak{p}_i} \xrightarrow{\sim} \bigoplus_{i=0}^{s} \mathrm{V}_{\mathfrak{p}_i}(A).$$

The component $\eta_{\mathfrak{p}_i} = \eta_{\mathfrak{q}_i} \oplus \eta_{\bar{\mathfrak{q}}_i}$ satisfies an equation

$$\psi(\xi_{\mathfrak{p}_i}(\lambda)v, w) = E^\lambda(\eta_{\mathfrak{q}_i}(v), \eta_{\bar{\mathfrak{q}}_i}(w)), \quad v \in V_{\mathfrak{q}_i},\ w \in V_{\bar{\mathfrak{q}}_i}.$$

We see that the data $\eta_{\mathfrak{q}_i}$ and $\eta_{\bar{\mathfrak{q}}_i}$ for $i = 0, \ldots, s$ determine the data $\eta_{\mathfrak{q}_i}$ and $\xi_{\mathfrak{p}_i}(\lambda)$ and vice versa. Therefore we obtain the S-valued point $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\xi_{\mathfrak{p}_i})_i)$ of (4.2).

Let $g \in G^\bullet(\mathbb{Q}_p)$. According to (4.4) we write it in the form $(\ldots, g_{\mathfrak{q}_i}, \ldots, a_{\mathfrak{p}_i}, \ldots)$, with $a_{\mathfrak{p}_i} = \mu_{\mathfrak{p}_i}(g)$. Then the Hecke operator $g : \mathcal{A}^\bullet_{\mathbf{K}^\bullet} \to \mathcal{A}^\bullet_{g^{-1}\mathbf{K}^\bullet g}$, sends (4.8) to the point

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\overline{\eta_{\mathfrak{q}_i} g_{\mathfrak{q}_i}})_i, (a_{\mathfrak{p}_i}\xi_{\mathfrak{p}_i})_i), \tag{4.9}$$

comp. (3.19).

It is convenient for us to introduce another action of $(F \otimes \mathbb{Q}_p)^\times$,

$$a_{|\xi} : \mathcal{A}^\bullet_{\mathbf{K}^\bullet} \to \mathcal{A}^\bullet_{\mathbf{K}^\bullet}, \quad a \in (F \otimes \mathbb{Q}_p)^\times. \tag{4.10}$$

We write $a = (\dots, a_{\mathfrak{p}}, \dots) \in (F \otimes \mathbb{Q}_p)^{\times} = \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}^{\times}$. Then $a_{|\xi}$ maps a point (4.8) to

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (a_{\mathfrak{p}}\xi_{\mathfrak{p}})_{\mathfrak{p}}).$$

For a fixed $\mathfrak{p}$ let $a_{\mathfrak{p}} \in F_{\mathfrak{p}}^{*}$. The we define $a_{\mathfrak{p}|\xi_{\mathfrak{p}}} := a_{|\xi}$ where a is the element with component $a_{\mathfrak{p}}$ at $\mathfrak{p}$ and with $a_{\mathfrak{p}'} = 1$ for $\mathfrak{p}' \neq \mathfrak{p}$.

The action of $a_{|\xi}$ of (4.10) coincides with the action of the Hecke operator $g = (\dots, g_{\mathfrak{q}_i}, g_{\bar{\mathfrak{q}}_i} \dots) \in G^{\bullet}(\mathbb{Q}_p)$, where $g_{\mathfrak{q}_i} = 1$ and $g_{\bar{\mathfrak{q}}_i} = a_{\mathfrak{p}_i} \in F_{\mathfrak{p}_i}^{\times} \cong K_{\bar{\mathfrak{q}}_i}^{\times}$ for $i = 0, \dots, s$.

We will denote by $U_p(F) \subset F^{\times}$ the set of all $a \in F^{\times}$ which are units in all fields $F_{\mathfrak{p}}$ with $\mathfrak{p}|p$.

Definition 4.3. Let $\mathbf{K}^{\bullet} = \mathbf{K}_p^{\bullet}\mathbf{K}^{\bullet,p} \subset G^{\bullet}(\mathbb{A}_f)$, with $\mathbf{K}_p^{\bullet}$ as in (4.6). We define a functor$\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet bis}$ on the category of E-schemes S. A point of $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet bis}(S)$ consists of the following data:

(a) An abelian scheme A over S up to isogeny prime to p with an action $\iota : O_{B,(p)} \to \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

(b) An $U_p(F)$-homogeneous polarization $\bar{\lambda}$ of A which is principal in p and which induces on B the involution $b \mapsto b^{\star}$.

(c) A class $\bar{\eta}^p$ modulo $\mathbf{K}^{\bullet,p}$ of $B \otimes \mathbb{A}_f^p$-module isomorphisms

$$\eta^p : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathrm{V}_f^p(A),$$

such that for each $\lambda \in \bar{\lambda}$ there is locally for the Zariski topology a constant $\xi^{(p)}(\lambda) \in (F \otimes \mathbb{A}_f^p)^{\times}(1)$ with

$$\psi(\xi^{(p)}(\lambda)v_1, v_2) = E^{\lambda}(\eta^p(v_1), \eta^p(v_2)).$$

(d) For each polarization $\lambda \in \bar{\lambda}$ and for each prime $\mathfrak{p}|p$ of O_F a section $\xi_{\mathfrak{p}}(\lambda) \in O_{F_{\mathfrak{p}}}^{\times}(1)/\mathbf{M}_{\mathfrak{p}}^{\bullet}$ such that $\xi_{\mathfrak{p}}(\lambda u) = u\xi_{\mathfrak{p}}(\lambda)$ for each $u \in U_p(F)$.

(e) For each $j = 1, \dots s$, a class $\bar{\eta}_{\mathfrak{q}_j}$ modulo $\mathbf{K}_{\mathfrak{q}_j}^{\bullet}$ of $O_{B_{\mathfrak{q}_j}}$-module isomorphisms

$$\eta_{\mathfrak{q}_j} : \Lambda_{\mathfrak{q}_j} \xrightarrow{\sim} T_{\mathfrak{q}_j}(A).$$

We require that the following condition (KC) holds,

$$\operatorname{char}(T, \iota(b) \mid \operatorname{Lie} A) = \prod_{\varphi : K \to \bar{\mathbb{Q}}} \varphi(\mathrm{Nm}_{B/K}^{o}(T - b))^{r_{\varphi}}.$$

Variant 4.4. We will also use a modified version of this Definition. We obtain a functor isomorphic to $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet bis}$ if we modify the items (b) and (d) of Definition 4.3 as follows.

(b') An F-homogeneous polarization $\bar{\lambda}$ on A which induces on B the involution $\star$ from (2.9).

(d') For each polarization $\lambda \in \bar{\lambda}$ and for each prime $\mathfrak{p}|p$ of O_F a section $\xi_{\mathfrak{p}}(\lambda) \in F_{\mathfrak{p}}^{\times}(1)/\mathbf{M}_{\mathfrak{p}}^{\bullet}$ such that $\xi_{\mathfrak{p}}(u\lambda) = u\xi_{\mathfrak{p}}(\lambda)$ for each $u \in F^{\times}$ and such that λ is principal in $\mathfrak{p}$ iff $\xi_{\mathfrak{p}}(\lambda) \in O_{F_{\mathfrak{p}}^{\times}}^{\times}(1)$.

Proposition 4.5. *Let $\mathbf{K}^{\bullet} = \mathbf{K}_p^{\bullet}\mathbf{K}^{\bullet,p} \subset G^{\bullet}(\mathbb{A}_f)$, with $\mathbf{K}_p^{\bullet}$ as in* (4.6)*. The functors $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet}$ and $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet bis}$ on the category of E-schemes are canonically isomorphic.*

Proof. The proof is an obvious modification of the proof of Proposition 3.4. But for later use we indicate the point of $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet}(S)$ which corresponds to a point $(A_0, \bar{\lambda}_0, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\xi_{\mathfrak{p}_i})_i)$ of $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet bis}(S)$ (recall from Definition 3.3 that the index i runs from $0, \dots, s$ and the index j from $1, \dots, s$). Since we work over E, the Tate module $T_{\mathfrak{q}_0}(A)$ makes sense. By our choice of $\mathbf{K}_{\mathfrak{q}_0}^{\bullet}$ there is a unique class of $O_{B_{\mathfrak{q}_0}}$-module isomorphisms $\eta_{\mathfrak{q}_0} : \Lambda_{\mathfrak{q}_0} \to T_{\mathfrak{q}_0}(A)$ modulo $\mathbf{K}_{\mathfrak{q}_0}^{\bullet}$. Therefore we may replace j in datum (e) by $i = 0, \dots s$ in Definition 4.3, without changing anything.

Let A be the isogeny class of A_0 and choose $\lambda_0 \in \bar{\lambda}_0$ and $\eta^p \in \bar{\eta}^p$ to construct a point of $\mathcal{A}_{\mathbf{K}^{\bullet}}^{\bullet}(S)$. For $i = 0, \dots, s$, the isomorphisms $\eta_{\mathfrak{q}_i} : \Lambda_{\mathfrak{q}_i} \to T_{\mathfrak{q}_i}(A_0)$ induce by duality (using ψ and E^{λ_0}) an isomorphism $T_{\bar{\mathfrak{q}}_i}(A_0) \to \Lambda_{\bar{\mathfrak{q}}_i}(1)$. If we multiply the inverse of this map by $\xi_{\mathfrak{p}_i}(\lambda_0)$ we obtain an isomorphism

$$\eta_{\bar{\mathfrak{q}}_i} : \Lambda_{\bar{\mathfrak{q}}_i} \xrightarrow{\sim} T_{\bar{\mathfrak{q}}_i}(A_0)$$

which satisfies

$$E^{\lambda_0}(\eta_{\mathfrak{q}_i}(v), \eta_{\bar{\mathfrak{q}}_i}(w)) = \psi(\xi_{\mathfrak{p}_i}(\lambda_0)v, w), \quad v \in \Lambda_{\mathfrak{q}_i},\ w \in \Lambda_{\bar{\mathfrak{q}}_i}. \tag{4.11}$$

We set $\eta_{\mathfrak{p}_i} = \eta_{\mathfrak{q}_i} \oplus \eta_{\bar{\mathfrak{q}}_i}$ and $\eta_p = \oplus_{i=0}^{s} \eta_{\mathfrak{p}_i}$. We denote by $\bar{\eta}_p$ the class modulo $\mathbf{K}_p$ of η_p, and by λ the $\mathbb{Q}$-homogeneous polarization which contains λ_0. Then $(A, \iota, \bar{\lambda}, \bar{\eta})$ is the corresponding point of $\mathcal{A}^{\bullet}_{\mathbf{K}^{\bullet}}(S)$. □

We reformulate the action of the Hecke operator (4.9) in terms of Variant 4.4 of Definition 4.3. This will not be used until section 6. We consider an element $g \in G^{\bullet}(\mathbb{Q}_p) \subset G^{\bullet}(\mathbb{A}_f)$. We write $g = (\ldots g_{\mathfrak{q}_i}, \bar{g}_{\mathfrak{q}_i}, \ldots)$ as before, where $i = 0, \ldots, s$. We consider an open compact subgroup $\mathbf{K}^{\bullet} = \mathbf{K}^{\bullet}_p \mathbf{K}^{\bullet,p} \subset G^{\bullet}(\mathbb{A}_f)$, with $\mathbf{K}^{\bullet}_p$ as in (4.6), such that $kg\Lambda_{\mathfrak{p}_i} = g\Lambda_{\mathfrak{p}_i}$, for $k \in \mathbf{K}^{\bullet}_{\mathfrak{p}_i}$.

Let

$$(A_0, \iota_0, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\xi_{\mathfrak{p}_i})_i) \in \mathcal{A}^{\bullet bis}_{\mathbf{K}^{\bullet}}(S) \tag{4.12}$$

be a point of Variant 4.4. We recall that there is a unique isomorphism of $O_{B_{\mathfrak{q}_0}}$-modules $\eta_{\mathfrak{q}_0} : \Lambda_{\mathfrak{q}_0} \to T_{\mathfrak{q}_0}(A_0)$ modulo $\mathbf{K}^{\bullet}_{\mathfrak{p}_0}$. This defines the unique class $\bar{\eta}_{\mathfrak{q}_0}$. First we use this class to describe the Hecke operator g.

A point

$$(A_1, \iota_1, \bar{\lambda}_1, \bar{\theta}^p, (\bar{\theta}_{\mathfrak{q}_j})_j, (\xi'_{\mathfrak{p}_i})_i) \in \mathcal{A}^{\bullet bis}_{g^{-1}\mathbf{K}^{\bullet}g}(S) \tag{4.13}$$

is the image of (4.12) by the Hecke operator g if the following conditions are fulfilled. There exists a quasi-isogeny

$$\alpha : (A_1, \iota_1) \to (A_0, \iota_0) \tag{4.14}$$

such that

$$\begin{aligned} &\alpha^*(\bar{\lambda}) = \bar{\lambda}_1, \quad \alpha^*(\bar{\eta}^p) = \bar{\theta}^p, \\ &\xi'_{\mathfrak{p}_i}(\alpha^*(\lambda)) = \mu_{\mathfrak{p}_i}(g)\xi_{\mathfrak{p}_i}(\lambda), \quad \text{for } \lambda \in \bar{\lambda}. \end{aligned} \tag{4.15}$$

Moreover we require that the data $\bar{\theta}_{\mathfrak{q}_i}$ and $\bar{\eta}_{\mathfrak{q}_i}$ for $i = 0, \ldots, s$ are connected by the following diagrams

$$\begin{array}{ccc} T_{\mathfrak{q}_i}(A_1) \otimes \mathbb{Q} & \xrightarrow{\alpha} & T_{\mathfrak{q}_i}(A_0) \otimes \mathbb{Q} \\ \uparrow \theta_{\mathfrak{q}_i} & & \uparrow \eta_{\mathfrak{q}_i} \\ \Lambda_{\mathfrak{q}_i} \otimes \mathbb{Q} & \xrightarrow[g_{\mathfrak{q}_i}]{} & \Lambda_{\mathfrak{q}_i} \otimes \mathbb{Q}, \end{array} \tag{4.16}$$

where $\theta_{\mathfrak{q}_i} \in \bar{\theta}_{\mathfrak{q}_i}$, $\eta_{\mathfrak{q}_i} \in \bar{\eta}_{\mathfrak{q}_i}$. The diagrams are required to be commutative after replacing $\eta_{\mathfrak{q}_i}$ by $\eta_{\mathfrak{q}_i} k$ for some $k \in \mathbf{K}^{\bullet}_{\mathfrak{q}_i}$.

We may reformulate the condition (4.16) for $i = 0$ without mentioning the classes $\bar{\eta}_{\mathfrak{q}_0}$ and $\bar{\theta}_{\mathfrak{q}_0}$. Recall that $G^{\bullet}_{\mathfrak{q}_0} \cong (B^{\text{opp}}_{\mathfrak{q}_0})^{\times} \cong D^{\times}_{\mathfrak{p}_0}$, cf. (4.3), (3.12). Let $m = \operatorname{ord} g_{\mathfrak{q}_0}$ be the valuation in the division algebra $D_{\mathfrak{p}_0}$. Let $\Pi \in O_{D_{\mathfrak{q}_0}}$ be a prime element which we regard also as an element of $O_{D^{opp}_{\mathfrak{q}_0}}$. We may define m by $g_{\mathfrak{q}_0}(\Lambda_{\mathfrak{q}_0}) = \Pi^m \Lambda_{\mathfrak{q}_0}$. Let X_0 be the p-divisible group of A_0 and X_1 the p-divisible group of A_1. The condition (4.16) for $i = 0$ is equivalent with the condition that the quasi-isogeny of p-divisible groups

$$\Pi^{-m}\alpha : (X_1)_{\mathfrak{q}_0} \to (X_0)_{\mathfrak{q}_0} \tag{4.17}$$

is an isomorphism.

We note that (4.15) means implicitly that a polarization $\lambda_1 \in \bar{\lambda}_1$ is $\mathfrak{p}_i$-principal if it differs from $\mu_{\mathfrak{p}_i}(g)^{-1}\alpha^*(\lambda)$ by a unit in $O^{\times}_{F_{\mathfrak{p}_j}}$, for a polarization $\lambda \in \bar{\lambda}$ of A_0 which is principal in p.

That (4.13) indeed describes the image by the Hecke operator given by $g \in G^{\bullet}(\mathbb{Q}_p)$ follows immediately from (4.9) if we pass from A_0 to its isogeny class as in Definition 4.2. Our conditions for A_1 only ensure that we obtain a point of Variant 4.4. This description of the Hecke operators will allow us to extend them in section 6 to a model of the functor Variant 4.4 over O_{E_ν}.

Using Proposition 4.5, we identify the functors $\mathcal{A}^{\bullet}_{\mathbf{K}^{\bullet}}$ and $\mathcal{A}^{\bullet bis}_{\mathbf{K}^{\bullet}}$ and use the notation $\mathcal{A}_{\mathbf{K}^{\bullet}}$ for this functor. Finally we can define a functor on the category of O_{E_ν}-schemes.

Definition 4.6. Let $\mathbf{K}^{\bullet} = \mathbf{K}^{\bullet}_p \mathbf{K}^{\bullet,p} \subset G^{\bullet}(\mathbb{A}_f)$, with $\mathbf{K}^{\bullet}_p$ as in (4.6). We define a functor $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}}$ on the category of O_{E_ν}-schemes S. An S-valued point consists of the data $(a), (b), (c), (e)$ as in Definition 4.3. But we replace (d) by the following datum,

(d^t) For each polarization $\lambda \in \bar{\lambda}$ and for each prime $\mathfrak{p}|p$ of O_F a section $\xi_{\mathfrak{p}}(\lambda) \in O^{\times}_{F_{\mathfrak{p}}}/\mathbf{M}^{\bullet}_{\mathfrak{p}}$ such that $\xi_{\mathfrak{p}}(u\lambda) = u\xi_{\mathfrak{p}}(\lambda)$ for each $u \in U_p(F)$.

Let $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}$ be the functor on the category of E-schemes S which is obtained by changing in Definition 4.2 the item (d) into

(d^t) For each polarization $\lambda \in \bar{\lambda}$ and for each prime $\mathfrak{p}|p$ of O_F a section $\xi_\mathfrak{p}(\lambda) \in F_\mathfrak{p}^\times/\mathbf{M}_\mathfrak{p}^\bullet$ such that $\xi_\mathfrak{p}(u\lambda) = u\xi_\mathfrak{p}(\lambda)$ for each $u \in F^\times$.

By changing (d) in Definition 4.3 in the same way (i.e., replacing $O_{F_\mathfrak{p}}^\times(1)/\mathbf{M}_\mathfrak{p}^\bullet$ by $O_{F_\mathfrak{p}}^\times/\mathbf{M}_\mathfrak{p}^\bullet$), we obtain another description of $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}$. We call this the t-version of Definition 4.3.

By the proof of Proposition 4.5, we have a canonical isomorphism

$$\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} E_\nu \cong \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} E_\nu. \tag{4.18}$$

Remark 4.7. For $g \in G^\bullet(\mathbb{A}_f)$ we have the Hecke operators $g : \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \to \mathcal{A}^{\bullet t}_{g^{-1}\mathbf{K}^\bullet g}$. For $g \in G^\bullet(\mathbb{A}_f^p)$ these extend obviously to $g : \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \to \tilde{\mathcal{A}}^{\bullet t}_{g^{-1}\mathbf{K}^\bullet g}$. Let $g \in G^\bullet(\mathbb{Q}_p)$, we have defined the Hecke operator by (4.9). In section 6 we will extend this Hecke operator to $g : \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \to \tilde{\mathcal{A}}^{\bullet t}_{g^{-1}\mathbf{K}^\bullet g}$, whenever both $\mathbf{K}_p^\bullet$ and $g^{-1}\mathbf{K}_p^\bullet g$ are as specified in (4.6), so that both source and target make sense. This will be done by the remark after Proposition 4.5. For the time being, we only need the extensions of Hecke operators defined by elements in $(K \otimes \mathbb{Q}_p)^\times \subset G^\bullet(\mathbb{Q}_p)$. For these Hecke operators we give an ad hoc definition, cf. (4.23).

Let $\zeta_{p^\infty} \in \bar{\mathbb{Q}}$ be a compatible system of primitive p^n-th roots of unity. It defines over $\bar{\mathbb{Q}}$ an isomorphism of étale sheaves for each $\mathfrak{p}$,

$$\kappa_\mathfrak{p} \colon O_{F_\mathfrak{p}}^\times/\mathbf{M}_\mathfrak{p}^\bullet \xrightarrow{\sim} O_{F_\mathfrak{p}}^\times(1)/\mathbf{M}_\mathfrak{p}^\bullet, \qquad \xi_\mathfrak{p} \longmapsto \zeta_{p^\infty}\xi_\mathfrak{p}. \tag{4.19}$$

It is defined over a finite abelian extension L/E which we choose independently of $\mathfrak{p}$. The isomorphism (4.19) defines an isomorphism of functors

$$\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L \xrightarrow{\sim} \mathcal{A}^{\bullet}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L. \tag{4.20}$$

Here, for an L-scheme S a point

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\xi_{\mathfrak{p}_i})_i) \in \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}(S) \tag{4.21}$$

is mapped by the isomorphism (4.20) to $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\zeta_{p^\infty}\xi_{\mathfrak{p}_i})_i)$. The isomorphism (4.20) is compatible with the Hecke operators $G^\bullet(\mathbb{A}_f)$ acting on both sides.

Proposition 4.8. *Let $\tau \in \operatorname{Gal}(L/E)$ be an automorphism. The isomorphism (4.20) fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L & \longrightarrow & \mathcal{A}^{\bullet}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L \\ {\scriptstyle \mathrm{id}\times\tau_c}\downarrow & & \downarrow{\scriptstyle \varsigma_{p^\infty}(\tau^{-1})_{|\xi}\times\tau_c} \\ \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L & \longrightarrow & \mathcal{A}^{\bullet}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} E} \operatorname{Spec} L. \end{array}$$

Here we take the composite of the cyclotomic character by the inclusion

$$\varsigma_{p^\infty} : \operatorname{Gal}(\bar{E}/E) \to \mathbb{Z}_p^\times \subset (O_F \otimes \mathbb{Z}_p)^\times.$$

See (4.10) for the definition of the automorphism $a_{|\xi}$ of $\mathcal{A}^{\bullet}_{\mathbf{K}^\bullet}$. A similiar definition applies to $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}$.

The proof coincides with that of Proposition 3.6. As a consequence we have an analogue of Proposition 3.7, i.e. there is a morphism of functors

$$\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{nr} \to \operatorname{Sh}_{\mathbf{K}^\bullet}(G, h\delta^{-1})_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{nr}. \tag{4.22}$$

The descent data relative to E_ν^{nr}/E_ν on both sides are compatible up to the factor $p_{|\xi}^{f_\nu}$ which can be expressed by a diagram similiar to that of Proposition 3.6. In contrast to Proposition 3.6, the morphism (4.22) is no longer an isomorphism since we are dealing with a coarse moduli scheme.

We will next show that the action of the group $(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \subset G^\bullet(\mathbb{Q}_p)$ on $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}$ by Hecke operators extends naturally to an action on the O_{E_ν}-scheme $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$. We write an element of that group as

$$z = (\ldots, a_i, b_i, \ldots) \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \cong \prod_{i=0}^{s} (K^\times_{\mathfrak{q}_i} \times K^\times_{\bar{\mathfrak{q}}_i}) \cong \prod_{i=0}^{s} (F^\times_{\mathfrak{p}_i} \times F^\times_{\mathfrak{p}_i}),$$

where $a_i, b_i \in F^\times_{\mathfrak{p}_i}$, for $i = 0, 1, \ldots, s$. We note that $\mu_{\mathfrak{p}_i}(z) = a_i b_i$.

We consider a point $(A, \iota, \bar{\lambda}, \bar{\eta})$ of $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}(S)$ as in Definition 4.1. We write $\eta = \eta^p \eta_p$ and

$$\eta_p = \oplus_{i=0}^{s} (\eta_{\mathfrak{q}_i} \oplus \eta_{\bar{\mathfrak{q}}_i}), \quad \eta_{\mathfrak{q}_i} : V_{\mathfrak{q}_i} \to \mathrm{V}_{\mathfrak{q}_i}(A),\ \eta_{\bar{\mathfrak{q}}_i} : V_{\bar{\mathfrak{q}}_i} \to \mathrm{V}_{\bar{\mathfrak{q}}_i}(A).$$

The Hecke operator $z : \mathcal{A}^\bullet_{\mathbf{K}^\bullet} \to \mathcal{A}^\bullet_{\mathbf{K}^\bullet}$ on S-valued points is given by

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\bar{\eta}_{\bar{\mathfrak{q}}_i})_i) \longmapsto (A, \iota, \bar{\lambda}, \bar{\eta}^p,\ (\bar{\eta}_{\mathfrak{q}_i} \cdot a_i)_i,\ (\bar{\eta}_{\bar{\mathfrak{q}}_i} \cdot b_i)_i). \tag{4.23}$$

Let $\mathbf{x} \in K^\times$. We write its image in $(K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p)^\times$ as

$$(\ldots, x_i, y_i, \ldots) \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \cong \prod_{i=0}^{s} (K^\times_{\mathfrak{q}_i} \times K^\times_{\bar{\mathfrak{q}}_i}) \cong \prod_{i=0}^{s} (F^\times_{\mathfrak{p}_i} \times F^\times_{\mathfrak{p}_i}),$$

where $x_i, y_i \in F^\times_{\mathfrak{p}_i}$. We note that $x_i y_i$ is the image of $\mathrm{Nm}_{K/F}\,\mathbf{x}$ in $F^\times_{\mathfrak{p}_i}$.

We consider the quasi-isogeny $\mathbf{x} : A \to A$ induced by multiplication by $\mathbf{x}$. The inverse image of the data $(A, \iota, \bar{\lambda}, \bar{\eta}^p,\ (\bar{\eta}_{\mathfrak{q}_i} \cdot a_i)_i,\ (\bar{\eta}_{\bar{\mathfrak{q}}_i} \cdot b_i)_i)$ by this quasi-isogeny is

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p \cdot \mathbf{x}^{-1},\ (\bar{\eta}_{\mathfrak{q}_i} \cdot a_i x_i^{-1})_i,\ (\bar{\eta}_{\bar{\mathfrak{q}}_i} \cdot b_i y_i^{-1})_i). \tag{4.24}$$

Therefore this is just another way to write the image under the Hecke operator (4.23).

We rewrite (4.23) in terms of the alternative Definition 4.2. In terms of this definition, the left hand side of (4.23) corresponds to $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\xi_{\mathfrak{p}_i})_i)$ and (4.24) corresponds to

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p \mathbf{x}^{-1}, (\eta_{\mathfrak{q}_i} a_i x_i^{-1})_i,\ (a_i b_i (\mathrm{Nm}_{K/F}\,\mathbf{x}^{-1}) \xi_{\mathfrak{p}_i})_i).$$

Summarizing, the Hecke operator $z : \mathcal{A}^\bullet_{\mathbf{K}^\bullet} \to \mathcal{A}^\bullet_{\mathbf{K}^\bullet}$ becomes in terms of Definition 4.2 the map

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_i})_i, (\xi_{\mathfrak{p}_i})_i) \longmapsto (A, \iota, \bar{\lambda}, \bar{\eta}^p \mathbf{x}^{-1}, (\bar{\eta}_{\mathfrak{q}_i} a_i x_i^{-1})_i,\ (a_i b_i (\mathrm{Nm}_{K/F}\,\mathbf{x}^{-1}) \xi_{\mathfrak{p}_i})_i). \tag{4.25}$$

In the same way z acts on $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}$. In fact, we are only interested in the latter functor. Let us choose $\mathbf{x} \in K^\times$, such that $a_i x_i^{-1}$ and $b_i y_i^{-1}$ are units in $O^\times_{F_{\mathfrak{p}_i}}$ for $i = 0, \ldots, s$. Assume we have chosen the left hand side of (4.25) in the form of the t-version of Definition 4.3. Note that we have added to the data of this definition the unique class of $O_{B_{\mathfrak{q}_0}}$-module isomorphisms $\bar{\eta}_{\mathfrak{q}_0} : \Lambda_{\mathfrak{q}_0} \xrightarrow{\sim} T_{\mathfrak{q}_0}(A)$ modulo $\mathbf{K}^\bullet_{\mathfrak{q}_0}$. We then see that the right hand side of (4.25) is also a point in the sense of Definition 4.3. For $i = 0$ we have the isomorphism $\bar{\eta}_{\mathfrak{q}_0} a_0 x_0^{-1} : \Lambda_{\mathfrak{q}_0} \xrightarrow{\sim} T_{\mathfrak{q}_0}(A)$ as required. Hence we may forget about $i = 0$ and obtain a definition of the Hecke operator in terms of the t-version of Definition 4.3,

$$z : \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet} \to \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet}.$$

This definition of z makes sense for the functor $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$. We define

$$\tilde{z} : \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \to \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \tag{4.26}$$

as follows. Let $(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\xi_{\mathfrak{p}_i})_i)$ be a point of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ with values in an O_{E_ν}-scheme S. We define the image by morphism (4.26) as

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p \mathbf{x}^{-1}, (\bar{\eta}_{\mathfrak{q}_j} a_j x_j^{-1})_j, (a_i b_i (\mathrm{Nm}_{K/F}\,\mathbf{x}^{-1}) \xi_{\mathfrak{p}_i})_i).$$

It is clear that this is an extension of z with respect to the isomorphism (4.18).

Recall from (1.1)

$$h_D \colon \mathbb{S} \to (D \otimes \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R}) \times \prod_{\chi \neq \chi_0} (D \otimes_{F,\chi} \mathbb{R})^\times, \quad z = a + b\mathbf{i} \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \times (1, \ldots, 1). \tag{4.27}$$

Moreover, we consider the composite

$$\begin{array}{cccc} h^\bullet_D : & \mathbb{S} & \to & (D \otimes \mathbb{R})^\times \times (K \otimes \mathbb{R})^\times & \to G^\bullet_{\mathbb{R}}, \\ & z & \longmapsto & h_D(z)\ \times\ 1 & \end{array} \tag{4.28}$$

cf. Lemma 2.1.

Recall from (1.11) the Shimura datum $(G^\bullet, h_D^\bullet)$. The next proposition relates the Shimura varieties $\mathrm{Sh}(G^\bullet, h)$ and $\mathrm{Sh}(G^\bullet, h_D^\bullet)$.

Proposition 4.9. *Let $\mathbf{K}^\bullet = \mathbf{K}_p^\bullet \mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}_p^\bullet$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O_{F_{\mathfrak{p}_0}}^\times$. Denote by f_ν the inertia index of $E_\nu/\mathbb{Q}_p$. Let $\pi_{\mathfrak{p}_0}$ be an arbitrary prime element of $F_{\mathfrak{p}_0}$. We consider the element*

$$\dot{z} = (\pi_{\mathfrak{p}_0}^{-1} p^{f_\nu}, p^{f_\nu}, \ldots, p^{f_\nu}) \in (F \otimes \mathbb{Q}_p)^\times = \prod_{i=0}^{s} F_{\mathfrak{p}_i}^\times.$$

Let $\tau \in \mathrm{Gal}(E_\nu^{nr}/E_\nu)$ be the Frobenius automorphism. Then there is a morphism of functors

$$\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} \to \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr}, \tag{4.29}$$

such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} \\
\Big\downarrow {\scriptstyle \dot{z}_{|\xi} \times \tau_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c} \\
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr},
\end{array}$$

Here, the right hand side of (4.29) is the coarse moduli scheme of the functor on the left hand side. See (4.10) for the definition of $\dot{z}_{|\xi}$.

We will show in Proposition 4.16 that $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr}$ is in fact the étale sheafification of $\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr}$.

Proof. Recall the morphism to the coarse moduli space

$$\mathcal{A}^{\bullet}_{\mathbf{K}^\bullet, E_\nu} \to \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h)_{E_\nu}. \tag{4.30}$$

Let $T^\bullet = (K \otimes \mathbb{Q})^\times \subset G^\bullet$ be the central torus. We consider $\delta : \mathbb{S} \to T^\bullet_{\mathbb{R}}$ cf. (3.30). The local reciprocity law $r_\nu(T^\bullet, \delta^{-1}) : E_\nu^\times \to T^\bullet(\mathbb{Q}_p)$ is the composite of the local reciprocity law $r_\nu(T, \delta^{-1})$ for T (given by (3.34)) with the inclusion $T(\mathbb{Q}_p) \subset T^\bullet(\mathbb{Q}_p)$. Let $e \in E_\nu^\times$ and let $\sigma \in \mathrm{Gal}(E_\nu^{ab}/E_\nu)$ be the automorphism which corresponds to it by local class field theory. If we twist the morphism (4.30) by $r_\nu(T^\bullet, \delta^{-1})$ we obtain as in the proof of Proposition 3.7 a commutative diagram

$$\begin{array}{ccc}
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h\delta^{-1})_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} \\
\Big\downarrow {\scriptstyle (p^{f_\nu \,\mathrm{ord}\, e})_{|\xi} \times \sigma_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \sigma_c} \\
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h\delta^{-1})_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab}.
\end{array} \tag{4.31}$$

We consider the homomorphism $\gamma : \mathbb{S} \to (K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ which in terms of the isomorphism (3.29) is defined by

$$\gamma : \mathbb{S} \to \mathbb{C}^\times \times \prod_{\varphi, r_\varphi = 2} \mathbb{C}^\times, \quad z \longmapsto (z, 1, \ldots, 1), \tag{4.32}$$

i.e., on the right hand side we have z at the factor which corresponds to $\bar{\varphi}_0$. We find that

$$h_D^\bullet = h\delta^{-1}\gamma. \tag{4.33}$$

Therefore we must twist the horizontal line of (4.31) by the local reciprocity law of γ. The one-parameter group μ_γ associated to the Shimura datum γ is

$$\mu_\gamma : \mathbb{C}^\times \to \prod_{\Phi} \mathbb{C}^\times, \quad z \longmapsto (1, \ldots, 1, z, 1, \ldots, 1),$$

where z is exactly at the place $\bar{\varphi}_0$. Since we are interested in the local reciprocity law we replace $\mathbb{C}$ by $\bar{\mathbb{Q}}_p$ cf. (8.11). The field of definition E_ν of μ_γ is the image of $\bar{\varphi}_0 : K_{\bar{\mathfrak{q}}_0} \to \bar{\mathbb{Q}}_p$. There is a canonical isomorphism of $K_{\bar{\mathfrak{q}}_0}$-algebras

$$K_{\bar{\mathfrak{q}}_0} \otimes_{\mathbb{Q}_p} E_\nu \cong K_{\bar{\mathfrak{q}}_0} \times C_{\bar{\mathfrak{q}}_0},$$

where the first factor corresponds to the compositum $K_{\bar{\mathfrak{q}}_0}$ of the fields $K_{\bar{\mathfrak{q}}_0}$ and E_ν, given by $\mathrm{id}_{K_{\bar{\mathfrak{q}}_0}}$ and $\bar{\varphi}_0^{-1}$.

We consider the homomorphism

$$E_\nu^\times \to (K_{\bar{\mathfrak{q}}_0} \otimes_{\mathbb{Q}_p} E_\nu)^\times \cong K_{\bar{\mathfrak{q}}_0}^\times \times C_{\bar{\mathfrak{q}}_0}^\times, \quad e \longmapsto \bar{\varphi}_0^{-1}(e) \times 1. \tag{4.34}$$

The one-parameter group μ_γ over E_ν is the homomorphism

$$E_\nu^\times \to (K \otimes_{\mathbb{Q}} E_\nu)^\times \cong \prod_i (K_{\mathfrak{q}_i} \otimes_{\mathbb{Q}_p} E_\nu)^\times \times \prod_i (K_{\bar{\mathfrak{q}}_i} \otimes_{\mathbb{Q}_p} E_\nu)^\times,$$

which is given by (4.34) on the factor $(K_{\bar{\mathfrak{q}}_0} \otimes_{\mathbb{Q}_p} E_\nu)^\times$ and is trivial on all other factors. The map

$$\mathrm{Nm}_{E_\nu/\mathbb{Q}_p} = \mathrm{Nm}_{K_{\bar{\mathfrak{q}}_0} \otimes_{\mathbb{Q}_p} E_\nu / K_{\bar{\mathfrak{q}}_0}} : K_{\bar{\mathfrak{q}}_0} \otimes_{\mathbb{Q}_p} E_\nu \to K_{\bar{\mathfrak{q}}_0}$$

becomes in terms of (4.34)

$$(a, c) \in K_{\bar{\mathfrak{q}}_0} \times C_{\bar{\mathfrak{q}}_0} \longmapsto a \, \mathrm{Nm}_{C_{\bar{\mathfrak{q}}_0}/K_{\bar{\mathfrak{q}}_0}}.$$

We conclude that the local reciprocity law associated to γ

$$r(T^\bullet, \gamma) : E_\nu^\times \to \prod_i K_{\mathfrak{q}_i}^\times \times \prod_i K_{\bar{\mathfrak{q}}_i}^\times \tag{4.35}$$

maps e to the element with component $\bar{\varphi}_{\bar{\mathfrak{q}}_0}^{-1}(e^{-1})$ at the factor $K_{\bar{\mathfrak{q}}_0}$ and with trivial component at all other factors.

By Corollary 8.5 and our remarks about the Hecke operators (4.9) we obtain from (4.31) a commutative diagram

$$\begin{array}{ccc}
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h\delta^{-1}\gamma)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} \\
\Big\downarrow {\scriptstyle (\varphi_{\mathfrak{p}_0}^{-1}(e^{-1})_{|\xi_{\mathfrak{p}_0}} (p^{f_\nu \operatorname{ord} e})_{|\xi} \times \sigma_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \sigma_c} \\
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h\delta^{-1}\gamma)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab}.
\end{array}$$

The $\xi_{\mathfrak{p}_0}$-part of the datum (d^t) in Definition 4.6 is a function with values in $F_{\mathfrak{p}_0}^\times / O_{F_{\mathfrak{p}_0}}^\times$. Therefore $(\varphi_{\mathfrak{p}_0}^{-1}(e^{-1})_{|\xi_{\mathfrak{p}_0}}$ acts on this datum exactly like $(\pi_{\mathfrak{p}_0}^{-\operatorname{ord} e})_{|\xi_{\mathfrak{p}_0}}$. This shows that for $e \in O_{E_\nu}^\times$ the vertical arrow on the left hand side in the above diagram is equal to $\mathrm{id} \times \sigma_c$. Therefore the horizontal arrow in this diagram is defined over E_ν^{nr}. The proposition follows. □

We use Proposition 4.9 to define a model of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}$ over O_{E_ν}. The group $T^\bullet(\mathbb{A}_f)/(\mathbf{K}^\bullet \cap T^\bullet(\mathbb{A}_f))$ acts through a finite quotient. Therefore the Hecke operator associated to $\dot{z}$ has finite order. It follows that the field E_ν^{nr} in Proposition 4.9 can be replaced by a finite unramified extension L/E_ν. We have extended $\dot{z}$ to an automorphism of the functor $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ over O_{E_ν}.

Definition 4.10. Let $\mathbf{K}^\bullet = \mathbf{K}_p^\bullet \mathbf{K}^{\bullet, p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}_p^\bullet$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O_{F_{\mathfrak{p}_0}}^\times$. We define $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)$ to be the O_{E_ν}-scheme given by the descent datum $\dot{z} \times \tau_c$ on $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_L$, where $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ is the coarse moduli scheme of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$.

The diagram of Proposition 4.9 becomes

$$\begin{array}{ccc}
\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E_\nu^{nr}} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E_\nu^{nr}} \\
\Big\downarrow {\scriptstyle \dot{z}_{|\xi} \times \tau_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c} \\
\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E_\nu^{nr}} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E_\nu^{nr}}.
\end{array} \tag{4.36}$$

Remark 4.11. Let us drop the assumption that $\mathbf{M}_{\mathfrak{p}_0} = O_{F_{\mathfrak{p}_0}}^\times$. We can write the diagram at the end of the proof of Proposition 4.9 in the form

$$\begin{array}{ccc}
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} \\
\Big\downarrow {\scriptstyle (\pi_{\mathfrak{p}_0}^{-\operatorname{ord} e})_{|\xi_{\mathfrak{p}_0}} (p^{f_\nu \operatorname{ord} e})_{|\xi} \times \sigma_c} & & \Big\downarrow {\scriptstyle (\pi_{\mathfrak{p}_0}^{-\operatorname{ord} e})_{|\xi_{\mathfrak{p}_0}} (\varphi_{\mathfrak{p}_0}^{-1}(e))_{|\xi_{\mathfrak{p}_0}} \times \sigma_c} \\
\mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\operatorname{Spec} E_\nu} \operatorname{Spec} E_\nu^{ab}.
\end{array}$$

As before e corresponds to σ by local class field theory. We define the Galois twist $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0})$ of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}$ by the commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0}) \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} \\ \Big\downarrow {\scriptstyle (\pi_{\mathfrak{p}_0}^{-\mathrm{ord}\, e})_{|\xi_{\mathfrak{p}_0}} (\varphi_{\mathfrak{p}_0}^{-1}(e))_{|\xi_{\mathfrak{p}_0}} \times \sigma_c} & & \Big\downarrow {\scriptstyle id \times \sigma_c} \\ \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0}) \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{ab}. \end{array}$$

Then we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0}) \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} \\ \Big\downarrow {\scriptstyle (\pi_{\mathfrak{p}_0}^{-\mathrm{ord}\, e})_{|\xi_{\mathfrak{p}_0}} (p^{f_\nu\, \mathrm{ord}\, e})_{|\xi} \times \sigma_c} & & \Big\downarrow {\scriptstyle (\mathrm{id} \times \sigma_c} \\ \mathcal{A}^{\bullet t}_{\mathbf{K}^\bullet, E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0}) \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, E_\nu^{nr}. \end{array}$$

In the same way as in Definition 4.6 we obtain a model $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)(\pi_{\mathfrak{p}_0})$ over O_{E_ν}. The diagram (4.36) continues to hold for arbitrary $\mathbf{K}^\bullet$ if we substitute $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)(\pi_{\mathfrak{p}_0})$ for $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)$. We note that the last two schemes are canonically identified if $\mathbf{K}^\bullet$ is of the type $\mathbf{M}_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$.

One could regard $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}(\pi_{\mathfrak{p}_0})$ as the twist of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)_{E_\nu}$ by the character of $\mathrm{Gal}(E_\nu^{ab}/E_\nu)$ associated to the Lubin-Tate group defined by $\pi_{\mathfrak{p}_0}$.

Our next aim is to compare the functors $\mathcal{A}_{\mathbf{K}}$ and $\mathcal{A}_{\mathbf{K}^\bullet}$. For this we need the following variant of a theorem of Chevalley [Che].

Proposition 4.12. *Let F be a totally real number field. We set $[F : \mathbb{Q}] = d = 2^h d'$ such that d' is odd. Let $M \geq 2$ be a natural number and let ℓ be a prime number such that*

$$\ell \equiv 2 \mod d', \qquad \ell \equiv 3 \mod 4.$$

For a natural number N, let $U_{N\ell}$ be the principal congruence subgroup of $(O_F \otimes \hat{\mathbb{Z}})^\times$,

$$U_{N\ell} = \{u \equiv 1 \mod N\ell(O_F \otimes \hat{\mathbb{Z}})\}.$$

For each natural number m there is a power N of M with the following property: for each element $f \in F^\times$ which is totally positive and such that $f \in U_{N\ell} \cdot \mathbb{A}_f^\times$, there is a unit $g \in O_F^\times$ such that

$$f = g^m q, \quad \text{for some } q \in \mathbb{Q}^\times,\ q > 0.$$

Proof. Set $U = U_{N\ell}$, where N will be determined in the proof. We write $f = u\alpha$ with $u \in U$, $\alpha \in \mathbb{A}_f^\times$. We find q such that $\alpha = q\beta$ and $\beta \in \hat{\mathbb{Z}}^\times$. Therefore we may assume that $f \in O_F^\times$ and hence $q = 1$. We obtain

$$f = u\alpha, \quad u \in U,\ \alpha \in \hat{\mathbb{Z}}^\times.$$

We note that $\mathrm{Nm}_{F/\mathbb{Q}}\, u \in U$. We find

$$f^d (\mathrm{Nm}_{F/\mathbb{Q}}\, f)^{-1} = u^d (\mathrm{Nm}_{F/\mathbb{Q}}\, u)^{-1} \alpha^d (\mathrm{Nm}_{F/\mathbb{Q}}\, \alpha)^{-1} = u^d (\mathrm{Nm}_{F/\mathbb{Q}}\, u)^{-1} \in U.$$

Since f is a totally positive unit $\mathrm{Nm}_{F/\mathbb{Q}}\, f = 1$ and therefore $f^d \in U$. By Chevalley [Che] there exists for a suitable N a unit $g \in O_F^\times$ such that $f^d = g^{md}$. Replacing m by a multiple, we may assume that m is even and that

$$g^m \equiv 1 \mod \ell(O_F \otimes \hat{\mathbb{Z}}).$$

We consider the d-th root of unity

$$f/g^m = \zeta.$$

Since $f \equiv \alpha \mod \ell(O_F \otimes \hat{\mathbb{Z}})$ we obtain

$$\zeta \equiv \alpha \mod \ell(O_F \otimes \hat{\mathbb{Z}}).$$

The right hand side is in $\mathbb{Z}/\ell\mathbb{Z} \subset O_F/\ell O_F$. This shows $\zeta^{\ell-1} \equiv 1$. Here and below, this is meant mod $\ell(O_F \otimes \hat{\mathbb{Z}})$. On the other hand, we have

$$\zeta^{2^h d'} \equiv 1.$$

Since $\ell - 1 \equiv 1 \mod d'$, we obtain that $\zeta^{2^h} \equiv 1$. If $h = 0$ we conclude from Serre's lemma that $\zeta = 1$. Let $h > 0$. By our assumption $(\ell - 1)/2$ is odd. Therefore the greatest common divisor of $\ell - 1$ and 2^h is 2. We conclude that $\zeta^2 \equiv 1$ and by Serre's lemma that $\zeta^2 = 1$. We obtain

$$f/g^m = \pm 1.$$

Since m is even, the left hand side is totally positive by assumption. This gives finally $f = g^m$. □

Let $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ be an open and compact subgroup. We set $\mathbf{K} = \mathbf{K}^\bullet \cap G(\mathbb{A}_f)$. For an open compact subgroup $U \subset (F \otimes \mathbb{A}_f)^\times$, we define

$$\mathbf{K}^\bullet_U = \{g \in \mathbf{K}^\bullet \mid \mu(g) \in U\hat{\mathbb{Z}}^\times\}.$$

Then

$$\mathbf{K} = \mathbf{K}^\bullet_U \cap G(\mathbb{A}_f). \tag{4.37}$$

Proposition 4.13. *We fix M and ℓ as in Proposition 4.12. Let $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ be an open compact subgroup. Then there exists a power N of M such that for the principal congruence subgroup $U = U_{N\ell} \subset (F \otimes \mathbb{A}_f)^\times$ of Proposition 4.12, the natural map of functors*

$$\mathcal{A}_{\mathbf{K}} \to \mathcal{A}^\bullet_{\mathbf{K}^\bullet_U} \tag{4.38}$$

is a monomorphism.

To show this, it is enough to check injectivity for points with values in $S = \operatorname{Spec} R$, where R is a noetherian E_ν-Algebra and $\operatorname{Spec} R$ is connected. We begin with two lemmas. Since the meaning of the class in the notation $(A, \bar{\lambda}, \bar{\eta})$ depends on whether this is an object of $\mathcal{A}_{\mathbf{K}}$ or of $\mathcal{A}^\bullet_{\mathbf{K}^\bullet}$, we use the notation $(A, \tilde{\lambda}, \tilde{\eta})$ in the latter case.

Lemma 4.14. *Let $(A, \bar{\lambda}, \bar{\eta}) \in \mathcal{A}_{\mathbf{K}}(R)$ with image $(A', \tilde{\lambda}', \tilde{\eta}') \in \mathcal{A}^\bullet_{\mathbf{K}^\bullet_U}(R)$. Then there is a polarization $\lambda' \in \tilde{\lambda}'$ and level structure $\eta' \in \tilde{\eta}'$ such that the point $(A, \bar{\lambda}, \bar{\eta})$ may be represented in the form $(A', \bar{\lambda}', \bar{\eta}')$.*

Proof. We start with arbitrary polarizations $\lambda' \in \tilde{\lambda}'$ and $\lambda \in \bar{\lambda}$ and arbitrary level structures $\eta' \in \tilde{\eta'}$, $\eta \in \bar{\eta}$. Since we have the same point in $\mathcal{A}^\bullet_{\mathbf{K}^\bullet_U}(R)$, there is an isogeny $\alpha : A' \to A$ such that $\alpha^*(\lambda) = f\lambda'$. Since we have chosen polarizations, $f \in F^\times$ must be totally positive. Moreover, α must respect η and η' up to a factor in $\mathbf{K}^\bullet_U$,

$$\alpha \circ \eta' \dot{c} = \eta, \quad \dot{c} \in \mathbf{K}^\bullet_U.$$

We claim that $(A', \overline{f\lambda'}, \overline{\eta'\dot{c}})$ is a point of $\mathcal{A}_{\mathbf{K}}(R)$. We have to check that the isomorphism

$$\eta_1 \dot{c} : V \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A')$$

respects the form ψ and the Riemann form $E^{f\lambda'}$, up to a factor $a \in \mathbb{A}_f^\times(1)$. But for $x, y \in V \otimes \mathbb{A}_f$ we have

$$\begin{aligned} E^{f\lambda'}(\eta'(\dot{c}x), \eta'(\dot{c}y)) &= E^{\alpha^*(\lambda)}(\alpha^{-1} \circ \eta(x), \alpha^{-1} \circ \eta(y)) = E^\lambda(\eta(x), \eta(y)) \\ &= a\psi(x, y) \end{aligned}$$

for some $a \in \mathbb{A}_f^\times(1)$, because $(A, \bar{\lambda}, \bar{\eta})$ is a point of $\mathcal{A}_{\mathbf{K}}(R)$.

It is obvious that

$$\alpha : (A', \overline{f\lambda'}, \overline{\eta'\dot{c}}) \to (A, \bar{\lambda}, \bar{\eta})$$

is an isomorphism and therefore both sides give the same point of $\mathcal{A}_{\mathbf{K}}(R)$. □

Lemma 4.15. *Let $(A_1, \bar{\lambda}_1, \bar{\eta}_1)$ and $(A_2, \bar{\lambda}_2, \bar{\eta}_2)$ be two points of $\mathcal{A}_{\mathbf{K}}(R)$ whose images in $\mathcal{A}^\bullet_{\mathbf{K}^\bullet_U}(R)$ by (4.38) are the same. Then there exists a totally positive $f \in F^\times$ and an element $\dot{c} \in \mathbf{K}^\bullet_U$, such that*

$$f(\dot{c}'\dot{c}) \in \mathbb{A}_f^\times,$$

and such that $(A_2, \bar{\lambda}_2, \bar{\eta}_2)$ is isomorphic to $(A_1, f\bar{\lambda}_1, \bar{\eta}_1\dot{c})$.

Proof. We choose arbitrary polarizations $\lambda_1 \in \bar{\lambda}_1$ and $\lambda_2 \in \bar{\lambda}_2$ and arbitrary level structures $\eta_1 \in \bar{\eta}_1$ and $\eta_2 \in \bar{\eta}_2$. We remark that for each $\dot{c} \in \mathbf{K}_U^\bullet$ the class $\eta_1 \dot{c}\mathbf{K}$ is invariant under the action of $\pi_1(\bar{s}, S)$ because $\mathbf{K} \subset \mathbf{K}_U^\bullet$ is a normal subgroup.

By the Lemma 4.14 we may assume that $(A_2, \bar{\lambda}_2, \bar{\eta}_2) = (A_1, \overline{f\lambda_1}, \overline{\eta_1 \dot{c}})$. We have factors $a_1, a_2 \in \mathbb{A}_f(1)$ such that for all $x, y \in V \otimes \mathbb{A}_f$

$$\begin{aligned} a_2\psi(x,y) = E^{f\lambda_1}(\eta_1(\dot{c}x), \eta_1(\dot{c}y)) = E^{\lambda_1}(\eta_1(\dot{c}x), \eta_1(\dot{c}fy)) = \\ = a_1\psi(\dot{c}x, \dot{c}fy) = a_1\psi(x, \dot{c}'\dot{c}fy). \end{aligned} \tag{4.39}$$

The assertion follows. □

Proof of Proposition 4.13. We may assume that $S = \operatorname{Spec} R$ is connected. We consider a point $(A, \bar{\lambda}, \bar{\eta}) \in \mathcal{A}_{\mathbf{K}}(R)$. Any other point with the same image by (4.38) is of the form

$$(A, f\bar{\lambda}, \bar{\eta}\dot{c}), \quad \text{such that } f \in F^\times,\ \dot{c} \in \mathbf{K}_U^\bullet,\ f\dot{c}'\dot{c} = a \in \mathbb{A}_f^\times. \tag{4.40}$$

Moreover f is totally positive. Replacing f by fq for some $q \in \mathbb{Q}^\times$, $q > 0$, does not change the point (4.40). Therefore we may assume that $a \in \hat{\mathbb{Z}}^\times$ and that f is a unit.

By Proposition 4.12, for each natural number m we find $U = U_{N\ell}$ in such a way that $f = g_m^{2m}$ for some $g_m \in O_F^\times$. Since $\mathbf{K}^\bullet \cap (F \otimes \mathbb{A}_f)^\times$ is open in $(F \otimes \mathbb{A}_f)^\times$ we may choose m such that $g_m^m \in \mathbf{K}^\bullet$. We set $g = g_m^m$. Since $f = g^2$, the multiplication isomorphism by g is an isomorphism

$$g : (A, f\bar{\lambda}, \bar{\eta}\dot{c}) \xrightarrow{\sim} (A, \bar{\lambda}, \bar{\eta}g\dot{c}).$$

We obtain

$$(g\dot{c})' \cdot (g\dot{c}) = g^2\dot{c}'\dot{c} = f\dot{c}'\dot{c} = a \in \mathbb{A}_f^\times,$$

and therefore $g\dot{c} \in G(\mathbb{A}_f) \cap \mathbf{K}^\bullet = \mathbf{K}$. We see that $(A, f\bar{\lambda}, \bar{\eta}\dot{c})$ and $(A, \bar{\lambda}, \bar{\eta})$ define the same point of $\mathcal{A}_{\mathbf{K}}$. □

We know that for $\mathbf{K} \subset G(\mathbb{A}_f)$ small enough the functor $\mathcal{A}_{\mathbf{K},E_\nu}$ is representable by the scheme $\operatorname{Sh}(G,h)_{\mathbf{K},E_\nu}$. In general the latter is a coarse moduli scheme.

Proposition 4.16. *Let $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ be an open and compact subgroup. We set $\mathbf{K} = G(\mathbb{A}_f) \cap \mathbf{K}^\bullet$. We assume that there is an O_K-lattice $\Gamma \subset V$ and an integer $m \geq 3$ such that for each $u \in \mathbf{K}^\bullet$ we have $u\Gamma \otimes \hat{\mathbb{Z}} \subset \Gamma \otimes \hat{\mathbb{Z}}$ and such that u acts trivially on $\Gamma/m\Gamma$, so that $\mathcal{A}_{\mathbf{K},E_\nu}$ is representable (this is the analogue of condition* (3.6) *for $\mathbf{K}^\bullet$ instaed of $\mathbf{K}$).*

Let U be as in Proposition 4.13. Then the étale sheafification of the presheaf $\mathcal{A}^\bullet_{\mathbf{K}_U^\bullet,E_\nu}$ on the big étale site is represented by $\operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet,E_\nu}$.

Proof. We begin with some general remarks on [De, Prop. 1.15] in our case. The morphism of schemes (not of finite type) $\operatorname{Sh}(G,h) \to \operatorname{Sh}(G^\bullet,h)$ is an open and closed immersion. More precisely, for any open compact subgroup $\mathbf{K}_1 \subset G(\mathbb{A}_f)$, there is an open compact subgroup $\mathbf{K}_1^\bullet \subset G^\bullet(\mathbb{A}_f)$ such that $\mathbf{K}_1^\bullet \cap G(\mathbb{A}_f) = \mathbf{K}_1$ and such that $\operatorname{Sh}(G,h)_{\mathbf{K}_1} \subset \operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet}$ is an open and closed immersion. Indeed, it is a closed immersion by [De, Prop. 1.15] and it is open because the local rings of these varieties are normal and have both the same constant dimension. If Z is a connected component of $\operatorname{Sh}(G,h)_{\mathbb{C}}$, then its image $Z^\bullet$ in $\operatorname{Sh}(G^\bullet,h)_{\mathbb{C}}$ is a connected component. For arbitrary open compact subgroups $\mathbf{K} \subset G(\mathbb{A}_f)$ resp. $\mathbf{K}^\bullet \subset G(\mathbb{A}_f)$ the image $Z_{\mathbf{K}}$ of Z in $\operatorname{Sh}(G,h)_{\mathbf{K},\mathbb{C}}$, resp. the image $Z^\bullet_{\mathbf{K}^\bullet}$ of $Z^\bullet$ in $\operatorname{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet,\mathbb{C}}$, is a connected component. For $\mathbf{K}_1$ and $\mathbf{K}_1^\bullet$ as above, the map $Z_{\mathbf{K}_1} \to Z^\bullet_{\mathbf{K}_1^\bullet}$ is an isomorphism. For $g \in G^\bullet(\mathbb{A}_f)$, the multiplication by g induces a map

$$g : \operatorname{Sh}(G^\bullet,h)_{g\mathbf{K}_1^\bullet g^{-1}} \to \operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet}.$$

Now $G^\bullet(\mathbb{A}_f)$ acts transitively on the connected components of $\operatorname{Sh}(G^\bullet,h)_{\mathbb{C}}$, cf. [De, Prop. 2.2]. Therefore the sets $gZ^\bullet_{g\mathbf{K}_1^\bullet g^{-1},\mathbb{C}}$ cover $\operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet,\mathbb{C}}$, as g runs through all elements of $G^\bullet(\mathbb{A}_f)$. We note that $g\mathbf{K}_1^\bullet g^{-1} \cap G(\mathbb{A}_f) = g\mathbf{K}_1 g^{-1}$ because $G(\mathbb{A}_f) \subset G^\bullet(\mathbb{A}_f)$ is a normal subgroup. We conclude that the images of the following composite maps cover $\operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet}$, as g varies in $G^\bullet(\mathbb{A}_f)$,

$$\varkappa_g : \operatorname{Sh}(G,h)_{g\mathbf{K}_1 g^{-1}} \to \operatorname{Sh}(G^\bullet,h)_{g\mathbf{K}_1^\bullet g^{-1}} \xrightarrow{g} \operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet}. \tag{4.41}$$

Now we turn to the proof of the proposition. We show that the morphism

$$\operatorname{Sh}(G,h)_{\mathbf{K}} \to \operatorname{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet} \tag{4.42}$$

is an open and closed immersion. Indeed by Proposition 4.13 we know that (4.42) induces an injection on the $\mathbb{C}$-valued points. By the remarks above there exists a open and compact subgroup $\mathbf{K}_1^\bullet \subset \mathbf{K}_U^\bullet$ such that $\mathbf{K}_1^\bullet \cap G(\mathbb{A}_f) = \mathbf{K}$ and such that $\mathrm{Sh}(G,h)_{\mathbf{K}} \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet}$ is an open and closed immersion. We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}(G,h)_{\mathbf{K}} & \longrightarrow & \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_1^\bullet} \\ & \searrow & \downarrow \\ & & \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet}. \end{array}$$

By our assumption on $\mathbf{K}^\bullet$, the vertical arrow is a finite étale morphism. Hence the same is true for the oblique arrow. Since by Proposition 4.13 its geometric fibres contain at most one element, the claim follows.

Let Y be the étale sheafification of $\mathcal{A}^\bullet_{\mathbf{K}_U^\bullet,E_\nu}$. We consider the preimage Y^o of $\mathrm{Sh}(G,h)_{\mathbf{K},E_\nu}$ by the natural morphism

$$Y \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet,E_\nu}.$$

Then $Y^o \subset Y$ is an open and closed subfunctor. We consider the natural morphism

$$\mathcal{A}_{\mathbf{K},E_\nu} \to \mathcal{A}^\bullet_{\mathbf{K}_U^\bullet,E_\nu} \to Y \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet,E_\nu}.$$

Since Y^o is a fibre product we obtain a factorization

$$\mathcal{A}_{\mathbf{K},E_\nu} \to Y^o \to \mathrm{Sh}(G,h)_{\mathbf{K},E_\nu}. \tag{4.43}$$

We claim that both arrows are isomorphisms. Since their composite is an isomorphism the first arrow is a monomorphism. Therefore it suffices to show that the first arrow is a surjection of étale sheaves. Since both functors $\mathcal{A}_{\mathbf{K},E_\nu}$ and $\mathcal{A}^\bullet_{\mathbf{K}_U^\bullet,E_\nu}$ commute with inductive limits, the stalks at a geometric point ξ of $\operatorname{Spec} R$ of the sheafifications are the points of these functors with values in the strict henselization R^{sh}_ξ. Therefore it is enough to show that

$$\mathcal{A}_{\mathbf{K},E_\nu}(R) \to Y^o(R) \tag{4.44}$$

is surjective for a strictly henselian local ring R. For an algebraically closed field R both sides have the same coarse moduli space. Therefore the map is bijective in this case. In general the residue field κ_R of R is algebraically closed, since we are in characteristic 0. We consider a point $(A,\tilde{\lambda},\tilde{\eta}) \in \mathcal{A}^\bullet_{\mathbf{K}_U^\bullet}(R) = Y(R)$ which is in $Y^o(R)$. Over κ_R this point is in the image of (4.44). By Lemma 4.14, the preimage by (4.44) has the form $(A_{\kappa_R},\bar{\lambda},\bar{\eta})$ for some $\lambda \in \tilde{\lambda}$ and $\eta \in \tilde{\eta}$. This is justified because the reduction to κ_R defines a bijection between the class $\tilde{\lambda}$ on A and its reduction on A_{κ_R}. The same applies to $\tilde{\eta}$. We must verify that (A,λ,η) defines a point of $\mathcal{A}_{\mathbf{K},E_\nu}(R)$. Since there is no difference of a rigidification η over κ_R or over R, this is already decided over κ_R. This proves that (4.44) is bijective. Consequently the arrows of (4.43) are isomorphism and therefore the functor Y^o is representable.

Now we deduce the representability of Y. Let $g \in G^\bullet(\mathbb{A}_f)$. We already noted that $g\mathbf{K}^\bullet g^{-1} \cap G(\mathbb{A}_f) = g\mathbf{K}g^{-1}$. The multiplication by g induces an isomorphism

$$\mathcal{A}^\bullet_{g\mathbf{K}_U^\bullet g^{-1},E_\nu} \xrightarrow{\sim} \mathcal{A}^\bullet_{\mathbf{K}_U^\bullet,E_\nu} \to Y. \tag{4.45}$$

We have shown that $\mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1},E_\nu}$ is an open and closed subfunctor of the sheafification of the left hand side of (4.45). (We note that the same $U = U_{N\ell}$ suffices for each $g \in G^\bullet(\mathbb{A}_f)$.) Taking the composite with (4.45), we obtain an open and closed immersion

$$\mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1},E_\nu} \to Y. \tag{4.46}$$

Its image is equal to the pullback of $\mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1},E_\nu} \xrightarrow{g} \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet,E_\nu}$ by the natural morphism $Y \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}_U^\bullet,E_\nu}$. Therefore (4.46) gives, for varying $g \in G^\bullet(\mathbb{A}_f)$, an open covering of Y by representable subfunctors. □

For later use we formulate a variant of the last argument.

Lemma 4.17. *Let* $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ *be an open compact subgroup and let* $\mathbf{K} = G(\mathbb{A}_f) \cap \mathbf{K}^\bullet$. *Assume that* $U \subset (F \otimes \mathbb{A}_f)^\times$ *is a principal congruence subgroup as constructed in the proof of Proposition 4.13. Then for all* $g \in G^\bullet(\mathbb{A}_f)$ *the canonical map*

$$\mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1}} \to \mathrm{Sh}(G^\bullet,h)_{g\mathbf{K}^\bullet_U g^{-1}} \tag{4.47}$$

is an open and closed immersion. The composite of this map with $g\colon \mathrm{Sh}(G^\bullet,h)_{g\mathbf{K}^\bullet_U g^{-1}} \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U}$ *gives an open and closed immersion,*

$$\varkappa_g : \mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1}} \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U}.$$

The maps $\varkappa_g$ *for varying* $g \in G^\bullet(\mathbb{A}_f^p)$ *are an open covering of* $\mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U}$.

If the group $\mathbf{K}^\bullet$ *satisfies the assumptions of Proposition 4.16, then the set of maps*

$$\{\varkappa_g : \mathrm{Sh}(G,h)_{g\mathbf{K}g^{-1}} \to \mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet}\}_{g\in G^\bullet(\mathbb{A}_f^p)}$$

is an étale covering by finite étale maps.

Proof. Only the last assertion remains to be proved. Let Z be a connected component of $\mathrm{Sh}(G,h)_\mathbb{C}$ and let $Z^\bullet \in \mathrm{Sh}(G^\bullet,h)$ be its image as in the proof of Proposition 4.16. As in that proof, it is enough to show that the sets $gZ^\bullet_{g\mathbf{K}^\bullet_U g^{-1}}$ cover $\mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U}$, as g runs through all elements of $G^\bullet(\mathbb{A}_f^p)$.

We consider $\tilde{G}^\bullet = \{b \in B^{\mathrm{opp}} \mid b'b \in F^\times\}$ as algebraic group over F. Then $\mathrm{Res}_{F/\mathbb{Q}}\tilde{G}^\bullet = G^\bullet$, cf. (2.2). We consider the homomorphisms

$$\begin{array}{cccccccc} \mu : \tilde{G}^\bullet & \to & F^\times, & & \det : \tilde{G}^\bullet & \to & K^\times \\ b & \longmapsto & b'b & & b & \longmapsto & \mathrm{Nm}^o_{B/K}. \end{array} \tag{4.48}$$

Let $\tilde{T}^\bullet$ be the algebraic torus over F given by

$$\tilde{T}^\bullet(F) = \{(f,k) \in F^\times \times K^\times \mid f^2 = k\bar{k}\}.$$

By (4.48) we obtain a homomorphism $\nu : \tilde{G}^\bullet \to \tilde{T}^\bullet$, $b \mapsto (\mu(b), \det b)$. Let $\tilde{H}^\bullet$ be the kernel of this map. One can check that $\tilde{H}^\bullet \times_{\mathrm{Spec}\, F} \mathrm{Spec}\,\mathbb{C} \cong \mathrm{SL}_2(\mathbb{C})$. Therefore we obtain an exact sequence

$$0 \to \tilde{G}^\bullet_{\mathrm{der}} \to \tilde{G}^\bullet \xrightarrow{\nu} \tilde{T}^\bullet \to 0,$$

where the derived group is simply connected. By [De, Thm. 2.4], we obtain a bijection

$$\pi_0(\mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U} \xrightarrow{\sim} \nu(\mathbf{K}^\bullet_\infty \times \mathbf{K}^\bullet_U)\backslash\tilde{T}^\bullet(\mathbb{A}_F)/\tilde{T}^\bullet(F). \tag{4.49}$$

The right hand side may also be written as $\nu(\mathbf{K}^\bullet_\infty \times \mathbf{K}^\bullet_U)\backslash T^\bullet(\mathbb{A})/T^\bullet(\mathbb{Q})$.

Because the cyclic extension K/F splits the torus $\tilde{T}^\bullet$, weak approximation holds for $\tilde{T}^\bullet$, cf. [V, Thm. 6.36]. In particular $\tilde{T}^\bullet(F)$ is dense in $\tilde{T}^\bullet(F \otimes_\mathbb{Q} \mathbb{R})\tilde{T}^\bullet(F \otimes_\mathbb{Q} \mathbb{Q}_p)$. This implies that

$$\tilde{T}^\bullet(F)\nu(\mathbf{K}^\bullet_\infty \times \mathbf{K}^\bullet_U)\tilde{T}^\bullet(\mathbb{A}^p_{F,f}) = \tilde{T}^\bullet(\mathbb{A}_F).$$

Hence $\tilde{T}^\bullet(\mathbb{A}^p_{F,f}) = T^\bullet(\mathbb{A}^p_f)$ acts transitively on the right hand side of (4.49). Since $G^\bullet(\mathbb{A}^p_f) \to T^\bullet(\mathbb{A}^p_f)$ is surjective, the sets $gZ^\bullet_{g\mathbf{K}^\bullet_U g^{-1}}$ for $g \in G^\bullet(\mathbb{A}^p_f)$ cover $\mathrm{Sh}(G^\bullet,h)_{\mathbf{K}^\bullet_U}$. Therefore the last assertion of the proposition follows as in the proof of Proposition 4.16. □

Let $\mathbf{K}^\bullet_p \subset G^\bullet(\mathbb{Q}_p)$ be the subgroup associated to a choice of Λ_p, $\mathbf{M}^\bullet$ and $\mathbf{K}_{\mathfrak{q}_i}$, for $i = 1,\dots,s$, cf. (4.6). We set $\mathbf{M} = \mathbb{Z}_p^\times \cap \mathbf{M}^\bullet$. We denote by $\mathbf{K}_p \subset G(\mathbb{Q}_p)$ the subgroup associated to the choice of Λ_p, $\mathbf{M}$ and $\mathbf{K}_{\mathfrak{q}_i}$, cf. (3.15). We see easily that

$$\mathbf{K}_p = \mathbf{K}^\bullet_p \cap G(\mathbb{Q}_p). \tag{4.50}$$

Under these hypotheses, we have an integral version of Proposition 4.13. It concerns $\widetilde{\mathcal{A}}^t_\mathbf{K}$ instead of $\mathcal{A}_\mathbf{K}$ and $\widetilde{\mathcal{A}}^{\bullet t}_\mathbf{K}$ instead of $\mathcal{A}^\bullet_\mathbf{K}$.

Proposition 4.18. *We fix* M *and* ℓ *as in Proposition 4.12, but we assume that both are prime to* p. *Let* $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p\mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, *with* $\mathbf{K}^\bullet_p$ *as in* (4.6), *where* $\mathbf{M}_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$. *Let* $\mathbf{K} = \mathbf{K}^\bullet \cap G^\bullet(\mathbb{A}_f)$.

Then there exists a power N *of* M *such that for the open compact subgroup* $U \subset (F \otimes \mathbb{A}_f)^\times$ *of Proposition 4.12, the natural map*

$$\widetilde{\mathcal{A}}^t_\mathbf{K} \to \widetilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U} \tag{4.51}$$

is a monomorphism of functors.

By assumption we have $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p}$ and $\mathbf{K}^\bullet_U = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p}_U$. The group $\mathbf{K} = \mathbf{K}^\bullet \cap G(\mathbb{A}_f) = \mathbf{K}^\bullet_U \cap G(\mathbb{A}_f)$ has a similiar decomposition and therefore the functor $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ makes sense. Similarly to the proof of Proposition 4.13, we need two lemmas which are analogous to Lemmas 4.14 and 4.15.

Lemma 4.19. *Let $(A, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, \xi_p)$ be a point of $\tilde{\mathcal{A}}^t_{\mathbf{K}}(R)$, with image $(A', \tilde{\lambda}', \tilde{\eta'}^p, (\bar{\eta'}_{\mathfrak{q}_j})_j, (\xi'_{\mathfrak{p}_i})_i)$ in $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(R)$. Then there is a polarization $\lambda' \in \tilde{\lambda}'$ and a level structure $\eta'^p \in \tilde{\eta}'^p$ and an element $\xi'_p(\lambda') \in \mathbb{Z}_p^\times$ such that for $i = 0, \ldots, s$*

$$\xi'_p(\lambda') \equiv \xi'_{\mathfrak{p}_i} \mod \mathbf{M}^\bullet_{\mathfrak{p}_i}$$

and such that the point $(A, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, \xi_p)$ is isomorphic to $(A', \bar{\lambda}', \bar{\eta'}^p, (\bar{\eta'}_{\mathfrak{q}_j})_j, \xi'_p(\lambda'))$. The function ξ'_p on $\bar{\lambda}'$ is given by $\xi'_p(u\lambda') = u\xi'_p(\lambda')$ for $u \in U_p(\mathbb{Q})$.

Proof. The proof is similar to that of Lemma 4.14. We may assume that $S = \operatorname{Spec} R$ is connected. Then we can argue over a geometric point $\bar{s}$ of S, as explained after Definition 3.1. We choose arbitrarily polarizations $\lambda' \in \tilde{\lambda}'$ and $\lambda \in \bar{\lambda}$, and prime-to-$p$ level structures $\eta'^p \in \tilde{\eta'}^p$ and $\eta^p \in \bar{\eta}^p$, and p-level structures $\eta'_{\mathfrak{q}_j} \in \bar{\eta'}_{\mathfrak{q}_j}$ and $\eta_{\mathfrak{q}_j} \in \bar{\eta}_{\mathfrak{q}_j}$. By assumption there exists an isogeny $\alpha : A' \to A$ of order prime to p such that

$$\alpha^*(\lambda) = f\lambda', \quad \alpha \circ \eta^p \dot{c}^p = \eta'^p, \quad \alpha \circ \eta'_{\mathfrak{q}_j} c_{\mathfrak{q}_j} = \eta_{\mathfrak{q}_j}, \quad \xi'_{\mathfrak{p}}(f\lambda')\varepsilon_{\mathfrak{p}} = \xi_p(\lambda),$$

where $f \in U_p(F)$ is totally positive, $\dot{c}^p \in G^\bullet(\mathbb{A}^p_f)$ and $c_{\mathfrak{q}_j} \in \mathbf{K}_{\mathfrak{q}_j}$ and $\varepsilon_{\mathfrak{p}} \in \mathbf{M}^\bullet_{\mathfrak{p}}$. From this the assertion follows easily. □

Lemma 4.20. *Let $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ and U as in Proposition 4.18. Let*

$$(A_1, \bar{\lambda}_1, \bar{\eta}^p_1, (\bar{\eta}_{1,\mathfrak{q}_j})_j, \xi_{1,p}),\ (A_2, \bar{\lambda}_2, \bar{\eta}^p_2, (\bar{\eta}_{2,\mathfrak{q}_j})_j, \xi_{2,p})$$

be two points of $\tilde{\mathcal{A}}^t_{\mathbf{K}}(R)$ which have the same image in $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(R)$. Then there exists a totally positive $f \in O_F^\times$, an element $\theta \in (O_F \otimes \hat{\mathbb{Z}})^\times$ such that $f\theta = a \in \hat{\mathbb{Z}}^\times$ and $\theta_p \in \prod_{i=0}^s \mathbf{M}^\bullet_{\mathfrak{p}_i}$, and an element $\dot{c} \in \mathbf{K}^{\bullet,p}_U$ with $\theta^p = \dot{c}'\dot{c}$, such that the point $(A_2, \bar{\lambda}_2, \bar{\eta}^p_2, (\bar{\eta}_{2,\mathfrak{q}_j})_j, \xi_{2,p})$ is isomorphic to

$$(A_1, \bar{\lambda}_1 f, \bar{\eta}^p_1 \dot{c}, (\bar{\eta}_{1,\mathfrak{q}_j})_j, \xi'_{1,p}).$$

Here the function $\xi'_{1,p}$ on $\bar{\lambda}_1 f$ is defined by

$$\xi'_{1,p}(\lambda_1 f) = a\xi_{1,p}(\lambda_1).$$

Proof. We fix a polarization $\lambda_1 \in \bar{\lambda}_1$ and $\eta_1 \in \bar{\eta}_1$. By Lemma 4.19, the point $(A_2, \bar{\lambda}_2, \bar{\eta}^p_2, (\bar{\eta}_{2,\mathfrak{q}_j})_j, \xi_{2,p})$ is isomorphic to a point of the form

$$(A_1, \bar{\lambda}_1 f, \eta_1^{\bar{p}} \dot{c}, (\bar{\eta}_{1,\mathfrak{q}_j})_j, \xi'_{1,p}).$$

The value $\xi'_{1,p}(f\lambda_1) \in \mathbb{Z}_p$ satisfies the following congruence in $O^\times_{F_{\mathfrak{p}_i}}$ for each $i = 0, \ldots s$,

$$\xi'_{1,p}(f\lambda_1) \equiv f\xi_{1,p}(\lambda_1) \mod \mathbf{M}^\bullet_{\mathfrak{p}_i}.$$

This implies that there is an element $\theta_p \in \prod_{i=1}^s \mathbf{M}^\bullet_{\mathfrak{p}_i}$ such that $f\theta_p = a_p \in \mathbb{Z}_p^\times$. Then we obtain

$$\xi'_{1,p}(f\lambda_1) \equiv a_p\xi_{1,p}(\lambda_1) \mod \mathbf{M}^\bullet_{\mathfrak{p}_i}.$$

Moreover an $\mathbb{A}^p_f$-version of (4.39) shows that there is an element $a^p \in (\mathbb{A}^p_f)^\times$ such that $f\dot{c}'\dot{c} = a^p$. We have the right to multiply f by an element of $U_p(\mathbb{Q})$. Therefore we may assume that $a^p \in \hat{\mathbb{Z}}^p$. The result follows by setting $a = a_p a^p$. □

Proof of Proposition 4.18. As in the proof of Proposition 4.13, it is enough to show that

$$\tilde{\mathcal{A}}^t_{\mathbf{K}}(R) \to \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(R)$$

is injective if $\operatorname{Spec} R$ is connected. Assume that we are given two points as in Lemma 4.20 which are mapped to the same point of $\tilde{\mathcal{A}}^\bullet_{\mathbf{K}^\bullet_U}(R)$. For suitable U we conclude as in the proof of Proposition 4.13 that $f = g^2$, for some $g \in O_F^\times \cap \mathbf{K}^\bullet$. If in the argument of that proof we choose

m big enough we may assume that $g \in \mathbf{M}^{\bullet}$. We obtain $a_p = f\theta_p \in \mathbf{M}^{\bullet} \cap \mathbb{Z}_p^{\times} = \mathbf{M}$ (see before (4.50)). The multiplication by g induces an isomorphism

$$g : (A_1, \bar{\lambda}_1 f, \bar{\eta}_1^p \dot{c}, (\bar{\eta}_{1,\mathfrak{q}_j})_j, \xi'_{1,p}) \to (A_1, \bar{\lambda}_1, \bar{\eta}_1^p, (\bar{\eta}_{1,\mathfrak{q}_j})_j, \xi_{1,p}). \tag{4.52}$$

Indeed, we have $g^*(\lambda_1) = \lambda_1 g^2$ and the morphism (4.52) respects the data $\bar{\eta}_1^p \dot{c}$ and $\bar{\eta}_1^p$, comp. the proof of Proposition 4.13. Furthermore, for $\lambda_1 \in \bar{\lambda}_1$ we obtain

$$g^*(\xi_{1,p}(\lambda_1 f)) = g^*(\xi_{1,p})(g^*(\lambda_1)) := \xi_{1,p}(\lambda_1) = a_p \xi_{1,p}(\lambda_1) = \xi'_{1,p}(\lambda_1 f).$$

The second to last equation holds because $a_p \in \mathbf{M}$. □

We recall that we assume that $D_{\mathfrak{p}_0}$ is a quaternion division algebra, cf. (3.12). In this case, we have the following integral version of Proposition 4.16.

Proposition 4.21. *Let $\mathbf{K}^{\bullet} = \mathbf{K}^{\bullet}_p \mathbf{K}^{\bullet,p} \subset G^{\bullet}(\mathbb{A}_f)$, with $\mathbf{K}^{\bullet}_p$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O^{\times}_{F_{\mathfrak{p}_0}}$. We set $\mathbf{K} = G(\mathbb{A}_f) \cap \mathbf{K}^{\bullet}$. We assume that there is an O_K-lattice $\Gamma \subset V$ and an integer $m \geq 3$ prime to p such that for each $g \in \mathbf{K}^{\bullet}$ we have $g\Gamma \subset \Gamma$ and such that g acts trivially on $\Gamma/m\Gamma$. (In this case $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ is representable.) Let U be as in Proposition 4.13.*

Let $\tilde{\mathsf{A}}^t_{\mathbf{K}}$ be the $\operatorname{Spec} O_{E_\nu}$-scheme which represents the functor $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ and let $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ be the coarse moduli scheme of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$. It is a normal scheme which is proper over $\operatorname{Spec} O_{E_\nu}$.

The canonical map $\tilde{\mathsf{A}}^t_{\mathbf{K}} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ is an open and closed immersion. The arrow $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ is the étale sheafification of the presheaf $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ on the big étale site.

Proof. The scheme $\tilde{\mathsf{A}}^t_{\mathbf{K}}$ is regular and the morphism $\tilde{\mathsf{A}}^t_{\mathbf{K}} \to \operatorname{Spec} O_{E_\nu}$ is generically smooth and proper. Its special fibre is a divisor with normal crossings. This follows from deformation theory because the p-divisible group $X_{\mathfrak{q}_0}$ is a special formal $O_{B_{\mathfrak{q}_0}}$-module in the sense of Drinfeld. The properness follows from a standard argument using that B is a division algebra, cf. [Dr, Prop. 4.1].

For the proof that a coarse moduli scheme $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ exists we refer to [Z1, 1.7 Satz]. Because this moduli scheme is obtained as a quotient of a normal scheme by a finite group, the coarse moduli scheme is normal. Since $\tilde{\mathsf{A}}^t_{\mathbf{K},E_\nu} \subset \tilde{\mathsf{A}}^t_{\mathbf{K}}$ is an open dense subset of a scheme which is locally integral we obtain a bijection between connected components

$$\pi_0(\tilde{\mathsf{A}}^t_{\mathbf{K},E_\nu}) \to \pi_0(\tilde{\mathsf{A}}^t_{\mathbf{K}}), \qquad Z \longmapsto \bar{Z}.$$

The same is true for the connected components of $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U, E_\nu}$ and $\tilde{\mathsf{A}}^{\bullet}_{\mathbf{K}^{\bullet t}_U}$. We claim that

$$\tilde{\mathsf{A}}^t_{\mathbf{K}} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U} \tag{4.53}$$

is an open and closed immersion. Indeed, the morphism (4.53) is proper because $\tilde{\mathsf{A}}^t_{\mathbf{K}}$ is proper over $\operatorname{Spec} O_{E_\nu}$. The general fiber over E_ν of this morphism coincides up to a Galois twist with (4.42) and is therefore an open and closed immersion. Let $Z \subset \tilde{\mathsf{A}}^t_{\mathbf{K},E_\nu}$ be a connected component which we also regard as a connected component of $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U, E_\nu}$. We consider the closures $\bar{Z} \subset \tilde{\mathsf{A}}^t_{\mathbf{K}}$ and $\bar{Z}^{\bullet} \subset \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ of Z. These are connected components and the morphism (4.53) induces an birational proper morphism $\bar{Z} \to \bar{Z}^{\bullet}$ of normal schemes. If we take the values in some algebraically closed field the last morphism becomes injective. This follows from Proposition 4.18 and the definition of a coarse moduli problem. Therefore $\bar{Z} \to \bar{Z}^{\bullet}$ is an isomorphism, and the claim is proved.

Let Y be the étale sheafification of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}$ on the big étale site. We consider the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{A}}^t_{\mathbf{K}} & \longrightarrow & \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U} \\ \downarrow & & \downarrow \\ Y^o & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{\mathsf{A}}^t_{\mathbf{K}} & \longrightarrow & \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^{\bullet}_U}. \end{array} \tag{4.54}$$

Here Y^o is defined to be the fiber product in the lower square. We know that the horizontal arrows are monomorphisms and the two lower ones are open and closed immersions. We will show that $\tilde{\mathcal{A}}^t_{\mathbf{K}} \to Y^o$ is an isomorphism of sheaves.

Let T be a noetherian scheme over $\operatorname{Spec} O_{E_\nu}$ and $\mathcal{O}^{sh}_{T,t}$ the strict henselization at a geometric point t of T. We have to show that the induced map of stalks

$$(\tilde{\mathcal{A}}^t_{\mathbf{K}})_{T,t} \to (Y^o)_{T,t}$$

is bijective. (The stalks are the same as the stalks of the restriction of both sheaves to the small étale site T_{et}.) Since $\tilde{\mathcal{A}}^t_{\mathbf{K}}$ commutes with direct limits, we obtain $(\tilde{\mathcal{A}}^t_{\mathbf{K}})_{T,t} = \tilde{\mathcal{A}}^t_{\mathbf{K}}(\mathcal{O}^{sh}_{T,t})$. The same is true for the presheaf $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}$. Therefore we obtain $(Y^o)_{T,t} \subset Y_{T,t} = \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(\mathcal{O}^{sh}_{T,t})$. This subset consists of the points on the right hand side which are mapped to $\tilde{\mathsf{A}}^t_{\mathbf{K}}(\mathcal{O}^{sh}_{T,t})$.

Let L the residue class field of $\mathcal{O}^{sh}_{T,t}$ which is separably closed. We firstly show that

$$\tilde{\mathcal{A}}^t_{\mathbf{K}}(L) \to Y^o(L) \tag{4.55}$$

is bijective. Equivalently we may show that $Y^o(L) \to \tilde{\mathsf{A}}^t_{\mathbf{K}}(L)$ is bijective. Clearly this map is surjective because $\tilde{\mathcal{A}}^t_{\mathbf{K}} \cong \tilde{\mathsf{A}}^t_{\mathbf{K}}$. Let $\theta_1, \theta_2 \in Y^o(L) \subset \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(L)$ be two elements with the same image in $\tilde{\mathsf{A}}^t_{\mathbf{K}}(L)$. By the properties of a coarse moduli scheme, the map $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(\bar{L}) \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(\bar{L})$ is bijective. We conclude that $\theta_{1,\bar{L}} = \theta_{2,\bar{L}}$ holds for the base change. Therefore we find a finite totally inseparable extension N of L such that $\theta_{1,N} = \theta_{2,N}$. If the last two points are represented by the data $(A_1, \tilde{\lambda}_1, \tilde{\eta}^p_1, (\bar{\eta}_{1,\mathfrak{q}_j})_j, (\xi_{1,\mathfrak{p}_i})_i)$ and $(A_2, \tilde{\lambda}_2, \tilde{\eta}^p_2, (\bar{\eta}_{2,\mathfrak{q}_j})_j, (\xi_{2,\mathfrak{p}_i})_i)$, we conclude by the rigidity of abelian varieties, applied to the nilimmersion $N \otimes_L N \to N$, that $\theta_1 = \theta_2$.

Now we consider the map

$$\tilde{\mathcal{A}}^t_{\mathbf{K}}(\mathcal{O}^{sh}_{T,t}) \to Y^o(\mathcal{O}^{sh}_{T,t}).$$

This map is clearly injective. We show that it is surjective. We consider a point

$$(A_1, \tilde{\lambda}_1, \tilde{\eta}^p_1, (\bar{\eta}_{1,\mathfrak{q}_j})_j, (\xi_{1,\mathfrak{p}_i})_i) \in Y^o(\mathcal{O}^{sh}_{T,t}) \subset \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}(\mathcal{O}^{sh}_{T,t}). \tag{4.56}$$

Over L this point is in the image of (4.55). By Lemma 4.19, the preimage has the form $(A_{1,L}, \bar{\lambda}_1, \bar{\eta}^p_1, (\bar{\eta}_{1,\mathfrak{q}_j})_j, (\xi_{2,\mathfrak{p}_i})_i)$. Since $\bar{\lambda}_1 \subset \tilde{\lambda}_1$, the polarizations in $\bar{\lambda}_1$ lift to polarizations of A_1 which are principal in p. Since there is no difference between a rigidification over $\mathcal{O}^{sh}_{T,t}$ and over the residue class field L, we see that $(A_1, \bar{\lambda}_1, \bar{\eta}^p_1, (\bar{\eta}_{1,\mathfrak{q}_j})_j, (\xi_{2,\mathfrak{p}_i})_i)$ is a point of $\tilde{\mathcal{A}}^t_{\mathbf{K}}(\mathcal{O}^{sh}_{T,t})$ which is mapped to the point (4.56). We have proved that the two vertical arrows on the left hand side of diagram (4.54) are isomorphisms.

To show that $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_U} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}$ is the étale sheafification, we can argue as in the proof of Proposition 4.16 if we substitute Lemma 4.17 by Lemma 4.22 below. $\square$

The group $G^\bullet(\mathbb{A}^p_f)$ acts on the projective system of the functors $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ for varying $\mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}^p_f)$ via the datum η^p of Definition 4.3. More explicitly, each $g \in G^\bullet(\mathbb{A}^p_f)$ induces by multiplication an isomorphism

$$g : \tilde{\mathcal{A}}^{\bullet t}_{g\mathbf{K}^\bullet g^{-1}} \to \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet},$$

which induces an isomorphism of the coarse moduli spaces.

Lemma 4.22. *Let $\mathbf{K}^\bullet$ be as in Proposition 4.21. Then there is an open subgroup $U \subset (F \otimes \mathbb{A}_f)^\times$ such that for each $g \in G^\bullet(\mathbb{A}^p_f)$ the natural morphism*

$$\tilde{\mathsf{A}}^t_{g\mathbf{K}g^{-1}} \to \tilde{\mathsf{A}}^{\bullet t}_{g\mathbf{K}^\bullet_U g^{-1}} \tag{4.57}$$

is an open and closed immersion. If we compose the immersion with the morphisms $g : \tilde{\mathsf{A}}^{\bullet t}_{g\mathbf{K}^\bullet_U g^{-1}} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}$, we obtain open and closed immersions

$$\varkappa_g : \tilde{\mathsf{A}}^t_{g\mathbf{K}g^{-1}} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}. \tag{4.58}$$

For varying $g \in G^\bullet(\mathbb{A}^p_f)$ the morphisms $\varkappa_g$ are an open covering of $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}$.

Proof. We have already seen that (4.57) is an open and closed immersion, cf. (4.53). Therefore the same is true of $\varkappa_g$. The general fibre of $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet_U}$ is up to a Galois twist $\mathrm{Sh}(G^\bullet, h)_{\mathbf{K}^\bullet_U, E_\nu}$. By Lemma 4.17 each connected component Z of $\mathrm{Sh}(G^\bullet, h)_{\mathbf{K}^\bullet_U, E_\nu}$ is in the image of $\varkappa_g$ for some $g \in G^\bullet(\mathbb{A}_f^p)$. Since $\tilde{\mathsf{A}}^\bullet_{\mathbf{K}^\bullet_U}$ is locally an integral scheme which is flat over O_{E_ν}, each connected component is of the form $\bar{Z}$. It is therefore in the image of the open and closed immersion $\varkappa_g$. Hence (4.58) is indeed a covering. □

In the sequel, only the Shimura varieties $\mathrm{Sh}(G^\bullet, h)$ and $\mathrm{Sh}(G^\bullet, h^\bullet_D)$ attached to the group $G^\bullet$ will play a role. These are *ramified* abelian Galois twists of each other, cf. Proposition 4.8. The following table summarizes the Shimura varieties and their relations to moduli functors.

Shimura variety	Moduli problem	First occurrence	Relation
$\mathrm{Sh}(G^\bullet, h)$	$\mathcal{A}^\bullet$, resp. $\mathcal{A}^{\bullet, bis}$	Def. 4.1/4.2, resp. Def. 4.3	coarse moduli scheme
$\mathrm{Sh}(G^\bullet, h^\bullet_D)$	$\mathcal{A}^{\bullet,t}$	Def. 4.6	coarse moduli sch. of unram. twist

The integral model $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$ of $\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$ over O_{E_ν} (defined if $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}^\bullet_p$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$) is defined by twisting back the integral extension $\tilde{\mathcal{A}}^{\bullet,t}$ of $\mathcal{A}^{\bullet,t}$.

5. The RZ-spaces

In this section, we discuss the RZ-spaces needed for the p-adic uniformization of the Shimura varieties of the last section.

We first discuss the banal case of a prime ideal $\mathfrak{p}_i$, for $i \neq 0$.

Definition 5.1. Let S be an O_{E_ν}-scheme. The category $\mathcal{P}_{\mathfrak{p}_i}(S)$ is the category of all triples $(Y, \iota, \bar{\lambda})$, where Y is a p-divisible group of height $8[F_{\mathfrak{p}_i} : \mathbb{Q}_p]$ over S, where $\iota : O_{B_{\mathfrak{p}_i}} \to \mathrm{End}\, Y$ is a $\mathbb{Z}_p$-Algebra homomorphism, and where $\bar{\lambda}$ is a $O^\times_{F_{\mathfrak{p}_i}}$-homogeneous polarization of Y such that each $\lambda \in \bar{\lambda}$ is principal. We demand that the Rosati involution associated to $\lambda \in \bar{\lambda}$ is compatible with the involution $b \mapsto b^\star$ on $B_{\mathfrak{p}_i}$ with respect to ι. The decomposition $O_{B_{\mathfrak{p}_i}} = O_{B_{\mathfrak{q}_i}} \times O_{B_{\bar{\mathfrak{q}}_i}}$ induces a composition $Y = Y_{\mathfrak{q}_i} \times Y_{\bar{\mathfrak{q}}_i}$. We demand moreover that the p-divisible group $Y_{\mathfrak{q}_i}$ is étale.

The definition implies that $\lambda = \lambda_{\mathfrak{q}_i} \oplus \lambda_{\bar{\mathfrak{q}}_i}$ where $\lambda_{\mathfrak{q}_i} : Y_{\mathfrak{q}_i} \to (Y_{\bar{\mathfrak{q}}_i})^\wedge$ and $\lambda_{\bar{\mathfrak{q}}_i} : Y_{\bar{\mathfrak{q}}_i} \to (Y_{\mathfrak{q}_i})^\wedge$ are isomorphisms to the dual p-divisible groups such that $\lambda_{\bar{\mathfrak{q}}_i} = -(\lambda_{\mathfrak{q}_i})^\wedge$ and such that for each $b_1 \in O_{B_{\mathfrak{q}_i}}$ and $b_2 \in O_{B_{\bar{\mathfrak{q}}_i}}$ the following diagrams are commutative,

$$\begin{array}{ccc} Y_{\bar{\mathfrak{q}}_i} & \xrightarrow{\iota(b_1^\star)} & Y_{\bar{\mathfrak{q}}_i} \\ {\scriptstyle\lambda_{\bar{\mathfrak{q}}_i}}\downarrow & & \downarrow{\scriptstyle\lambda_{\bar{\mathfrak{q}}_i}} \\ (Y_{\mathfrak{q}_i})^\wedge & \xrightarrow[\iota(b_1)^\wedge]{} & (Y_{\mathfrak{q}_i})^\wedge, \end{array} \qquad \begin{array}{ccc} Y_{\mathfrak{q}_i} & \xrightarrow{\iota(b_2^\star)} & Y_{\mathfrak{q}_i} \\ {\scriptstyle\lambda_{\mathfrak{q}_i}}\downarrow & & \downarrow{\scriptstyle\lambda_{\mathfrak{q}_i}} \\ (Y_{\bar{\mathfrak{q}}_i})^\wedge & \xrightarrow[\iota(b_2)^\wedge]{} & (Y_{\bar{\mathfrak{q}}_i})^\wedge. \end{array} \tag{5.1}$$

Since one of these diagrams is the dual of the other it is enough to require the commutativity of one of these diagrams.

We construct an object of $\mathcal{P}_{\mathfrak{p}_i}(\mathrm{Spec}\, \bar{\kappa}_{E_\nu})$ as follows. Let us denote the action of the Frobenius endomorphism on $W(\bar{\kappa}_{E_\nu})$ by σ. Recall from (3.13) the lattices $\Lambda_{\mathfrak{q}_i}$ and $\Lambda_{\bar{\mathfrak{q}}_i}$. We endow $\Lambda_{\mathfrak{q}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu})$ with the structure of a Dieudonné module by defining the action of the Frobenius F on this module by

$$F(u_1 \otimes \xi_1) = p u_1 \otimes \sigma(\xi_1), \quad u_1 \in \Lambda_{\mathfrak{q}_i},\ \xi_1 \in W(\bar{\kappa}_{E_\nu}),$$

and we endow $\Lambda_{\bar{\mathfrak{q}}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu})$ with a structure of a Dieudonné module by defining the action of the Frobenius on this module by

$$F(u_2 \otimes \xi_2) = u_2 \otimes \sigma(\xi_2), \quad u_2 \in \Lambda_{\bar{\mathfrak{q}}_i},\ \xi_2 \in W(\bar{\kappa}_{E_\nu}).$$

The direct sum of these Dieudonné modules defines a Dieudonné module structure on $\Lambda_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu})$. We consider the perfect alternating $W(\bar{\kappa}_{E_\nu})$-bilinear form

$$\psi_W : \Lambda_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu}) \times \Lambda_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu}) \to W(\bar{\kappa}_{E_\nu}), \tag{5.2}$$

cf. (3.13). One checks easily that this is a bilinear form of Dieudonné modules. By covariant Dieudonné theory, $(\Lambda_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_{E_\nu}), \psi_W)$ corresponds to a principally polarized p-divisible group $(\Lambda^{pd}_{\mathfrak{p}_i}, \lambda_\psi)$. We have the decomposition $\Lambda^{pd}_{\mathfrak{p}_i} = \Lambda^{et}_{\mathfrak{q}_i} \oplus \Lambda^{mult}_{\bar{\mathfrak{q}}_i}$, where the first factor is an étale p-divisible group and the second factor is multiplicative. The action of $O_{B_{\mathfrak{p}_i}}$ on $\Lambda_{\mathfrak{p}_i}$ defines an action of $O_{B_{\mathfrak{p}_i}}$ on $\Lambda^{pd}_{\mathfrak{p}_i}$. We see that $(\Lambda^{pd}_{\mathfrak{p}_i}, \bar{\lambda}_\psi)$ is an object of the category $\mathcal{P}_{\mathfrak{p}_i}(\bar{\kappa}_{E_\nu})$ and that each other object in this category is isomorphic to it.

Recall the group $G^\bullet_{\mathfrak{p}_i}$, cf. (4.1). An element $g \in G^\bullet_{\mathfrak{p}_i}$ induces a quasi-isogeny of the p-divisible $O_{B_{\mathfrak{p}_i}}$-module $\Lambda^{pd}_{\mathfrak{p}_i}$ which respects the polarization λ_ψ up to a factor in $F^\times_{\mathfrak{p}_i}$. In particular we conclude that

$$\mathbf{K}^\bullet_{\mathfrak{p}_i} \subset \operatorname{Aut}_{\mathcal{P}_{\mathfrak{p}_i}}(\Lambda^{pd}_{\mathfrak{p}_i}, \bar{\lambda}_\psi).$$

We consider schemes S over $\operatorname{Spf} O_{E_\nu}$ or equivalently O_{E_ν}-schemes S where p is locally nilpotent. We set $\bar{S} = S \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} \kappa_{E_\nu}$.

Definition 5.2. Let S be a scheme over $\operatorname{Spf} O_{\breve{E}_\nu}$ so that $\bar{S}$ is a scheme over $\bar{\kappa}_{E_\nu}$. We denote by $\Lambda^{et}_{\mathfrak{q}_i,\bar{S}} = \Lambda^{et}_{\mathfrak{q}_i} \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S}$ the base change. The unique lift to an étale p-divisible group over S is denoted by $\Lambda^{et}_{\mathfrak{q}_i,S}$. This is a constant étale p-divisible group.

A rigidification of an object $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_i}(S)$ modulo $\mathbf{K}^\bullet_{\mathfrak{p}_i}$ consists of a class $\bar{\eta}_{\mathfrak{q}_i}$ of isomorphisms of p-divisible $O_{B_{\mathfrak{q}_i}}$-modules

$$\eta_{\mathfrak{q}_i} : \Lambda^{et}_{\mathfrak{q}_i,S} \xrightarrow{\sim} Y_{\mathfrak{q}_i} \qquad \operatorname{mod} \mathbf{K}^\bullet_{\mathfrak{q}_i}$$

and a class $\bar{\xi}_{\mathfrak{p}_i}$ of maps $\xi_{\mathfrak{p}_i} : \bar{\lambda} \to O^\times_{F_{\mathfrak{p}_i}}/\mathbf{M}_{\mathfrak{p}_i}$ such that $\xi_{\mathfrak{p}_i}(\lambda u) = \xi_{\mathfrak{p}_i}(\lambda)u$, for $u \in O^\times_{F_{\mathfrak{p}_i}}$, $\lambda \in \bar{\lambda}$.

Equivalently we could replace $\bar{\eta}_{\mathfrak{q}_i}$ by a class of $O_{B_{\mathfrak{q}_i}}$-module homomorphisms of p-adic étale sheaves $\eta_{\mathfrak{q}_i} : \Lambda^{et}_{\mathfrak{q}_i,S} \to T_p(Y_{\mathfrak{q}_i})$ modulo $\mathbf{K}^\bullet_{\mathfrak{q}_i}$. We will use this definition only in the case where $Y_{\mathfrak{q}_i}$ is a constant étale p-divisible group. We denote the category of objects of $\mathcal{P}_{\mathfrak{p}_i}(S)$ with an rigidification by $\mathcal{P}_{\mathfrak{p}_i}(S)_{\mathbf{K}^\bullet_{\mathfrak{p}_i}}$.

We reformulate the definition of an rigidified object $(Y, \iota, \bar{\lambda}, \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i})$. To each $\lambda \in \bar{\lambda}$ we associate an $O_{B_{\bar{\mathfrak{q}}_i}}$-module isomorphism of p-divisible groups $\eta_{\bar{\mathfrak{q}}_i}$ by the following commutative diagram

$$\begin{array}{ccc} \Lambda^{et}_{\mathfrak{q}_i,S} & \xrightarrow{\eta_{\mathfrak{q}_i}} & Y_{\mathfrak{q}_i} \\ \Big\downarrow{\scriptstyle \xi_{\mathfrak{p}_i}(\lambda)\lambda_\psi} & & \Big\downarrow{\scriptstyle \lambda_{\bar{\mathfrak{q}}_i}} \\ (\Lambda^{mult}_{\bar{\mathfrak{q}}_i})^\wedge_S & \xleftarrow{\eta^\wedge_{\bar{\mathfrak{q}}_i}} & (Y_{\bar{\mathfrak{q}}_i})^\wedge. \end{array} \tag{5.3}$$

Then a rigidification modulo $\mathbf{K}^\bullet_{\mathfrak{p}_i}$ of $(Y, \iota, \bar{\lambda})$ is equivalently described by a class $\bar{\eta}_{\mathfrak{p}_i}$ of isomorphisms in the category $\mathcal{P}_{\mathfrak{p}_i}(S)$:

$$\eta_{\mathfrak{p}_i} : (\Lambda^{pd}_{\mathfrak{p}_i}, \bar{\lambda}_\psi)_S \xrightarrow{\sim} (Y, \iota, \bar{\lambda}) \qquad \operatorname{mod} \mathbf{K}^\bullet_{\mathfrak{p}_i}. \tag{5.4}$$

We see that, for S connected, $\eta_{\mathfrak{p}_i}$ is given by its value at a geometric point ω of S, where $(\eta_{\mathfrak{p}_i})_\omega$ is invariant modulo $\mathbf{K}^\bullet_{\mathfrak{p}_i}$. This makes sense because, by the diagram (5.3) above, everything comes down to a morphism between p-adic étale sheaves.

We indicate how this allows to extend a Hecke operator $g \in G^\bullet_{\mathfrak{p}_i} \subset G^\bullet(\mathbb{A}_f)$ from the generic fiber $\tilde{\mathcal{A}}^{\bullet t}_{E_\nu}$ to the whole functor $\tilde{\mathcal{A}}^{\bullet t}$. We consider a congruence subgroup $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ such that $kg\Lambda_{\mathfrak{p}_i} = g\Lambda_{\mathfrak{p}_i}$ for $k \in \mathbf{K}^\bullet_{\mathfrak{p}_i}$. We consider a point

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\bar{\xi}_{\mathfrak{p}_j})_j) \in \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(S).$$

Note here that, by $\mathbf{M}^\bullet_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$, the choice of $\xi_{\mathfrak{p}_0}$ is redundant, so we drop it from the notation. Let $Y_{\mathfrak{p}_i}$ be the $\mathfrak{p}_i$-part of the p-divisible group of A. It inherits the structure of an rigidified object $(Y_{\mathfrak{p}_i}, \iota, \bar{\lambda}, \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i})$ of $\mathcal{P}_{\mathfrak{p}_i}(S)$. We choose $\eta_{\mathfrak{q}_i} \in \bar{\eta}_{\mathfrak{q}_i}$ and write a commutative diagram

$$\begin{array}{ccc} (\Lambda^{pd}_{\mathfrak{p}_i,S}, \bar{\lambda}_\psi)_S & \xrightarrow{\eta_{\mathfrak{p}_i}} & (Y_{\mathfrak{p}_i}, \iota, \bar{\lambda}) \\ \uparrow g & & \uparrow a \\ (\Lambda^{pd}_{\mathfrak{p}_i,S}, \bar{\lambda}_\psi)_S & \xrightarrow{\eta'_{\mathfrak{p}_i}} & (Y'_{\mathfrak{p}_i}, \iota', \bar{\lambda}'). \end{array}$$

Here the maps a and g are quasi-isogenies and $\eta_{\mathfrak{p}_i}$ and $\eta'_{\mathfrak{p}_i}$ are understood as explained after (5.4). For $\lambda \in \bar{\lambda}$ we define $\lambda' = \mu^{-1}_{\mathfrak{p}_i}(g)a^*(\lambda)$. This is a principal polarization because $g^*(\lambda_\psi) = \mu_{\mathfrak{p}_i}(g)\lambda_\psi$. We define $\xi^{'}(a^*(\lambda)) = \mu_{\mathfrak{p}_i}(g)\xi(\lambda)$. This gives a rigidified object

$$(Y'_{\mathfrak{p}_i}, \iota', \bar{\lambda}', \bar{\eta}'_{\mathfrak{q}_i}, \bar{\xi}'_{\mathfrak{p}_i}). \tag{5.5}$$

We find a quasi-isogeny of abelian varieties $\alpha : (A', \iota') \to (A, \iota)$ which induces on the $\mathfrak{p}_j$-parts of the p-divisible groups an isomorphism for $j \neq i$ and the map a on the $\mathfrak{p}_i$-parts. Then (A', ι') inherits the data $(\bar{\eta}^p)'$, $\bar{\eta}'_{\mathfrak{q}_j}$, $\bar{\xi}'_{\mathfrak{p}_j}$ for $j \neq i$ by pull back via α. The data $\bar{\eta}'_{\mathfrak{q}_i}$, $\bar{\xi}'_{\mathfrak{p}_i}$ are inherited from (5.5). The $U_p(F)$-homogeneous polarization of A' consists of all $\lambda' \in \alpha^*(\bar{\lambda})$ which are principal in p. We define the image by the Hecke operator g as

$$(A', \iota', \bar{\lambda}', (\bar{\eta}^p)', (\bar{\eta}'_{\mathfrak{q}_j})_j, (\bar{\xi}'_{\mathfrak{p}_j})_j) \in \tilde{\mathcal{A}}^{\bullet t}_{g^{-1}\mathbf{K}^\bullet g}(S).$$

It follows from the discussion after the proof of Proposition 4.5 that this defines an extension of the Hecke operators over the generic fiber $\tilde{\mathcal{A}}^{\bullet t}_{E_\nu}$.

We fix an object $(\mathbb{X}, \iota_{\mathbb{X}}, \bar{\lambda}_{\mathbb{X}})$ of the category $\mathcal{P}_{\mathfrak{p}_i}(\bar{\kappa}_{E_\nu})$, e.g. $(\Lambda^{pd}_{\mathfrak{p}_i}, \bar{\lambda}_\psi)$. We choose $\lambda_{\mathbb{X}} \in \bar{\lambda}_{\mathbb{X}}$ and call $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ the framing object. We define $(\mathbb{X}, \iota_{\mathbb{X}}, \bar{\lambda}_{\mathbb{X}})_S$ in the same way as in Definition 5.2. The RZ-space is defined as follows:

Definition 5.3. We denote by $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}$ the functor on the category of schemes S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$, where a point of $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}(S)$ is given by the following data up to isomorphism:

(1) an object $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_i}(S)$,
(2) a rigidification ($\bar{\eta}_{\mathfrak{q}_i}$ modulo $\mathbf{K}^\bullet_{\mathfrak{p}_i}, \bar{\xi}_{\mathfrak{p}_i}$ modulo $\mathbf{M}^\bullet_{\mathfrak{p}_i}$) of $(Y, \iota, \bar{\lambda})$,
(3) a quasi-isogeny of p-divisible $O_{B_{\mathfrak{p}_i}}$-modules $\rho : (Y, \iota) \to (\mathbb{X}, \iota_{\mathbb{X}})_S$ which respects the polarizations on both sides up to a factor in $F^\times_{\mathfrak{p}_i}$.

It follows from (5.4) that we can represent a point of $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}(S)$ by a class $\bar{\rho}$ of quasi-isogenies

$$\rho : (\Lambda^{pd}_{\mathfrak{p}_i}, \bar{\lambda}_\psi)_S \to (\mathbb{X}, \iota_{\mathbb{X}}, \bar{\lambda}_{\mathbb{X}}) \text{ modulo } \mathbf{K}^\bullet_{\mathfrak{p}_i}\ , \tag{5.6}$$

which respects the polarizations of both sides up to a factor in $F^\times_{\mathfrak{p}_i}$.

We choose an isomorphism

$$(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \xrightarrow{\sim} (\Lambda^{pd}_{\mathfrak{p}_i}, \lambda_\psi) \tag{5.7}$$

which respects the polarizations. Then we see from (5.6) that a point of $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}(S)$ is (locally) represented by an element $g \in G^\bullet_{\mathfrak{p}_i}$. Therefore we obtain

Proposition 5.4. *The choice of an isomorphism (5.7) defines an isomorphism*

$$\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}} \xrightarrow{\sim} G^\bullet_{\mathfrak{p}_i}/\mathbf{K}^\bullet_{\mathfrak{p}_i},$$

where the right hand side denotes the constant sheaf.

Let $g \in G^\bullet_{\mathfrak{p}_i}$. If we represent a point of $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}(S)$ in the form (5.6), the assignement $\rho \mapsto \rho g$ defines a functor morphism

$$g : \mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}} \to \mathrm{RZ}_{\mathfrak{p}_i,g^{-1}\mathbf{K}^\bullet_{\mathfrak{p}_i}g}. \tag{5.8}$$

We call this a Hecke operator. Note that (5.8) is only defined if $\mathbf{K}^\bullet_{\mathfrak{p}_i}$ is sufficiently small with respect to g, i.e. if $g^{-1}\mathbf{K}^\bullet_{\mathfrak{p}_i}g\Lambda_{\mathfrak{p}_i} \subset \Lambda_{\mathfrak{p}_i}$. We could define $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}$ for an arbitrary open compact subgroup $\mathbf{K}^\bullet_{\mathfrak{p}_i} \subset G_{\mathfrak{p}_0}$ by making the Hecke operators part of the definition.

Now we discuss the case of the prime ideal $\mathfrak{p}_0$.

We first define the category $\mathcal{P}_{\mathfrak{p}_0}(S)$ for a scheme S over $\mathrm{Spf}\, O_{E_\nu}$. Because of the isomorphism $\varphi_0 : O_{F_{\mathfrak{p}_0}} \to O_{E_\nu}$ it makes sense to speak of a special formal $O_{B_{\mathfrak{q}_0}}$-module in the sense of Drinfeld over S, cf. [Dr] or [KRZ, §5.1]. We consider p-divisible $O_{B_{\mathfrak{p}_0}}$-modules (Y,ι). The decomposition $O_{B_{\mathfrak{p}_0}} = O_{B_{\mathfrak{q}_0}} \times O_{B_{\bar{\mathfrak{q}}_0}}$ induces a decomposition

$$Y = Y_{\mathfrak{q}_0} \times Y_{\bar{\mathfrak{q}}_0}.$$

We consider principal polarizations λ on Y which induce on $O_{B_{\mathfrak{p}_0}}$ the given involution $\star$. They are given by two isomorphisms $\lambda_{\mathfrak{q}_0} : Y_{\mathfrak{q}_0} \to (Y_{\bar{\mathfrak{q}}_0})^\wedge$ and $\lambda_{\bar{\mathfrak{q}}_0} : Y_{\bar{\mathfrak{q}}_0} \to (Y_{\mathfrak{q}_0})^\wedge$ such that $\lambda_{\bar{\mathfrak{q}}_0} = -\lambda_{\mathfrak{q}_0}^\wedge$. Moreover we have commutative diagrams like (5.1) for $i = 0$.

Definition 5.5. An object $(Y, \iota, \bar{\lambda})$ of the category $\mathcal{P}_{\mathfrak{p}_0}(S)$ consists of the following data:

(1) A p-divisible $O_{B_{\mathfrak{p}_0}}$-module (Y,ι) over S such that $Y_{\mathfrak{q}_0}$ is a special formal $O_{B_{\mathfrak{q}_0}}$-module.

(2) An $O_{F_{\mathfrak{p}_0}}^\times$-homogeneous polarization $\bar{\lambda}$ of Y, such that each $\lambda \in \bar{\lambda}$ is principal and such that the Rosati involution of λ induces on $O_{B_{\mathfrak{p}_0}}$ the involution $\star$.

We note that the functor $(Y, \iota, \bar{\lambda})$ from $\mathcal{P}_{\mathfrak{p}_0}(S)$ to the category of special formal $O_{B_{\mathfrak{q}_0}}$-modules over S it not faithful. But it would be an equivalence of categories is we replace $\mathcal{P}_{\mathfrak{p}_0}(S)$ by the category of triples (Y, ι, λ) with a given polarization λ as in the Definition above. We fix an object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\operatorname{Spec} \bar{\kappa}_{E_\nu}$. We call this the framing object. We keep the notation of Definition 5.2.

Definition 5.6. We denote by $\mathrm{RZ}_{\mathfrak{p}_0}$ the functor on the category of schemes S over $\operatorname{Spf} O_{\breve{E}_\nu}$ such that a point of $\mathrm{RZ}_{\mathfrak{p}_0}(S)$ consists of the following data up to isomorphism:

(1) an object $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_0}(S)$,

(2) a quasi-isogeny of p-divisible $O_{B_{\mathfrak{p}_0}}$-modules

$$\rho : (Y, \iota)_{\bar{S}} \to (\mathbb{X}, \iota_{\mathbb{X}}) \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S}$$

which respects the polarizations on both sides up to a factor in $F_{\mathfrak{p}_0}^\times$.

Let (G, ι) be a p-divisible $O_{F_{\mathfrak{p}_0}}$-module over an $O_{F_{\mathfrak{p}_0}}$-Algebra R. Let $F_{\mathfrak{p}_0}^t \subset F_{\mathfrak{p}_0}$ be the maximal subfield which is unramified over $\mathbb{Q}_p$ and let $\sigma \in \operatorname{Gal}(F_{\mathfrak{p}_0}^t/\mathbb{Q}_p)$ be the Frobenius. Let $\varepsilon : O_{F_{\mathfrak{p}_0}} \to R$ be the structure morphism. There is the natural decomposition

$$O_{F_{\mathfrak{p}_0}} \otimes_{\mathbb{Z}_p} R \cong \prod_{i=0}^{f-1} O_{F_{\mathfrak{p}_0}} \otimes_{O_{F_{\mathfrak{p}_0}^t}, \sigma^i \varepsilon} R,$$

where $f = [F_{\mathfrak{p}_0}^t : \mathbb{Q}_p]$ is the index of inertia of $F_{\mathfrak{p}_0}$. This decomposition induces a corresponding decomposition of the R-module given by the Lie algebra of G. We set

$$\operatorname{height}_{F_{\mathfrak{p}_0}} G := \operatorname{height}_{F_{\mathfrak{p}_0}}(\pi \mid G) = [F_{\mathfrak{p}_0} : \mathbb{Q}_p]^{-1} \operatorname{height} G,$$

where π is a prime element of $F_{\mathfrak{p}_0}$.

Let $\alpha : G_1 \to G_2$ be an isogeny which is an $O_{F_{\mathfrak{p}_0}}$-module homomorphism such that $\operatorname{rank}_R \operatorname{Lie}_i G_1 = \operatorname{rank}_R \operatorname{Lie}_i G_2$ for $i = 0, \dots, f-1$. Then $\operatorname{height} \alpha$ is divisible by $[F^t : \mathbb{Q}_p]$. In this case we define

$$\operatorname{height}_{F_{\mathfrak{p}_0}} \alpha = [F_{\mathfrak{p}_0}^t : \mathbb{Q}_p]^{-1} \operatorname{height} \alpha,$$

If $\alpha : X \to X'$ is a quasi-isogeny of special formal $O_{B_{\mathfrak{p}_0}}$-modules, the integer $\operatorname{height}_{F_{\mathfrak{p}_0}} \alpha$ is divisible by 2, because there is a quadratic unramified extension of $F_{\mathfrak{p}_0}$ which is contained in $B_{\mathfrak{p}_0}$.

We consider a point of $\mathrm{RZ}_{\mathfrak{p}_0}(S)$ as in Definition 5.6. The quasi-isogeny ρ is the direct sum of two quasi-isogenies

$$\rho_{\mathfrak{q}_0} : Y_{\mathfrak{q}_0, \bar{S}} \to \mathbb{X}_{\mathfrak{q}_0} \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S}, \quad \rho_{\bar{\mathfrak{q}}_0} : Y_{\bar{\mathfrak{q}}_0, \bar{S}} \to \mathbb{X}_{\bar{\mathfrak{q}}_0} \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S}. \tag{5.9}$$

For each pair of integers (a, b) such that $a + b \equiv 0 \mod 2$, we define an open and closed subfunctor of $\mathrm{RZ}_{\mathfrak{p}_0}$

$$\mathrm{RZ}_{\mathfrak{p}_0}(a,b)(S) = \{(Y, \iota, \bar{\lambda}, \rho) \mid \operatorname{height}_{F_{\mathfrak{p}_0}} \rho_{\mathfrak{q}_0} = 2a, \ \operatorname{height}_{F_{\mathfrak{p}_0}} \rho_{\bar{\mathfrak{q}}_0} = 2b\} \tag{5.10}$$

We remark that for a point $(Y, \iota, \bar{\lambda}, \rho) \in \mathrm{RZ}_{\mathfrak{p}_0}(S)$ the sum $(\mathrm{height}_{F_{\mathfrak{p}_0}} \rho_{\mathfrak{q}_0} + \mathrm{height}_{F_{\mathfrak{p}_0}} \rho_{\bar{\mathfrak{q}}_0})$ is always divisible by 4. Indeed, by definition there is an $f \in F_{\mathfrak{p}_0}^\times$ such that the following diagram of quasi-isogenies is commutative,

$$\begin{array}{ccc} Y_{\mathfrak{q}_0,\bar{S}} & \xrightarrow{\rho_{\mathfrak{q}_0}} & \mathbb{X}_{\mathfrak{q}_0} \times_{\mathrm{Spec}\, \bar{\kappa}_{E_\nu}} \bar{S} \\ {\scriptstyle \lambda f}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathbb{X}}} \\ (Y_{\bar{\mathfrak{q}}_0,\bar{S}})^\wedge & \xleftarrow{\rho_{\bar{\mathfrak{q}}_0}^\wedge} & (\mathbb{X}_{\bar{\mathfrak{q}}_0})^\wedge \times_{\mathrm{Spec}\, \bar{\kappa}_{E_\nu}} \bar{S}. \end{array} \tag{5.11}$$

Since λ and $\lambda_{\mathbb{X}}$ are isomorphisms, we obtain that

$$\mathrm{height}_{F_{\mathfrak{p}_0}} \rho_{\mathfrak{q}_0} + \mathrm{height}_{F_{\mathfrak{p}_0}} \rho_{\bar{\mathfrak{q}}_0} = \mathrm{height}_{F_{\mathfrak{p}_0}} (f \mid Y_{\mathfrak{q}_0,\bar{S}}).$$

The right hand side is divisible by 4 because $\mathrm{height}_{F_{\mathfrak{p}_0}} Y_{\mathfrak{q}_0,\bar{S}} = 4$.

We conclude that

$$\mathrm{RZ}_{\mathfrak{p}_0} = \coprod_{a+b\equiv 0 \bmod 2} \mathrm{RZ}_{\mathfrak{p}_0}(a,b). \tag{5.12}$$

We introduce Hecke operators acting on $\mathrm{RZ}_{\mathfrak{p}_0}$. Let $(Y_{\mathfrak{q}_0}, \iota_{\mathfrak{q}_0})$ be a p-divisible $O_{B_{\mathfrak{q}_0}}$-module. For $u_1 \in B_{\mathfrak{q}_0}^\times$ we define

$$\iota_{\mathfrak{q}_0}^{u_1} : O_{B_{\mathfrak{q}_0}} \to \mathrm{End}\, Y_{\mathfrak{q}_0}, \quad \iota_{\mathfrak{q}_0}^{u_1}(b) = \iota_{\mathfrak{q}_0}(u_1^{-1} b u_1).$$

We set $Y_{\mathfrak{q}_0}^{u_1} = Y_{\mathfrak{q}_0}$ and write $(Y_{\mathfrak{q}_0}^{u_1}, \iota_{\mathfrak{q}_0}^{u_1})$. We use the same definition for a p-divisible $O_{B_{\bar{\mathfrak{q}}_0}}$-module.

Let $u = (u_1, u_2) \in B_{\mathfrak{p}_0}^\times = B_{\mathfrak{q}_0}^\times \times B_{\bar{\mathfrak{q}}_0}^\times$. For a p-divisible $O_{B_{\mathfrak{p}_0}}$-module (Y, ι) we set $\iota^u(b) = \iota(u^{-1} b u)$, $b \in O_{B_{\mathfrak{p}_0}}$.

Lemma 5.7. *Let $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_0}(S)$. Let $u = (u_1, u_2) \in B_{\mathfrak{p}_0}^\times$ such that $u_2^\star u_1 \in F_{\mathfrak{p}_0}^\times$. Then for each $\lambda \in \bar{\lambda}$ the Rosati involution of λ on $\mathrm{End}\, Y^u$ induces via*

$$\iota^u : O_{B_{\mathfrak{p}_0}} \to \mathrm{End}\, Y^u$$

the involution $\star$ on $O_{B_{\mathfrak{p}_0}}$. In particular $(Y^u, \iota^u, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_0}(S)$.

Proof. We must verify the commutativity of the following diagram,

$$\begin{array}{ccc} Y_{\mathfrak{q}_0}^{u_1} & \xrightarrow{\iota^{u_1}(b_2^\star)} & Y_{\bar{\mathfrak{q}}_0}^{u_1} \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ (Y_{\bar{\mathfrak{q}}_0}^{u_2})^\wedge & \xrightarrow[\iota^{u_2}(b_2)^\wedge]{} & (Y_{\bar{\mathfrak{q}}_0}^{u_2})^\wedge. \end{array}$$

Indeed,

$$\lambda^{-1} \iota^{u_2}(b_2)^\wedge \lambda = \lambda^{-1} \iota(u_2^{-1} b_2 u_2)^\wedge \lambda = \iota(u_2^\star b_2^\star (u_2^\star)^{-1}) = \iota(u_1^{-1} b_2^\star u_1) = \iota^{u_1}(b_2^\star).$$

The second equation holds because the Rosati involution of λ induces via ι the involution $\star$ on $O_{B_{\mathfrak{p}_0}}$. The third equation holds because $u_2^\star u_1 \in F_{\mathfrak{p}_0}$ implies that

$$u_1^{-1} b_2^\star u_1 = u_2^\star u_1 u_1^{-1} b_2^\star u_1 (u_2^\star u_1)^{-1} = u_2^\star b_2^\star (u_2^\star)^{-1}.$$

□

Let $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_0}(S)$. Then

$$\iota(u) : (Y^u, \iota^u) \to (Y, \iota) \tag{5.13}$$

is a quasi-isogeny of p-divisible $O_{B_{\mathfrak{p}_0}}$-modules.

Lemma 5.8. *Let $\lambda \in \bar{\lambda}$. We assume that for $u = (u_1, u_2)$ we have $u_2^\star u_1 \in F_{\mathfrak{p}_0}^\times$. Then the quasi-isogeny (5.13) respects the polarization λ on both sides of (5.13) up to a factor in $F_{\mathfrak{p}_0}^\times$.*

Proof. We must show that there exists $f \in O_{F_{\mathfrak{p}_0}}^\times$ such that the following diagram in commutative.

$$\begin{array}{ccc} Y_{\mathfrak{q}_0} & \xrightarrow{\lambda\iota(f)} & (Y_{\bar{\mathfrak{q}}_0})^\wedge \\ {\scriptstyle \iota(u_1)}\uparrow & & \downarrow{\scriptstyle \iota(u_2)^\wedge} \\ Y_{\mathfrak{q}_0}^{u_1} & \xrightarrow[\lambda]{} & (Y_{\bar{\mathfrak{q}}_0}^{u_2})^\wedge. \end{array}$$

The commutativity is equivalent to the first of the following equations,

$$\lambda = \iota(u_2)^\wedge \lambda\iota(f)\iota(u_1) = \lambda\lambda^{-1}\iota(u_2)^\wedge\lambda\iota(f)\iota(u_1) = \lambda\iota(u_2^\star)\iota(f)\iota(u_1) = \lambda\iota(fu_2^\star u_1).$$

Therefore we obtain a commutative diagram if we choose $fu_2^\star u_1 = 1$. □

We define the group

$$\mathcal{H}_{\mathfrak{p}_0} = \{u \in B_{\mathfrak{p}_0} \,| u^\star u \in F_{\mathfrak{p}_0}^\times\,\} \subset B_{\mathfrak{p}_0}^\times$$

Note that for $u = (u_1, u_2)$ as above, the conditions $u^\star u \in F_{\mathfrak{p}_0}^\times$, resp. $u_2^\star u_1 \in F_{\mathfrak{p}_0}^\times$, resp. $u_1 u_2^\star \in F_{\mathfrak{p}_0}^\times$ are equivalent.

The group $\mathcal{H}_{\mathfrak{p}_0}$ acts from the left on the functor $\mathrm{RZ}_{\mathfrak{p}_0}$. Let $(Y, \iota, \bar{\lambda}, \rho) \in \mathrm{RZ}_{\mathfrak{p}_0}(S)$. For $u \in \mathcal{H}_{\mathfrak{p}_0}$ we define the Hecke correspondence

$$\mathfrak{h}(u) : \mathrm{RZ}_{\mathfrak{p}_0} \to \mathrm{RZ}_{\mathfrak{p}_0}, \quad \mathfrak{h}(u)((Y, \iota, \bar{\lambda}, \rho)) = (Y^u, \iota^u, \bar{\lambda}, \rho\iota(u)). \tag{5.14}$$

This definition makes sense because of Lemmas 5.7 and 5.8. We note that for $v \in \mathcal{H}_{\mathfrak{p}_0}$ we obtain $(Y^u)^v = Y^{vu}$, $\iota(u)\iota^u(v) = \iota(vu)$. The map $\mathfrak{h}(u)$ is the identity on $\mathrm{RZ}_{\mathfrak{p}_0}$ if $u \in O_{B_{\mathfrak{p}_0}}^\times$ because $\iota(u) : (Y^u, \iota^u, \bar{\lambda}) \to (Y, \iota, \bar{\lambda})$ is then an isomorphism.

If $u = (u_1, u_2) \in \mathcal{H}_{\mathfrak{p}_0}$, then the Hecke operator induces maps

$$\mathfrak{h}((u_1, u_2)) : \mathrm{RZ}_{\mathfrak{p}_0}(a, b) \to \mathrm{RZ}_{\mathfrak{p}_0}(a + \mathrm{ord}_{B_{\mathfrak{q}_0}} u_1,\ b + \mathrm{ord}_{B_{\bar{\mathfrak{q}}_0}} u_2). \tag{5.15}$$

We conclude that $\mathcal{H}_{\mathfrak{p}_0}$ acts transitively on the set of subspaces $\{\mathrm{RZ}_{\mathfrak{p}_0}(a, b)\}$ in the decomposition (5.12). Indeed, if $c + d$ is an even sum of integers we can find $u_1 \in B_{\mathfrak{q}_0}^\times$ and $f \in F_{\mathfrak{p}_0}^\times$ such that $\mathrm{ord}_{B_{\mathfrak{q}_0}} u_1 = c$ and $\mathrm{ord}_{B_{\mathfrak{q}_0}} f = c + d$. We define u_2 by the equation $u_2^\star u_1 = f$. Then the right hand side of (5.15) becomes $\mathrm{RZ}_{\mathfrak{p}_0}(a + c, b + d)$.

If we use the right action of $B_{\mathfrak{p}_0}$ on $V_{\mathfrak{p}_0}$, we can write

$$G_{\mathfrak{p}_0}^\bullet = \{g \in B_{\mathfrak{p}_0}^{\mathrm{opp}} \mid gg' \in F_{\mathfrak{p}_0}^\times\,\}.$$

The anti-isomorphism

$$\begin{array}{ccc} B_{\mathfrak{q}_0} \times B_{\bar{\mathfrak{q}}_0} & \to & B_{\mathfrak{q}_0}^{\mathrm{opp}} \times B_{\bar{\mathfrak{q}}_0}^{\mathrm{opp}} \\ (b_1, b_2) & \longmapsto & (b_1, (b_2^\star)') \end{array} \tag{5.16}$$

defines an anti-isomorphism $\mathcal{H}_{\mathfrak{p}_0} \to G_{\mathfrak{p}_0}^\bullet$. Therefore $G_{\mathfrak{p}_0}^\bullet$ acts from the right on $\mathrm{RZ}_{\mathfrak{p}_0}$. We write this action

$$(Y, \iota, \bar{\lambda}, \rho) \longmapsto (Y, \iota, \bar{\lambda}, \rho)_{|g}, \quad g \in G_{\mathfrak{p}_0}^\bullet. \tag{5.17}$$

From the properties of the action of $\mathcal{H}_{\mathfrak{p}_0}$ above, we conclude that $\mathbf{K}_{\mathfrak{p}_0}^\bullet \subset G_{\mathfrak{p}_0}^\bullet$ acts trivially on $\mathrm{RZ}_{\mathfrak{p}_0}$. Therefore the group $G_{\mathfrak{p}_0}^\bullet/\mathbf{K}_{\mathfrak{p}_0}^\bullet$ acts on $\mathrm{RZ}_{\mathfrak{p}_0}$. This group isomorphic to $\mathbb{Z}^2$ and acts simply transitively on the set of subspaces $\{\mathrm{RZ}_{\mathfrak{p}_0}(a, b)\}$ in the decomposition (5.12).

We denote by $\hat{\Omega}_{F_{\mathfrak{p}_0}}^2$ the Drinfeld upper half plane over $\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}$. This is a regular formal scheme of dimension 2, comp. [Dr], [RZ], [KRZ, §5.1]. The formal scheme $\hat{\Omega}_{F_{\mathfrak{p}_0}}^2 \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}$ represents the Drinfeld functor $\mathcal{M}_{\mathrm{Dr}}(0)$ whose points with values in a scheme S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$ are given by pairs $(Y_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0})$, where $Y_{\mathfrak{q}_0}$ is a special formal $O_{B_{\mathfrak{q}_0}}$-module over S and ρ a quasi-isogeny of special formal $O_{B_{\mathfrak{q}_0}}$-modules of height zero,

$$\rho_{\mathfrak{q}_0} : Y_{\mathfrak{q}_0, \bar{S}} \to \mathbb{X}_{\mathfrak{q}_0} \times_{\mathrm{Spec}\, \bar{\kappa}_{E_\nu}} \bar{S}.$$

We denote by $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ the functor whose points are given by pairs $(Y_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0})$ where $\rho_{\mathfrak{q}_0}$ is allowed to have arbitrary height. Note that $\mathrm{height}_{O_{F_{\mathfrak{p}_0}}} \rho_{\mathfrak{q}_0} = 2a$ is automatically even. We obtain the decomposition

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} = \coprod_{a \in \mathbb{Z}} \mathcal{M}_{\mathrm{Dr}}(a),$$

cf. [KRZ] §5.1. Let $\mathbf{J}_{\mathfrak{q}_0}$ be the group of all quasi-isogenies $\delta \in \mathrm{End}^o_{B_{\mathfrak{q}_0}} \mathbb{X}_{\mathfrak{q}_0}$. Then $\mathbf{J}_{\mathfrak{q}_0} \cong \mathrm{GL}_2(F_{\mathfrak{p}_0})$, cf. [Dr]. This group acts from the left on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ by changing $\rho_{\mathfrak{q}_0}$ to $\delta\rho_{\mathfrak{q}_0}$.

Let $\Pi \in O_{B_{\mathfrak{q}_0}}$ be a prime element. The Hecke operator $\mathfrak{h}(\Pi)$ in the sense of [KRZ] (5.1.16) acts on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ as

$$\mathfrak{h}(\Pi) : \mathcal{M}_{\mathrm{Dr}}(a) \xrightarrow{\sim} \mathcal{M}_{\mathrm{Dr}}(a+1), \quad (Y_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0}) \longmapsto (Y^{\Pi}_{\mathfrak{q}_0}, \iota_{\mathbb{X}_{\mathfrak{q}_0}}(\Pi) \circ \rho^{\Pi}_{\mathfrak{q}_0}). \tag{5.18}$$

We define an action of $\mathbf{J}_{\mathfrak{q}_0}$ on $\mathcal{M}_{\mathrm{Dr}}(0)$. For $\delta \in \mathbf{J}_{\mathfrak{q}_0}$, we set

$$\mathrm{pr}(\delta)(Y_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0}) = \mathfrak{h}(\Pi)^{-\operatorname{ord}_{\mathfrak{p}_0} \det \delta} \circ \delta(Y_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0}). \tag{5.19}$$

Because the Hecke operators commute with the action of $\mathbf{J}_{\mathfrak{q}_0}$ this is an action of the group $\mathbf{J}_{\mathfrak{q}_0}$. One can easily see that this action of $\mathcal{M}_{\mathrm{Dr}}(0)$ factors through $\mathbf{J}_{\mathfrak{q}_0} = \mathrm{GL}_2(F_{\mathfrak{p}_0}) \to \mathrm{PGL}_2(F_{\mathfrak{p}_0})$. Using (5.18) as identifications we obtain an isomorphism

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} \cong \mathcal{M}_{\mathrm{Dr}}(0) \times \mathbb{Z} \cong (\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times \mathbb{Z}. \tag{5.20}$$

Proposition 5.9. *The isomorphism (5.20) does not depend on the choice of the prime element Π. The action of $\mathbf{J}_{\mathfrak{q}_0}$ on the left hand side induces on the right hand side*

$$\delta(\omega, m) = (\mathrm{pr}(\delta)\omega,\ \operatorname{ord}_{\mathfrak{p}_0} \det \delta + m), \quad \delta \in \mathbf{J}_{\mathfrak{q}_0},\ \omega \in \mathcal{M}_{\mathrm{Dr}}(0),\ m \in \mathbb{Z},$$

cf. (5.19).

In the next section we will write $\delta\omega := \mathrm{pr}(\delta)\omega$.

Proof. This is clear. □

Lemma 5.10. *There is a canonical isomorphism of functors over* $\mathrm{Spf}\, O_{\breve{E}_\nu}$

$$\mathrm{RZ}_{\mathfrak{p}_0}(0,0) \xrightarrow{\sim} \hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}.$$

Proof. We begin with a general remark which is useful later on. We consider the isomorphism of rings

$$\begin{array}{rcl} B_{\mathfrak{p}_0} = B_{\mathfrak{q}_0} \times B_{\bar{\mathfrak{q}}_0} & \xrightarrow{\sim} & B_{\mathfrak{q}_0} \times B^{\mathrm{opp}}_{\mathfrak{q}_0}. \\ (b, c) & \longmapsto & (b, c^\star) \end{array} \tag{5.21}$$

The involution $\star$ on $B_{\mathfrak{p}_0}$ induces on the right hand side the involution $(b_1, b_2) \mapsto (b_2, b_1)$. The maximal orders defined on each side (cf. (3.12)) are mapped isomorphically to each other. Consider a point $(Y, \iota, \bar{\lambda}) \in \mathcal{P}_{\mathfrak{p}_0}(S)$. We choose $\lambda \in \bar{\lambda}$. It defines an isomorphism $\lambda : Y_{\mathfrak{q}_0} \to (Y_{\bar{\mathfrak{q}}_0})^\wedge$. It becomes an isomorphism of $O_{B_{\mathfrak{q}_0}}$-modules if we consider $(Y_{\bar{\mathfrak{q}}_0})^\wedge$ as an $O_{B_{\mathfrak{q}_0}}$-module via the isomorphism (5.21), cf. (5.1). By the choice of λ we may identify $Y_{\bar{\mathfrak{q}}_0}$ with the p-divisible $O^{\mathrm{opp}}_{B_{\mathfrak{q}_0}}$-module $(Y_{\mathfrak{q}_0})^\wedge$. The $O^\times_{F_{\mathfrak{p}_0}}$-homogeneous polarization on $Y \cong Y_{\mathfrak{q}_0} \oplus (Y_{\mathfrak{q}_0})^\wedge$ becomes the polarization induced by $\mathrm{id} : Y_{\mathfrak{q}_0} \to ((Y_{\mathfrak{q}_0})^\wedge)^\wedge$. We call this the canonical polarization. Let us denote by $B^o_{\mathfrak{p}_0}$ the right hand side of (5.21) with its involution. Then we may describe an object of $\mathcal{P}_{\mathfrak{p}_0}(S)$ as a triple $(Y = Y_{\mathfrak{q}_0} \oplus Y_{\mathfrak{q}_0}, \iota, \bar{\lambda})$, where $Y_{\mathfrak{q}_0}$ is a special formal $O_{B_{\mathfrak{q}_0}}$-module, where $\iota : O_{B^o_{\mathfrak{q}_0}} \to \mathrm{End}\, Y$ is the natural action and where $\bar{\lambda}$ is the $O^\times_{F_{\mathfrak{p}_0}}$-homogeneous class of the canonical polarization.

Now we remark that for a point $(Y, \iota, \bar{\lambda}, \rho)$ of $\mathrm{RZ}_{\mathfrak{p}_0}(0,0)(S)$ there is a unique $\lambda \in \bar{\lambda}$ which makes the diagram (5.11) commute with $f = 1$. Indeed, one notes that in this diagram it follows that $f \in O^\times_{F_{\mathfrak{p}_0}}$ if $\rho_{\mathfrak{q}_0}$ and $\rho_{\bar{\mathfrak{q}}_0}$ are quasi-isogenies of height zero.

Therefore the point is uniquely determined by $(Y_{\mathfrak{q}_0}, \iota_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0})$. This proves the Lemma. □

We introduce the group $\mathbf{I}_{\mathfrak{p}_0}$ of all quasi-isogenies $\gamma\colon (\mathbb{X}, \iota_{\mathbb{X}}) \to (\mathbb{X}, \iota_{\mathbb{X}})$ which respect the polarization $\lambda_{\mathbb{X}}$ up to a factor in $F^\times_{\mathfrak{p}_0}$. If we denote by $\gamma \mapsto \gamma'$ the involution on $\mathrm{End}^o\, \mathbb{X}$ induced by $\lambda_{\mathbb{X}}$, we obtain

$$\mathbf{I}_{\mathfrak{p}_0} = \{\gamma \in \mathrm{End}^o_{B_{\mathfrak{p}_0}} \mathbb{X} \mid \gamma'\gamma \in F^\times_{\mathfrak{p}_0}\}. \tag{5.22}$$

The group $\mathbf{I}_{\mathfrak{p}_0}$ acts from the left on the functor $\mathrm{RZ}_{\mathfrak{p}_0}$,

$$(Y, \iota, \bar{\lambda}, \rho) \longmapsto (Y, \iota, \bar{\lambda}, \gamma\rho), \qquad \gamma \in \mathbf{I}_{\mathfrak{p}_0}.$$

This action commutes with the action of $G^\bullet_{\mathfrak{p}_0}$.

We make this action more explicit by using the description of the category $\mathcal{P}_{\mathfrak{p}_0}(S)$ given in the proof of Lemma 5.10. Recall the group $\mathbf{J}_{\mathfrak{q}_0}$ of all quasi-isogenies $\delta \in \mathrm{End}^o_{B_{\mathfrak{q}_0}} \mathbb{X}_{\mathfrak{q}_0}$. Then an element $\delta_2 \in \mathbf{J}^{\mathrm{opp}}_{\mathfrak{q}_0}$ acts on $\mathbb{X}^\wedge_{\mathfrak{q}_0}$ by $\iota_{\mathbb{X}_{\mathfrak{q}_0}}(\delta_2)^\wedge$. We conclude that

$$\mathbf{I}_{\mathfrak{p}_0} = \{(\delta_1, \delta_2) \in \mathbf{J}_{\mathfrak{q}_0} \times \mathbf{J}^{\mathrm{opp}}_{\mathfrak{q}_0} \mid \delta_1\delta_2 \in F^\times_{\mathfrak{p}_0}\}.$$

If we replace in (5.15) (u_1, u_2) by any other $(v_1, v_2) \in \mathcal{H}_{\mathfrak{p}_0}$ such that $\mathrm{ord}_{B_{\mathfrak{q}_0}} u_i = \mathrm{ord}_{B_{\mathfrak{q}_0}} v_i$ for $i = 1, 2$, we obtain the same morphism. Using this morphism as an identification of both sides of (5.15), we obtain an isomorphism

$$\mathrm{RZ}_{\mathfrak{p}_0} = \mathrm{RZ}_{\mathfrak{p}_0}(0, 0) \times \Lambda, \tag{5.23}$$

where $\Lambda = \{(a, b) \in \mathbb{Z}^2 \,| a + b \equiv 0 \mod 2\ \}$. Combining this with Lemma 5.10, we obtain

Proposition 5.11. *There is an isomorphism of functors*

$$(\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times G^\bullet_{\mathfrak{p}_0}/\mathbf{K}^\bullet_{\mathfrak{p}_0} \xrightarrow{\sim} \mathrm{RZ}_{\mathfrak{p}_0}$$

which is equivariant with respect to the actions of $G^\bullet_{\mathfrak{p}_0}/\mathbf{K}^\bullet_{\mathfrak{p}_0}$ on both sides.

The right hand side of (5.23) can be written as $\mathcal{M}_{\mathrm{Dr}}(0) \times \Lambda$. An element $(\delta_1, \delta_2) \in \mathbf{I}_{\mathfrak{p}_0}$ then sends a point $(\omega, (m_1, m_2))$ to $(\mathrm{pr}(\delta_1)\omega,\ (m_1 + \mathrm{ord}_{\mathfrak{p}_0} \det \delta_1,\ m_2 + \mathrm{ord}_{\mathfrak{p}_0} \det \delta_2))$.

As noted above we will write in the next section $\delta\omega := \mathrm{pr}(\delta)\omega$.

Proof. Only the last assertion needs a proof. We consider a point $(x, (m_1, m_2))$ from the right hand side of (5.23), where x corresponds to $\omega = (Y_{\mathfrak{q}_0}\iota_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0}) \in \mathcal{M}_{\mathrm{Dr}}(0)$. The image of a point $(x, (m_1, m_2))$ under the action of (δ_1, δ_2) is computed by looking at $Y_{\mathfrak{q}_0}$ only. By the description of $\mathrm{pr}(\delta_1)$, this shows the result. □

6. The p-adic uniformization of Shimura curves

In this section, we prove the p-adic uniformization of the integral model $\tilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$, cf. Definition 4.10. Here $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p\mathbf{K}^{\bullet,p}$, with $\mathbf{K}^\bullet_p \subset G^\bullet(\mathbb{Q}_p)$ defined by (4.6), where $\mathbf{M}^\bullet_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$. From this, Cherednik uniformization, i.e., Theorem 1.1 will follow. We stress that the Shimura varieties $\mathrm{Sh}(G, h)$ and $\mathrm{Sh}(G^\bullet, h)$ from the previous sections will not reappear.

We consider the functor

$$\mathrm{RZ}_{p,\mathbf{K}^\bullet_p} = \mathrm{RZ}_{\mathfrak{p}_0} \times \prod_{i=1}^{s} \mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}},$$

cf. Definitions 5.6 and 5.3. Each of these factors is defined by a choice of a framing object which we denote by $(\mathbb{X}_i, \iota_{\mathbb{X}_i}, \lambda_{\mathbb{X}_i})$, for $i = 0, \dots, s$. We choose the framing objects as follows. We fix a point

$$(A_o, \iota_o, \bar{\lambda}_o, \bar{\eta}^p_o, (\bar{\eta}_{\mathfrak{q}_j,o})_j, (\xi_{\mathfrak{p}_j,o})_j) \in \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(\bar{\kappa}_{E_\nu}). \tag{6.1}$$

The last two data of the point are for $j = 1, \dots, s$. Indeed, by $\mathbf{M}^\bullet_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$, the choice of $\xi_{\mathfrak{p}_0}$ is redundant. We choose an element $\eta^p_o \in \bar{\eta}^p_o$. We denote by $\mathbb{X}$ the p-divisible group of A_o. We set $\mathbb{X}_i = \mathbb{X}_{\mathfrak{p}_i}$, with its action $\iota_{\mathbb{X}_i}$ from ι_o and a polarization $\lambda_{\mathbb{X}_i}$ from some element of $\bar{\lambda}_o$.

We denote by $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet/\,\mathrm{Spf}\, O_{E_\nu}}$, resp. $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet/\,\mathrm{Spf}\, O_{\breve{E}_\nu}}$ the restriction of the functor $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ to the category of schemes over $\mathrm{Spf}\, O_{E_\nu}$, resp. $\mathrm{Spf}\, O_{\breve{E}_\nu}$. We define the uniformization morphism of functors on the category of schemes S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$ (the definition depends on the choice of the tuple (6.1)),

$$\tilde{\Theta}^\bullet : \mathrm{RZ}_{p,\mathbf{K}^\bullet_p} \times G^\bullet(\mathbb{A}^p_f)/(\mathbf{K}^\bullet)^p \to \tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet/\,\mathrm{Spf}\, O_{\breve{E}_\nu}}. \tag{6.2}$$

For the definition we recall that a point with values in S of the functor on the left hand side consists the following data

(1) a point $(Y_0, \iota_0, \bar{\lambda}_0, \rho_0)$ of $\mathrm{RZ}_{\mathfrak{p}_0}(S)$, cf. Definition 5.6,

(2) a point $(Y_j, \iota_j, \bar{\lambda}_j, \bar{\eta}_{\mathfrak{q}_j}, \bar{\xi}_{\mathfrak{p}_j}, \rho_j)$ of $\mathrm{RZ}_{\mathfrak{p}_j,\mathbf{K}^\bullet_{\mathfrak{p}_j}}(S)$ for $j = 1, \dots s$, cf. Definition 5.3,

(3) an element $g \in G^\bullet(\mathbb{A}^p_f)$.

Here

$$\rho_i : (Y_i, \iota_i)_{\bar{S}} \to (\mathbb{X}_i, \iota_{\mathbb{X}_i}) \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S} \tag{6.3}$$

is a quasi-isogeny for $i = 0, \dots, s$ which respects the polarizations on both sides up to a factor in $F_{\mathfrak{p}_i}^\times$. We define as follows an abelian scheme $(\bar{A}, \bar{\iota}_A)$ over $\bar{S}$ and an isogeny

$$(\bar{A}, \iota_{\bar{A}}) \to (A_o, \iota_o) \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S}. \tag{6.4}$$

Let us denote by $\bar{Y}$ the p-divisible group of $\bar{A}$. Then $(\bar{Y}_{\mathfrak{p}_i}, \iota_i)$ is identified with $(Y_{i,\bar{S}}, \iota_{i,\bar{S}})$ and the quasi-isogeny (6.4) induces on the p-divisible groups the quasi-isogenies (6.3). We choose $\lambda_o \in \bar{\lambda}_o$ and consider the inverse image θ of λ_o on $\bar{A}$ by (6.4). Let $\bar{\theta} = F^\times \theta$ be the F-homogeneous polarization it generates. By the definition of the RZ-spaces, the polarization induced by θ on the p-divisible group $Y_{i,\bar{S}}$ differs from $\lambda_{i,\bar{S}}$ by a factor in $F_{\mathfrak{p}_i}^\times$. We then define $\bar{\lambda}_{\bar{A}}$ to be the $U_p(F)$-homogeneous polarization on $\bar{A}$ which consists of all elements of $\bar{\theta}$ which on the p-divisible groups $\bar{Y}_i$ differ from $\lambda_{i,\bar{S}}$ by a factor in $O_{F_{\mathfrak{p}_i}}^\times$.

Since a lifting of the p-divisible group of $\bar{A}$ with these extra structures is given by the data (1) and (2) above, we obtain by the Serre-Tate theorem a lifting A of $\bar{A}$ over S with extra structures ι_A and $\bar{\lambda}_A$. We obtain a point

$$(A, \iota_A, \bar{\lambda}_A, \bar{\eta}_A^p, (\bar{\eta}_{\mathfrak{q}_j,A})_j, (\bar{\xi}_{\mathfrak{p}_j})_j),$$

where the last three items are defined as follows. We take the inverse image of η_o^p by (6.4) to obtain $\eta_{\bar{A}}^p$. Since we have étale sheaves, this gives η_A^p and then its class $\bar{\eta}_A^p$. The last two items are deduced directly from the item (2) above. Indeed $T_{\mathfrak{q}_j}(A) = T_p(Y_{j,\mathfrak{q}_j})$ and a rigidification in the sense of Definition 5.2 is equivalent to

$$\eta_{\mathfrak{q}_j} : \Lambda_{\mathfrak{q}_j} \xrightarrow{\sim} T_p(Y_{j,\mathfrak{q}_j}) \mod \mathbf{K}_{\mathfrak{q}_j}^\bullet,$$

and a function

$$\xi_{\mathfrak{p}_j} : \bar{\lambda}_A \to O_{F_{\mathfrak{p}_j}}^\times \mod \mathbf{M}_{\mathfrak{p}_j}^\bullet.$$

This last function is induced by the injection $\bar{\lambda}_A \to \bar{\lambda}_j$ from (2).

We define the image under $\tilde{\Theta}^\bullet$ in (6.2) of the point given by the data (1), (2), (3) to be

$$(A, \iota_A, \bar{\lambda}_A, \eta_A^p g, (\bar{\eta}_{\mathfrak{q}_j,A})_j, (\bar{\xi}_{\mathfrak{p}_j})_j). \tag{6.5}$$

For this we have used our choice of η_o^p.

Proposition 6.1. *Recall from (5.17) the Hecke operator action of $G_{\mathfrak{p}_0}^\bullet$ on $\mathrm{RZ}_{\mathfrak{p}_0}$ and from (5.8) the Hecke operator action of $G_{\mathfrak{p}_i}^\bullet$ on $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}_{\mathfrak{p}_i}^\bullet}$. Together we obtain an action of $G^\bullet(\mathbb{A}_f)$ by Hecke operators on the left hand side of (6.2).*

There is an extension of the Hecke operators $G^\bullet(\mathbb{A}_f)$ from the tower $\mathcal{A}_{\mathbf{K}^\bullet}^{\bullet t}$ to the tower $\tilde{\mathcal{A}}_{\mathbf{K}^\bullet}^{\bullet t}$ such that the uniformization morphism $\tilde{\Theta}^\bullet$ is compatible with the actions of Hecke operators on both sides.

Proof. This is trivial for the action of $G^\bullet(\mathbb{A}_f^p)$. The proof for elements in $G^\bullet(\mathbb{Q}_p)$ is based on the description of the Hecke operators after the proof of Proposition 4.5, comp. Remark 4.7. For $j = 1, \dots, s$, the local component at $\mathfrak{p}_j$ of a Hecke correspondence is described after (5.4). The local component at $\mathfrak{p}_0$ of a Hecke correspondence is described by (5.14). We write here the argument only for the action of $g \in G_{\mathfrak{p}_0}^\bullet \subset G(\mathbb{A}_f)$. We consider a point on the left hand side of (6.2) defined by the data (1), (2), (3). We may assume that the element of (3) is 1. We take the image by $\tilde{\Theta}^\bullet$,

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p, (\bar{\eta}_{\mathfrak{q}_j})_j, (\bar{\xi}_{\mathfrak{p}_j})_j) \in \tilde{\mathcal{A}}_{\mathbf{K}^\bullet}^{\bullet t}(S), \tag{6.6}$$

cf. (6.5). Let $(Y_0, \iota_0, \bar{\lambda}_0, \rho_0)$ be the datum (1). Then $Y_{\mathfrak{p}_0}$ is the $\mathfrak{p}_0$-part of the p-divisible group of A with the induced action ι_0 and polarization $\bar{\lambda}_0$. Let $u = (u_1, u_2) \in B_{\mathfrak{q}_0} \times B_{\bar{\mathfrak{q}}_0} = B_{\mathfrak{p}_0}$ be the element which corresponds to g by the anti-isomorphism (5.16). The Hecke operator g on the left hand side of (6.2) is given by $(Y_0^u, \iota_0^u, \bar{\lambda}_0^u, \rho_0 \circ \iota_0(u))$. Note that for the underlying polarized p-divisble groups we have $(Y_0^u, \lambda_0^u) = (Y_0, \lambda_0)$. We consider the quasi-isogeny

$$\iota_0(u) : (Y_0^u, \iota_0^u, \bar{\lambda}_0^u) \to (Y_0, \iota_0, \bar{\lambda}_0). \tag{6.7}$$

We note that $\iota_0(u)^*(\lambda) = \lambda^u \circ \iota_0(u_2^\star u_1) = \lambda^u \circ \mu(g)$. Therefore, applying g on the left hand side of (6.2) leads on the right hand side to the following point. There is a quasi-isogeny of abelian varieties

$$\alpha : (A', \iota') \to (A, \iota)$$

which induces on the $\mathfrak{p}_0$-parts of the p-divisible groups the map (6.7) and is an isomorphism on the $\mathfrak{p}_j$-parts for $j > 0$. Looking at the p-divisible groups, we see that, for the given p-principal polarization λ on A, the polarization $\mu_{\mathfrak{p}_0}(g)^{-1}\alpha^*(\lambda)$ on the $\mathfrak{p}_0$-part of the p-divisible group is principal and for $j > 0$ the $\mathfrak{p}_j$-parts of $\alpha^*(\lambda)$ is principal. We define $\bar{\lambda}'$ on A' as the class of all polarizations in $F^\times \alpha^*(\lambda)$ which are p-principal. We define all other data $(\bar{\eta}')^p$, $\bar{\eta}'_{\mathfrak{q}_j}$, $\bar{\xi}_j$ for $j > 0$ by pull back via α. Then A' with the extra structure just introduced is a point of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(S)$. To see that this point represents for an E_ν-scheme S the Hecke operator, we need the functions $\xi_{\mathfrak{p}_0}$ and $\xi'_{\mathfrak{p}_0}$ with values in $F^\times_{\mathfrak{p}_0}/O^\times_{F_{\mathfrak{p}_0}}$. So far it was not necessary to mention them because they have value 1 for a polarization which is principal in p. Therefore we have

$$1 = \xi_{\mathfrak{p}_0}(\lambda) = \xi'_{\mathfrak{p}_0}(\mu_{\mathfrak{p}_0}(g)^{-1}\alpha^*(\lambda)).$$

But then (4.9) shows that A' gives the Hecke operator of $g \in G^\bullet_{\mathfrak{p}_0}$. □

By [RZ, 6.29] the following maps are isomorphisms,

$$\begin{aligned} \operatorname{End}^o_B(A_o) \otimes_{\mathbb{Q}} \mathbb{A}^p_f &\xrightarrow{\sim} \operatorname{End}^o_{B\otimes\mathbb{A}^p_f} \mathrm{V}^p(A_o) \\ \operatorname{End}^o_B(A_o) \otimes_{\mathbb{Q}} \mathbb{Q}_p &\xrightarrow{\sim} \operatorname{End}^o_{B\otimes\mathbb{Q}_p} \mathbb{X}. \end{aligned} \tag{6.8}$$

Here, as above, $\mathbb{X} = \prod_{i=0}^s \mathbb{X}_i$ is the p-divisible group of A_o. We obtain

$$\operatorname{End}^o_{B\otimes\mathbb{Q}_p} \mathbb{X} \cong \prod_{i=0}^{s} \operatorname{End}^o_{B_{\mathfrak{p}_i}} \mathbb{X}_i$$

We denote by $\mathbf{I}$ the algebraic group of all B-linear quasi-isogenies $A_o \to A_o$ which respect the polarization λ_o up to a constant in $F^\times$. Let $\gamma \mapsto \gamma'$ be the Rosati involution on $\operatorname{End}^o_B A_o$ induced by λ_o. We can write

$$\mathbf{I}(\mathbb{Q}) = \{\alpha \in \operatorname{End}^o_B A_o \mid \alpha'\alpha \in F^\times\}. \tag{6.9}$$

In the framing object (6.1) we choose $\eta^p_o \in \bar{\eta}^p_o$. This isomorphism $\eta^p_o : \mathrm{V}(A_o) \xrightarrow{\sim} V \otimes \mathbb{A}^p_f$ induces by (6.8) an isomorphism

$$\mathbf{I}(\mathbb{A}^p_f) \xrightarrow{\sim} \{\gamma \in \operatorname{End}^o_{B\otimes\mathbb{A}^p_f} \mathrm{V}^p(A_o) \mid \gamma'\gamma \in (F \otimes \mathbb{A}_f)^\times\} \xrightarrow{\sim} G^\bullet(\mathbb{A}^p_f). \tag{6.10}$$

We also denote by $\gamma \mapsto \gamma'$ the involution on $\operatorname{End}^o_{B\otimes\mathbb{Q}_p} \mathbb{X}$ induced by the polarization $\lambda_{\mathbb{X}}$. We define

$$\mathbf{I}_{\mathfrak{p}_0} = \{\gamma \in \operatorname{End}^o_{B_{\mathfrak{p}_0}} \mathbb{X}_0 \mid \gamma'\gamma \in F^\times_{\mathfrak{p}_o}\},$$

cf. (5.22). If $j > 0$ we can take $(\Lambda^{pd}_{\mathfrak{p}_j}, \lambda_\psi)$ for the framing object $\mathbb{X}_j$, cf. (5.7). Using the definition (5.2) of λ_ψ we obtain an isomorphism

$$\begin{aligned} \mathbf{I}_{\mathfrak{p}_j} &\cong \{\gamma \in \operatorname{End}_{B_{\mathfrak{p}_j}} V_{\mathfrak{p}_i} \mid \psi(\gamma v_1, \gamma v_2) = \psi(f v_1, v_2), \text{ for some } f \in F^\times_{\mathfrak{p}_j}\} \\ &= G^\bullet_{\mathfrak{p}_j}. \end{aligned} \tag{6.11}$$

By (6.8) we obtain

$$\mathbf{I}(\mathbb{Q}_p) = \prod_{i=0}^{s} \mathbf{I}_{\mathfrak{p}_i}.$$

The group $\mathbf{I}_{\mathfrak{p}_i}$ acts from the left on the functor $\mathrm{RZ}_{\mathfrak{p}_i}$, for $i = 0, \ldots, s$ by

$$\alpha_{\mathfrak{p}_i} \in \mathbf{I}_{\mathfrak{p}_i} : \quad (Y_i, \iota_i, \bar{\lambda}_i, \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i}, \rho_i) \longmapsto (Y_i, \iota_i, \bar{\lambda}_i \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i}, \alpha_{\mathfrak{p}_i}\rho_i).$$

This makes sense because $\alpha_{\mathfrak{p}_i} : (\mathbb{X}_i, \iota_i) \to (\mathbb{X}_i, \iota_i)$ is a quasi-isogeny which respects the polarization $\lambda_{\mathbb{X}_i}$ up to constant. Note that for $i = 0$ the data $\bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i}$ are absent.

The group $\mathbf{I}(\mathbb{Q})$ acts on the left hand side of (6.2). If $\alpha \in \mathbf{I}(\mathbb{Q})$, with components $\alpha_{\mathfrak{p}_i} \in \mathbf{I}_{\mathfrak{p}_i}$ and $\alpha^p \in \mathbf{I}(\mathbb{A}^p_f) \cong G^\bullet(\mathbb{A}^p_f)$, then a point of the left hand side given by the data 1, 2, 3 is mapped to the data

$$\big((Y_i, \iota_i, \bar{\lambda}_i \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i}, \alpha_{\mathfrak{p}_i}\rho_i),\ \alpha^p g\big). \tag{6.12}$$

With respect to this action the morphism (6.2) is equivariant. To see this, we consider the morphism derived from (6.4)

$$(\bar{A}, \iota_{\bar{A}}) \to (A_o, \iota_o) \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S} \xrightarrow{\alpha} (A_o, \iota_o) \times_{\operatorname{Spec} \bar{\kappa}_{E_\nu}} \bar{S} \tag{6.13}$$

This composite may be used to compute the image of the data (6.12) by (6.2). One can easily see that this gives the same point in $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}(S)$ as for the original data.

Proposition 6.2. *Let $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}^\bullet_p$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$. Let $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ be the coarse moduli scheme of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet}$. The morphism (6.2) induces an isomorphism of formal schemes*

$$\Theta^\bullet : \mathbf{I}(\mathbb{Q})\backslash(\mathrm{RZ}_{p,\mathbf{K}^\bullet_p} \times G^\bullet(\mathbb{A}^p_f)/(\mathbf{K}^\bullet)^p) \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet / \operatorname{Spf} O_{\breve{E}_\nu}}. \tag{6.14}$$

The morphism is compatible with the Weil descent data on both sides as spelled out in the proof.

Proof. This is a variant of the general uniformization theorem [RZ, 6.30]. By Proposition 4.21, if $\mathbf{K}^\bullet$ is small enough, the morphism

$$\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet} \to \tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet}$$

is the etale sheafification. Therefore we can use deformation theory as in [RZ, 6.23] to show that $\Theta^\bullet$ is étale. For this one needs that the action of $\mathbf{I}(\mathbb{Q})$ is fixpoint free, if $\mathbf{K}^{\bullet,p}$ is small enough. We refer to the argument in loc.cit. for details. In addition to the arguments given in [RZ], one needs that $\Theta^\bullet$ is surjective on the $\bar{\kappa}_{E_\nu}$-valued points. This follows from the Hasse principle for $G^\bullet$ as explained in [KRZ, Prop. 7.1.11, Prop. 7.3.2]. We omit the details. If we drop the smallness assumption on $\mathbf{K}^{\bullet,p}$, it follows from Proposition 4.21 that $\Theta^\bullet$ is an isomorphism for the normal subgroup $\mathbf{K}^\bullet_U \subset \mathbf{K}^\bullet$. Dividing by the action of $\mathbf{K}^\bullet$ we obtain that $\Theta^\bullet$ is an isomorphism.

Both sides of (6.14) are endowed with a Weil descent datum relative to the extension $O_{\breve{E}_\nu}/O_{E_\nu}$. Because the right hand side is obtained by a base change $\operatorname{Spf} O_{\breve{E}_\nu} \to \operatorname{Spf} O_{E_\nu}$, we have there the natural Weil descent datum. On the left hand side the Weil descent datum is induced by a Weil descent datum of $\mathrm{RZ}_{p,\mathbf{K}^\bullet_p}$. It is defined on each $\mathrm{RZ}_{\mathfrak{p}_i}$ as follows. Let $\tau \in \operatorname{Gal}(\breve{E}_\nu/E_\nu)$ be the Frobenius. Let R be $O_{\breve{E}_\nu}$-Algebra such that p is nilpotent in R. We give the descent datum as a functorial map

$$\omega_{\mathfrak{p}_i}(R) : \mathrm{RZ}_{\mathfrak{p}_i}(R) \to \mathrm{RZ}_{\mathfrak{p}_i}(R_{[\tau]}). \tag{6.15}$$

For this we write $\bar{R} = R \otimes_{O_{\breve{E}_\nu}} \bar{\kappa}_{E_\nu}$ and we denote by $\varepsilon : \bar{\kappa}_{E_\nu} \to \bar{R}$ the structure morphism. Then (6.15) maps a point $(Y_i, \iota_i, \bar{\lambda}_i, \bar{\eta}_{\mathfrak{q}_i}, \bar{\xi}_{\mathfrak{p}_i}, \rho_i)$ from the right hand side of (6.15) to the point given by the same data except the ρ_i is replaced by the composite

$$Y_i \times_{\operatorname{Spec} R} \operatorname{Spec} \bar{R} \xrightarrow{\rho_i} \varepsilon_* \mathbb{X}_i \xrightarrow{\varepsilon_* F_{\mathbb{X},\tau}} \varepsilon_* \tau_* \mathbb{X}_i.$$

We denote here by $F_{\mathbb{X},\tau}$ the Frobenius morphism of the p-divisible group relative to κ_{E_ν}. The compatibility of the Weil descent data is explained in the proof of [KRZ, Lem. 7.3.1]. $\square$

For the following one should keep in mind that in (6.15) we have used τ to describe the Weil descent datum and not τ^{-1} as e.g. in Proposition 4.9. Recall from Definition 4.10 that the scheme $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)$ over O_{E_ν} is a Galois twist of $\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet}$ according to the following diagram

$$\begin{array}{ccc}
\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} \\
\Big\downarrow {\scriptstyle \dot{z}_{|\xi} \times \tau_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c} \\
\tilde{\mathsf{A}}^{\bullet t}_{\mathbf{K}^\bullet} \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} O_{E^{nr}_\nu},
\end{array} \tag{6.16}$$

where the horizontal arrows are isomomorphisms. Indeed, the morphism (4.29) becomes an isomorphism if we replace on the left hand side $\mathcal{A}$ by A. Therefore it follows from Proposition 4.9 that

$$\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D) \times_{\operatorname{Spec} O_{E_\nu}} \operatorname{Spec} E_\nu \cong \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h^\bullet_D)_{E_\nu}. \tag{6.17}$$

Theorem 6.3. *Let $\mathbf{K}^\bullet = \mathbf{K}^\bullet_p \mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f)$, with $\mathbf{K}^\bullet_p$ as in (4.6), where $\mathbf{M}_{\mathfrak{p}_0} = O^\times_{F_{\mathfrak{p}_0}}$. Let $\Pi \in D_{\mathfrak{p}_0}$ be a prime element of this division algebra. It acts on $V_{\mathfrak{p}_0} = D^{\mathrm{opp}}_{\mathfrak{p}_0} \otimes_{F_{\mathfrak{p}_0}} K_{\mathfrak{p}_0}$ by multiplication with $\Pi \otimes 1$ from the right. This defines an element of $G^\bullet_{\mathfrak{p}_0}$ which we denote simply* *by* Π *and we use this notation also for its image by* $G^{\bullet}_{\mathfrak{p}_0} \subset G^{\bullet}(\mathbb{A}_f)$. *The action of the Hecke operator is denoted by* $|_{\Pi}$.

Let $\widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)$ *be the integral model over* O_{E_ν} *of Definition 4.10 of the Shimura variety* $\mathrm{Sh}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)$. *We denote by* $\widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{E_\nu}}$ *the p-adic completion and we set*

$$\widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{\breve{E}_\nu}} = \widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{E_\nu}} \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu}.$$

Then there is an isomorphism of formal schemes

$$\mathbf{I}(\mathbb{Q})\backslash(\hat{\Omega}^2_{E_\nu} \times G^{\bullet}(\mathbb{A}_f)/\mathbf{K}^{\bullet}) \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu} \xrightarrow{\sim} \widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{\breve{E}_\nu}} \tag{6.18}$$

For varying $\mathbf{K}^{\bullet}$ *this morphism is compatible with the action of* $G^{\bullet}(\mathbb{A}_f)$ *by Hecke operators on both sides.*

Let $\tau \in \mathrm{Gal}(\breve{E}_\nu/E_\nu)$ *the Frobenius and* $\tau_c = \operatorname{Spf} \tau^{-1} : \operatorname{Spf} O_{\breve{E}_\nu} \to \operatorname{Spf} O_{\breve{E}_\nu}$. *The canonical Weil descent datum on the right hand side of (6.18) is given on the left hand side by the commutative diagram*

$$\begin{array}{ccc}
\mathbf{I}(\mathbb{Q})\backslash(\hat{\Omega}^2_{E_\nu} \times G^{\bullet}(\mathbb{A}_f)/\mathbf{K}^{\bullet}) \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{\breve{E}_\nu}} \\
\Big\downarrow {\scriptstyle |_{\Pi^{-1}} \times \mathrm{id} \times \tau_c} & & \Big\downarrow {\scriptstyle \mathrm{id} \times \tau_c} \\
\mathbf{I}(\mathbb{Q})\backslash(\hat{\Omega}^2_{E_\nu} \times G^{\bullet}(\mathbb{A}_f)/\mathbf{K}^{\bullet}) \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu} & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{\breve{E}_\nu}}
\end{array}$$

Proof. From the morphism $\Theta^{\bullet}$ in (6.14) and the definition and the horizontal line of the diagram (6.16) we obtain an isomorphism of formal schemes over $\operatorname{Spf} O_{\breve{E}_\nu}$

$$\mathbf{I}(\mathbb{Q})\backslash(\mathrm{RZ}_{p,\mathbf{K}^{\bullet}_p} \times G^{\bullet}(\mathbb{A}^p_f)/(\mathbf{K}^{\bullet})^p) \xrightarrow{\sim} \widetilde{\mathrm{Sh}}_{\mathbf{K}^{\bullet}}(G^{\bullet}, h^{\bullet}_D)^{\wedge}_{/\operatorname{Spf} O_{\breve{E}_\nu}}. \tag{6.19}$$

We obtain the isomorphism (6.18) if we rewrite the left hand side using the Propositions 5.4 and 5.11. We consider an R and ε as in the definition (6.15) of $\omega_{\mathfrak{p}_i}$. Let us denote the functor on the left hand side of (6.19) by $\mathcal{F}$. It is endowed with its natural Weil descent datum $\mathcal{F}(R) \to \mathcal{F}(R_{[\tau]})$ given by the $\omega_{\mathfrak{p}_i}$, $i = 0, \ldots, s$. By (6.16), the morphism (6.19) becomes compatible with the Weil descent data if we multiply the natural Weil descent datum on $\mathcal{F}$ by the operator $\dot{z}^{-1}_{|\xi}$. The exponent -1 appears because we use here τ instead of τ^{-1} as in the statement of the theorem. By the explanation after (4.10) this means that we have to replace $\omega_{\mathfrak{p}_o}$ by $\omega_{\mathfrak{p}_0}(1, \pi_{\mathfrak{p}_0} p^{-f_\nu})$, resp. $\omega_{\mathfrak{p}_j}$ by $\omega_{\mathfrak{p}_j}(1, p^{-f_\nu})$, where $(1, \pi_{\mathfrak{p}_0} p^{-f_\nu}) \in G^{\bullet}_{\mathfrak{p}_0}$ and $(1, p^{-f_\nu}) \in G^{\bullet}_{\mathfrak{p}_j}$, cf. Proposition 4.9.

We first check what this modified Weil descent datum does on $\mathrm{RZ}_{\mathfrak{p}_0}$. Let $(Y, \iota, \bar{\lambda}, \rho) \in \mathrm{RZ}_{\mathfrak{p}_0}(R)$. The action of the Hecke operator $(1, \pi_{\mathfrak{p}_0} p^{-f_\nu})$ is the same as $\mathfrak{h}((1, \pi_{\mathfrak{p}_0} p^{-f_\nu}))$, where we regard $u := (1, \pi_{\mathfrak{p}_0} p^{-f_\nu})$ as an element of $\mathcal{H}_{\mathfrak{p}_0}$, cf. (5.14). We note that $\iota^u = \iota$ because u lies in the center. Therefore the action of the Hecke operator $\mathfrak{h}(u)$ is

$$(Y, \iota, \bar{\lambda}, \rho) \longmapsto (Y, \iota, \bar{\lambda}, \rho \circ \iota(u)).$$

The image of $(Y, \iota, \bar{\lambda}, \rho)$ by the map

$$\omega_{\mathfrak{p}_0} \mathfrak{h}(u) : \mathrm{RZ}_{\mathfrak{p}_0}(R) \to \mathrm{RZ}_{\mathfrak{p}_0}(R[\tau])$$

is $(Y, \iota, \bar{\lambda}, \rho')$, where ρ' is given by

$$(Y_{\mathfrak{q}_0})_{\bar{R}} \times (Y_{\bar{\mathfrak{q}}_0})_{\bar{R}} \xrightarrow{\rho_{\mathfrak{q}_0} \times \rho_{\bar{\mathfrak{q}}_0}} \varepsilon_* \mathbb{X}_{\mathfrak{q}_0} \times \varepsilon_* \mathbb{X}_{\bar{\mathfrak{q}}_0} \xrightarrow{\varepsilon_* F_{\mathbb{X}_{\mathfrak{q}_0}, \tau} \times \varepsilon_* \pi_{\mathfrak{p}_0} p^{-f_\nu} F_{\mathbb{X}_{\bar{\mathfrak{q}}_0}, \tau}} \varepsilon_* \tau_* \mathbb{X}_{\mathfrak{q}_0} \times \varepsilon_* \tau_* \mathbb{X}_{\bar{\mathfrak{q}}_0}.$$

We note that the Weil descent datum commutes with all Hecke operators. It is straightforward to compute the following heights,

$$\mathrm{height}_{F_{\mathfrak{p}_0}} F_{\mathbb{X}_{\mathfrak{q}_0}, \tau} = 2, \quad \mathrm{height}_{F_{\mathfrak{p}_0}} \pi_{\mathfrak{p}_0} p^{-f_\nu} F_{\mathbb{X}_{\bar{\mathfrak{q}}_0}, \tau} = 2.$$

Therefore $\omega_{\mathfrak{p}_0} \mathfrak{h}(u)$ is of degree $(1, 1)$,

$$\omega_{\mathfrak{p}_0} \mathfrak{h}(u) : \mathrm{RZ}_{\mathfrak{p}_0}(a, b) \to \mathrm{RZ}_{\mathfrak{p}_0}(a + 1, b + 1).$$

The Hecke operator $|_\Pi$ has also degree $(1,1)$. We write $\omega_{\mathfrak{p}_0}\mathfrak{h}(u) =|_\Pi (|_\Pi)^{-1}\omega_{\mathfrak{p}_0}\mathfrak{h}(u)$. The Weil descent datum $(|_\Pi)^{-1}\omega_{\mathfrak{p}_0}\mathfrak{h}(u)$ is of degree $(0,0)$ and defines therefore a Weil descent datum on $\mathrm{RZ}_{\mathfrak{p}_0}(0,0)$. By the isomorphism of Lemma 5.10, it induces a Weil descent datum on

$$\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}. \tag{6.20}$$

But this isomorphism is just the projection to the $\mathfrak{q}_0$-part. Therefore the induced Weil descent datum maps an R-valued point $(Y_{\mathfrak{q}_0}, \iota_{\mathfrak{q}_0}, \rho_{\mathfrak{q}_0})$ to the point $(Y^{\Pi^{-1}}_{\mathfrak{q}_0}, \iota^{\Pi^{-1}}_{\mathfrak{q}_0}, \rho'_{\mathfrak{q}_0})$ where $\rho'_{\mathfrak{q}_0}$ is the following composite

$$(Y^{\Pi^{-1}}_{\mathfrak{q}_0}) \xrightarrow{\rho_{\mathfrak{q}_0}} \varepsilon_*\mathbb{X}^{\Pi^{-1}}_{\mathfrak{q}_0} \xrightarrow{\varepsilon_* F_{\mathbb{X}_{\mathfrak{q}_0},\tau}} \varepsilon_*\tau_*\mathbb{X}^{\Pi^{-1}}_{\mathfrak{q}_0} \xrightarrow{\iota(\Pi^{-1})} \varepsilon_*\tau_*\mathbb{X}_{\mathfrak{q}_0}.$$

But by the proof of [KRZ, Prop. 5.1.7], this is exactly the natural Weil descent datum on the base change $\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}$. Therefore the Weil descent datum on this factor is as claimed.

Now we consider the Weil descent data $\omega_{\mathfrak{p}_j}(1,p^{-f_\nu})$ on $\mathrm{RZ}_{\mathfrak{p}_j,\mathbf{K}^\bullet_{\mathfrak{p}_j}}$. We have to show that under the isomorphism of Proposition 5.4

$$\mathrm{RZ}_{\mathfrak{p}_j,\mathbf{K}^\bullet_{\mathfrak{p}_j}} \xrightarrow{\sim} G^\bullet_{\mathfrak{p}_j}/\mathbf{K}^\bullet_{\mathfrak{p}_j} \times_{\mathrm{Spf}\, O_{E_\nu},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}, \tag{6.21}$$

the Weil descent datum $\omega_{\mathfrak{p}_j}(1,p^{-f_\nu})$ induces on the right hand side the natural descent datum. The formal group $(\mathbb{X}_j, \iota_j)$ which was used to define (6.21) is defined over the field κ_{E_ν}, as we see from (5.7). Therefore $\mathrm{RZ}_{\mathfrak{p}_i,\mathbf{K}^\bullet_{\mathfrak{p}_i}}$ is defined over κ_{E_ν} and Proposition 5.4 holds over that field. We obtain a natural isomorphism $\mathbb{X}_i \cong \tau_*\mathbb{X}_i$. With this identification, the morphism

$$\mathbb{X}_{\mathfrak{q}_i} \times \mathbb{X}_{\bar{\mathfrak{q}}_i} \xrightarrow{F_{\mathbb{X}_{\mathfrak{q}_i},\tau} \times p^{-f_\nu} F_{\mathbb{X}_{\bar{\mathfrak{q}}_i},\tau}} \tau_*\mathbb{X}_{\mathfrak{q}_i} \times \tau_*\mathbb{X}_{\bar{\mathfrak{q}}_i} \cong \mathbb{X}_{\mathfrak{q}_i} \times \mathbb{X}_{\bar{\mathfrak{q}}_i}$$

becomes the identity. We see that $\omega_{\mathfrak{p}_i}(1,p^{-f_\nu})$ induces on the right hand side of (6.21) the natural descent datum. This proves the commutativity of the last diagram in the theorem. ☐

We now turn to the p-adic uniformization of the quaternionic Shimura curve. We first recall the following well-known fact.

Lemma 6.4. *Let K/F be separable quadratic extension of fields. Let L be a quaternion algebra with center K and let $l \mapsto l'$ be an involution of the second kind of L. Then there exist a quaternion algebra C with center F and an isomorphism of K-algebras*

$$L \cong C \otimes_F K,$$

such that the involution $l \mapsto l'$ induces on the right hand side the map $c \otimes k \mapsto c^\iota \otimes \sigma(k)$, where c^ι is the main involution and σ is the nontrivial element in $\mathrm{Gal}(K/F)$.

Proof. We consider the main involution $l \mapsto l^\iota$ of L over K. It is characterized by $l + l^\iota = \mathrm{Tr}^o_{L/K}$. Since the reduced trace is respected by an isomorphism of F-algebras, one verifies easily that

$$(l')^\iota = (l^\iota)'.$$

Therefore $\rho(l) := (l^\iota)'$ is a σ-linear isomorphism $L \to L$ such that $\rho^2 = \mathrm{id}_L$. The invariants define C by Galois descent. ☐

As an example we consider $(A_o, \iota_o, \bar{\lambda}_o)$, cf. (6.1). The ring $L = \mathrm{End}^o_B A_o$ is by (6.8) a quaternion algebra with center K. Let $l \mapsto l'$ be the Rosati involution induced by λ_o. We define the Cherednik twist of D,

$$\check{D} = \{l \in \mathrm{End}^o_B A_o \mid l' = l^\iota\}. \tag{6.22}$$

This is a quaternion algebra over F. Since $ll^\iota \in F$, we obtain by (6.9) that

$$\check{D}^\times \subset \mathbf{I}(\mathbb{Q}). \tag{6.23}$$

Because the Rosati involution is positive, the main involution is positive on $\check{D}$. It follows that at each infinite place of F, the localization of $\check{D}$ is isomorphic to the Hamilton quaternions. By (6.8) we find

$$\mathrm{End}^o_B A_o \otimes \mathbb{A}^p_f \cong B^{\mathrm{opp}} \otimes \mathbb{A}^p_f \cong D \otimes_F K.$$

Since the Riemann form of λ_o induces via η_o^p on $V \otimes \mathbb{A}_f^p$ the form ψ (up to a constant), we see that the induced involution on $B^{\mathrm{opp}} \cong D \otimes_F K$ is by (2.10) the involution $d \otimes k \mapsto d^\iota \otimes \bar{k}$. We obtain the isomorphism

$$\check{D} \otimes \mathbb{A}_f^p \cong D \otimes \mathbb{A}_f^p. \tag{6.24}$$

The data $\eta_{\mathfrak{q}_j,o}$ for $j = 1, \dots, s$ in Definition 4.6 provide isomorphisms

$$(\Lambda_{\mathfrak{p}_j}^{pd}, \bar{\lambda}_\psi) \xrightarrow{\sim} (\mathbb{X}_j, \iota_{\mathbb{X}_j}, \bar{\lambda}_{\mathbb{X}_j}).$$

From this we obtain

$$\mathrm{End}_B^o A_o \otimes_F F_{\mathfrak{p}_j} = \mathrm{End}_{B_{\mathfrak{p}_j}} V_{\mathfrak{p}_j} = B_{\mathfrak{q}_j}^{\mathrm{opp}} \times B_{\bar{\mathfrak{q}}_j}^{\mathrm{opp}}.$$

The Rosati involution of λ_o induces on the right hand side the map $(b_1, b_2) \mapsto (b_2', b_1')$. We obtain

$$\check{D}_{\mathfrak{p}_j} = \{(b_1, b_2) \in B_{\mathfrak{q}_j}^{\mathrm{opp}} \times B_{\bar{\mathfrak{q}}_j}^{\mathrm{opp}} \mid (b_2', b_1') = (b_1^\iota, b_2^\iota)\} \cong B_{\mathfrak{q}_j}^{\mathrm{opp}} = D_{\mathfrak{p}_j}, \tag{6.25}$$

where the last isomorphism is given by the projection. Comparing with (4.2) gives the embedding

$$\check{D}_{\mathfrak{p}_j}^\times \subset G_{\mathfrak{p}_j}^\bullet, \quad j = 1, \dots, s, \tag{6.26}$$

which coincides with the inclusion $\check{D}_{\mathfrak{p}_j}^\times \subset \mathbf{I}_{\mathfrak{p}_j} = G_{\mathfrak{p}_j}^\bullet$ via (6.11).

We obtain in the same way

$$\mathrm{End}_B^o A_o \otimes_F F_{\mathfrak{p}_0} \cong \mathrm{End}_{B_{\mathfrak{p}_0}} \mathbb{X}_0 \cong \mathrm{End}_{B_{\mathfrak{q}_0}} \mathbb{X}_{\mathfrak{q}_0} \times \mathrm{End}_{B_{\bar{\mathfrak{q}}_0}} \mathbb{X}_{\bar{\mathfrak{q}}_0},$$

and therefore

$$\check{D}_{\mathfrak{p}_0} = \{(\gamma_1, \gamma_2) \in \mathrm{End}_{B_{\mathfrak{q}_0}} \mathbb{X}_{\mathfrak{q}_0} \times \mathrm{End}_{B_{\bar{\mathfrak{q}}_0}} \mathbb{X}_{\bar{\mathfrak{q}}_0} \mid \gamma_2' = \gamma_1^\iota \}. \tag{6.27}$$

The projection to the first factor defines an isomorphism

$$\check{D}_{\mathfrak{p}_0} \cong \mathrm{End}_{B_{\mathfrak{q}_0}} \mathbb{X}_{\mathfrak{q}_0} \cong \mathrm{M}_2(F_{\mathfrak{p}_0}). \tag{6.28}$$

Altogether we find that the quaternion algebras D and $\check{D}$ over F have the same invariants for all places except for $\mathfrak{p}_0$ and the infinite place $\chi_0 = \varphi_{0|F}$. For the last two places they have opposite invariants.

We denote by H the multiplicative group of D considered as an algebraic group over $\mathbb{Q}$. The Shimura curve $\mathrm{Sh}(H, h_D)$ is defined over $E(H, h_D) = \chi_0(F)$. We have $E(H, h_D) \subset E$ and $E(H, h_D) \to E_\nu$ is a p-adic place of $E(H, h_D)$, cf. (2.17), resp. (2.19).

Let $\mathbf{K}_{\mathfrak{p}_0} = O_{D_{\mathfrak{p}_0}}^\times \subset D_{\mathfrak{p}_0}^\times$ be the unique maximal compact subgroup. We choose for $i = 1, \dots, s$ arbitrary open compact subgroups $\mathbf{K}_{\mathfrak{p}_i} \subset D_{\mathfrak{p}_i}^\times$ and set $\mathbf{K}_p = \prod_{i=0}^s \mathbf{K}_{\mathfrak{p}_i}$. Finally, we choose an arbitrary open compact subgroup $\mathbf{K}^p \subset (D \otimes \mathbb{A}_f^p)^\times$ and set

$$\mathbf{K} = \mathbf{K}_p \mathbf{K}^p \subset H(\mathbb{A}_f). \tag{6.29}$$

(Note that, since the group G plays no role anymore, we can recycle the notation used in (3.16).) The natural embedding $D \to D \otimes_F K = B^{\mathrm{opp}}$ induces by (2.2) an embedding $H \subset G^\bullet$. In terms of the exact sequence of Lemma 2.1 this embedding is obtained from $D^\times \to D^\times \times K^\times \to G^\bullet(\mathbb{Q})$.

We find isomorphisms

$$\begin{aligned} D_{\mathfrak{p}_i}^\times &\cong (B_{\mathfrak{q}_i}^{\mathrm{opp}})^\times = \mathrm{Aut}_{B_{\mathfrak{q}_i}} V_{\mathfrak{q}_i} \\ D_{\mathfrak{p}_i}^\times &\cong (B_{\bar{\mathfrak{q}}_i}^{\mathrm{opp}})^\times = \mathrm{Aut}_{B_{\bar{\mathfrak{q}}_i}} V_{\bar{\mathfrak{q}}_i}. \end{aligned} \tag{6.30}$$

We define $\mathbf{K}_p^\bullet \subset G^\bullet(\mathbb{Q}_p)$ in the form (4.6) using the isomorphism above such that

$$\mathbf{K}_{\mathfrak{q}_i}^\bullet = \mathbf{K}_{\mathfrak{p}_i}, \quad \mathbf{M}_{\mathfrak{p}_i}^\bullet = O_{F_{\mathfrak{p}_i}}^\times, \quad i = 0, \cdots, s. \tag{6.31}$$

Then $\mathbf{K}_p = H(\mathbb{Q}_p) \cap \mathbf{K}_p^\bullet$. By Proposition 8.6 below, there is an open compact subgroup $\mathbf{K}^{\bullet,p} \subset G^\bullet(\mathbb{A}_f^p)$ such that for $\mathbf{K}^\bullet = \mathbf{K}_p^\bullet \mathbf{K}^{\bullet,p}$ we have

$$\mathbf{K} = H(\mathbb{A}_f) \cap \mathbf{K}^\bullet,$$

and the natural morphism

$$\mathrm{Sh}_{\mathbf{K}}(H, h_D) \times_{\mathrm{Spec}\, E(H,h_D)} \mathrm{Spec}\, E \to \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)$$

is an open and closed immersion. It follows for example by Theorem 6.3 that $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet)$ is a flat and integral scheme over O_{E_ν}. Therefore the inclusion of the generic fiber

$$\mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, E_\nu \subset \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet, h_D^\bullet) \tag{6.32}$$

induces a bijection between the sets of connected components of these schemes. The connected components of the right hand side are the closures of the connected components of the left hand side.

Definition 6.5. We define the scheme $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H,h_D)$ over $\operatorname{Spec} O_{E_\nu}$ as the Zariski closure of the open and closed subscheme $\mathrm{Sh}_{\mathbf{K}}(H,h_D)\times_{\operatorname{Spec} E(G,h_D)}\operatorname{Spec} E_\nu$ of the left hand side of (6.32) in $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet,h_D^\bullet)$. Hence $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H,h_D)$ is a union of connected components of $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet}(G^\bullet,h_D^\bullet)$.

We consider the diagonal embedding

$$D_{\mathfrak{p}_0}^\times \subset G_{\mathfrak{p}_0}^\bullet \subset B_{\mathfrak{q}_0}^{\mathrm{opp},\times}\times B_{\bar{\mathfrak{q}}_0}^{\mathrm{opp},\times},$$

cf. (4.2), (6.30). It defines an open and closed embedding

$$(\hat{\Omega}^2_{F_{\mathfrak{p}_0}}\times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0}\operatorname{Spf} O_{\breve{E}_\nu})\times D_{\mathfrak{p}_0}^\times/\mathbf{K}_{\mathfrak{p}_0}\subset(\hat{\Omega}^2_{F_{\mathfrak{p}_0}}\times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0}\operatorname{Spf} O_{\breve{E}_\nu})\times G_{\mathfrak{p}_0}^\bullet/\mathbf{K}_{\mathfrak{p}_0}^\bullet=\mathrm{RZ}_{\mathfrak{p}_0}.$$

In terms of the decomposition (5.12), the left hand side is

$$\coprod_{a\in\mathbb{Z}}\mathrm{RZ}_{\mathfrak{p}_0}(a,a).$$

This is invariant by the action of $\breve{D}_{\mathfrak{p}_0}^\times\subset\mathbf{I}_{\mathfrak{p}_0}$ on $\mathrm{RZ}_{\mathfrak{p}_0}$. Therefore $\breve{D}_{\mathfrak{p}_0}^\times$ acts on

$$(\hat{\Omega}^2_{F_{\mathfrak{p}_0}}\times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0}\operatorname{Spf} O_{\breve{E}_\nu})\times D_{\mathfrak{p}_0}^\times/\mathbf{K}_{\mathfrak{p}_0}. \tag{6.33}$$

We now combine the actions of $\breve{D}^\times$ in its prime-to-p component (6.24), its $\mathfrak{p}_0$-component (6.33) and its $\mathfrak{p}_j$-components (6.25).

Proposition 6.6. *Let $\mathbf{K}_p\subset(D\otimes\mathbb{Q}_p)^\times$ as in (6.29). By (6.31) this defines a subgroup $\mathbf{K}_p^\bullet\subset G^\bullet(\mathbb{Q}_p)$. Let $\mathbf{K}^{\bullet,p}\subset G^\bullet(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup. We set $\mathbf{K}^p=\mathbf{K}^{\bullet,p}\cap(D\otimes\mathbb{A}_f^p)^\times$. Let $\mathbf{K}=\mathbf{K}_p\mathbf{K}^p$ and $\mathbf{K}^\bullet=\mathbf{K}_p^\bullet\mathbf{K}^{\bullet,p}$ as above. Then the morphism*

$$\breve{D}^\times\backslash((\hat{\Omega}^2_{F_{\mathfrak{p}_0}}\times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0}\operatorname{Spf} O_{\breve{E}_\nu})\times D^\times(\mathbb{A}_f)/\mathbf{K})\longrightarrow \mathbf{I}(\mathbb{Q})\backslash((\hat{\Omega}^2_{E_\nu}\times_{\operatorname{Spf} O_{E_\nu}}\operatorname{Spf} O_{\breve{E}_\nu})\times G^\bullet(\mathbb{A}_f)/\mathbf{K}^\bullet)$$

is an open and closed immersion.

Proof. The obvious sequence of algebraic groups over $\mathbb{Q}$ which is induced by (6.23)

$$0\to F^\times\to\breve{D}^\times\times K^\times\xrightarrow{\kappa}\mathbf{I}\to 0 \tag{6.34}$$

is exact. By Hilbert 90 the sequence remains exact if we take the $\mathbb{Q}$-valued points $0\to F^\times\to\breve{D}^\times\times K^\times\xrightarrow{\kappa}\mathbf{I}(\mathbb{Q})\to 0$. To check the exactness, we can make a base change from $\mathbb{Q}$ to an algebraically closed field where there is no difference to Lemma 2.1. We will regard (6.34) also as a sequence of algebraic groups over F. Note that $\mathbf{I}$ is the Weil restriction $\operatorname{Res}_{F/\mathbb{Q}}\tilde{\mathbf{I}}$ where $\tilde{\mathbf{I}}$ is an algebraic group over F which is defined in terms of the F-algebra $\operatorname{End}^o_B A_o$, cf. (6.9). Recall that $\tilde{\mathbf{I}}(F)=\mathbf{I}(\mathbb{Q})$. For $i=0,\dots s$ we write the group $\tilde{\mathbf{I}}(F_{\mathfrak{p}_i})$ as follows

$$\begin{array}{ccccc}\breve{D}_{\mathfrak{p}_i}^\times\times\breve{D}_{\mathfrak{p}_i}^\times & \supset & \breve{D}_{\mathfrak{p}_i}^\times\times^{F_{\mathfrak{p}_i}}(K_{\mathfrak{q}_i}\times K_{\bar{\mathfrak{q}}_i}) & \xrightarrow{\sim} & \breve{D}_{\mathfrak{p}_i}^\times\times F_{\mathfrak{p}_i}^\times=\tilde{\mathbf{I}}(F_{\mathfrak{p}_i})\\ (df_1,df_2) & \hookleftarrow & (d,(f_1,f_2)) & \longmapsto & (df_1,f_1^{-1}f_2)\end{array} \tag{6.35}$$

The canonical embedding $\breve{D}_{\mathfrak{p}_i}^\times\subset\tilde{\mathbf{I}}(F_{\mathfrak{p}_i})$ becomes $\breve{D}_{\mathfrak{p}_i}^\times\to\breve{D}_{\mathfrak{p}_i}^\times\times F_{\mathfrak{p}_i}^\times$, $d\mapsto(d,1)$.

For $j\geq 1$, $\mathbf{K}_{\mathfrak{p}_j}\subset D_{\mathfrak{p}_j}^\times=\breve{D}_{\mathfrak{p}_j}^\times$ is the arbitrary given compact open subgroup and

$$\mathbf{K}_{\mathfrak{p}_j}^\bullet=\mathbf{K}_{\mathfrak{p}_j}\times O_{F_{\mathfrak{p}_j}}^\times\subset D_{\mathfrak{p}_j}^\times\times F_{\mathfrak{p}_j}^\times=\breve{D}_{\mathfrak{p}_j}^\times\times F_{\mathfrak{p}_j}^\times=\tilde{\mathbf{I}}(F_{\mathfrak{p}_j}). \tag{6.36}$$

We see again that $\mathbf{K}_{\mathfrak{p}_j}^\bullet\cap\breve{D}_{\mathfrak{p}_j}^\times=\mathbf{K}_{\mathfrak{p}_j}$. We introduce the groups $C_{F,\mathfrak{p}_j}=\mathbf{K}_{\mathfrak{p}_j}\cap F_{\mathfrak{p}_j}^\times$ and $C_{K,\mathfrak{p}_j}=C_{F,\mathfrak{p}_j}\times O_{\mathfrak{p}_j}^\times\subset K_{\mathfrak{p}_j}^\times$ in the sense of the right hand side of (6.36). Then we obtain

$$\mathbf{K}_{\mathfrak{p}_j}^\bullet=\mathbf{K}_{\mathfrak{p}_j}\times^{C_{F,\mathfrak{p}_j}}C_{K,\mathfrak{p}_j}\subset\breve{D}_{\mathfrak{p}_j}^\times\times^{F_{\mathfrak{p}_j}^\times}K_{\mathfrak{p}_j}^\times. \tag{6.37}$$

For the proof we may assume that $\mathbf{K}^{\bullet,p}$ is of the following type. Let $C_F^p=\mathbf{K}^p\cap(\mathbb{A}_{F,f}^p)^\times$. We choose an arbitrary open compact subgroup $C_K^p\subset(K\otimes_F\mathbb{A}_{F,f}^p)^\times$ such that $C_K^p\cap(\mathbb{A}_{F,f}^p)^\times=C_F^p$. We define $\mathbf{K}_C^{\bullet,p}$ as the image of $\mathbf{K}^p\times C_K^p$ by the homomorphism

$$(\breve{D}\otimes_F\mathbb{A}_{F,f}^p)^\times\times(K\otimes_F\mathbb{A}_{F,f}^p)^\times\to\tilde{\mathbf{I}}(\mathbb{A}_{F,f}^p). \tag{6.38}$$

We set

$$\mathbf{K}^{\mathfrak{p}_0} = (\prod_{j=1}^{s} \mathbf{K}_{\mathfrak{p}_j})\mathbf{K}^p \subset (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times, \quad \mathbf{K}_C^{\bullet,\mathfrak{p}_0} = (\prod_{j=1}^{s} \mathbf{K}_{\mathfrak{p}_j}^{\bullet})\mathbf{K}_C^{\bullet,p} \subset \tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0}).$$

Moreover we set $\mathcal{Z} = \hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}}, \varphi_0} \operatorname{Spf} O_{\breve{E}_\nu}$ and $\Lambda = \{(a,b) \in \mathbb{Z}^2 \mid a + b \equiv 0 \mod 2\}$. Then we may write the morphism of the Proposition as follows,

$$\check{D}^\times \backslash (\mathcal{Z} \times \mathbb{Z}) \times (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times / \mathbf{K}^{\mathfrak{p}_0} \longrightarrow \tilde{\mathbf{I}}(F) \backslash (\mathcal{Z} \times \Lambda) \times \tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0}) / \mathbf{K}_C^{\bullet,\mathfrak{p}_0}. \tag{6.39}$$

The group $\check{D}^\times$ acts on $\mathcal{Z}$ via $\check{D}^\times \to \check{D}^\times_{\mathfrak{p}_0} \to \mathrm{PGL}_2(F_{\mathfrak{p}_0})$, as we have explained at the end of section 5. We will denote the image in $\mathrm{PGL}_2(F_{\mathfrak{p}_0})$ of an element g by the last map by $\bar{g}$. An element $g \in \check{D}^\times_{\mathfrak{p}_0}$ acts on $\mathcal{Z} \times \mathbb{Z}$ by

$$g[\omega, m] = [\bar{g}\omega, m + \operatorname{ord}_{\mathfrak{p}_0} \det g], \quad [\omega, m] \in \mathcal{Z} \times \mathbb{Z}.$$

An element $(g,f) \in \check{D}^\times_{\mathfrak{p}_0} \times F^\times_{\mathfrak{p}_0} = \tilde{\mathbf{I}}(F_{\mathfrak{p}_0})$ acts on $\mathcal{Z} \times \Lambda$ by

$$(g,f)[\omega,(a,b)] = [\bar{g}\omega, (a + \operatorname{ord}_{\mathfrak{p}_0} \det g, b + \operatorname{ord}_{\mathfrak{p}_0} \det g + 2 \operatorname{ord}_{\mathfrak{p}_0} f], \quad [\omega,(a,b)] \in \mathcal{Z} \times \Lambda.$$

The morphism (6.39) induces the diagonal $\mathbb{Z} \to \Lambda \subset \mathbb{Z}^2$.

We fix an element $m \times g \in \mathbb{Z} \times (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times$. The image of $\mathcal{Z} \times m \times g$ in the left hand side of (6.39) is of the form

$$\bar{\Gamma}_g \backslash \mathcal{Z},$$

where $\bar{\Gamma}_g$ is the image of the group $\{d \in \check{D} \cap g\mathbf{K}^{\mathfrak{p}_0} g^{-1} \mid \operatorname{ord}_{\mathfrak{p}_0} \det d = 0\}$ in $\mathrm{PGL}_2(F_{\mathfrak{p}_0})$. Here $\det d \in F^\times$ denotes the reduced norm of d.

Let J be the projective group of inner automorphisms of the F-algebra $\check{D}$ considered as an algebraic group over F. We denote by $\bar{\mathbf{K}}^{\mathfrak{p}_0}$ the image of $\mathbf{K}^{\mathfrak{p}_0}$ in $J(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$. This is an open and compact subgroup. To see this one notes that for each place w of F the map $(\check{D} \otimes_F F_w)^\times \to J(F_w)$ is open because $\check{D}^\times \to J$ is a smooth morphism of algebraic varieties. Because J is compact at each archimedian places of F the subgroup $J(F) \subset J(\mathbb{A}_{F,f})$ is discrete. It follows that $\Gamma'_g := J(F) \cap \bar{g}\bar{\mathbf{K}}^{\mathfrak{p}_0}\bar{g}^{-1} \subset J(F_{\mathfrak{p}_0})$ is a discrete subgroup. If $\mathbf{K}^p$ is sufficiently small, Γ'_g acts without fixed points on the Bruhat-Tits building of $\mathrm{PGL}_2(F_{\mathfrak{p}_0})$. Then each point of $\mathcal{Z}(\bar{\kappa}(\mathfrak{p}_0))$ has a Zariski neighbourhood U such that $\gamma U \cap U = \emptyset$ for $\gamma \in \Gamma'_g$ and $\gamma \neq 1$. This also holds for $\bar{\Gamma}_g \subset \Gamma'_g$. By our considerations, we can write the left hand side of (6.39) as

$$\coprod_i \bar{\Gamma}_{g_i} \backslash \mathcal{Z},$$

for a suitable choice of elements $g_i \in (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times$. In the same way we can write the right hand side of (6.39) as

$$\coprod_j \bar{\Gamma}_{h_j} \backslash \mathcal{Z},$$

with a suitable choice of elements $h_j \in \tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$. To show the Proposition, it is sufficient to show that (6.39) induces an injection on the $\bar{\kappa}(\mathfrak{p}_0)$-valued points. Indeed, the restriction of (6.39) to $\bar{\Gamma}_{g_i} \backslash \mathcal{Z}$ induces a morphism

$$\bar{\Gamma}_{g_i} \backslash \mathcal{Z} \to \bar{\Gamma}_{h_j} \backslash \mathcal{Z} \tag{6.40}$$

for a suitable j, which is injective on $\bar{\kappa}(\mathfrak{p}_0)$-valued points. Up to isomorphism we obtain the same map if we replace h_j by the image

$$g_i \times 1 \in (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times \times (K \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times$$

in $\tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$. By the injectivity of (6.40) and from the fact that the actions of the Γ-groups on both sides of (6.40) are fixed point free on the sets of $\bar{\kappa}(\mathfrak{p}_0)$-valued points, one obtains that the groups $\bar{\Gamma}_{g_i}$ and $\bar{\Gamma}_{h_j}$ coincide on both sides of (6.40) and that this morphism is an isomorphism. From this the Proposition easily follows.

It remains to prove the injectivity. We consider two elements $[\omega_i, h_i] \in \mathcal{Z} \times \mathbb{Z} \times (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times$, $i = 1, 2$, with $\omega_i \in \mathcal{Z}(\bar{\kappa}(\mathfrak{p}_0))$ and $h_i \in \mathbb{Z} \times (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times$, which represent the same element on the right hand side of (6.39). We will show that they also represent the same element on the left hand side.

By assumption there exists $g^\bullet \in \tilde{\mathbf{I}}(F)$ and $k^\bullet \in \mathbf{K}_C^{\bullet,\mathfrak{p}_0}$ such that

$$[\omega_1, h_1] = g^\bullet[\omega_2, h_2]k^\bullet.$$

We write $g^\bullet = g\lambda \in \check{D}^\times \times^{F^\times} K^\times$ with $g \in \check{D}^\times$ and $\lambda \in K^\times$. We set $C_K^{\mathfrak{p}_0} = (\prod_{j=1}^s C_{K,\mathfrak{p}_j})C_K^p$. By (6.37) and (6.38) we can write $k^\bullet = kc$ with $k \in \mathbf{K}^{\mathfrak{p}_0}$ and $c \in C_K^{\mathfrak{p}_0}$. Replacing $[\omega_2, h_2]$ by $g[\omega_2, h_2]k$ we may assume that

$$[\omega_1, h_1] = \lambda[\omega_2, h_2]c. \tag{6.41}$$

This implies $\omega_1 = \omega_2$ and

$$h_2^{-1}h_1 = \lambda c. \tag{6.42}$$

This equation takes place in $\Lambda \times \tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$. The Λ-part of (6.42) is equivalent with

$$\mathrm{ord}_{\mathfrak{q}_0} \lambda = \mathrm{ord}_{\bar{\mathfrak{q}}_0} \lambda. \tag{6.43}$$

Next we consider the $\tilde{\mathbf{I}}(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$-part of (6.42). We obtain

$$\lambda c = h_2^{-1}h_1 \in (\check{D} \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times \cap (K \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times = (\mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times. \tag{6.44}$$

We consider the torus $S = K^\times/F^\times$ over F. This torus is compact at all infinite places of F. Therefore $S(F) \subset S(\mathbb{A}_{F,f})$ is discrete and the group of units of S is finite. The equation (6.44) tells us that λ is a unit in $S(F_w)$ for all finite places $w \neq \mathfrak{p}_0$ of F, because λ is in the image $\bar{C}_K^{\mathfrak{p}_0}$ of the compact open subgroup $C_K^{\mathfrak{p}_0}$ by the morphism $(K \otimes_F \mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times \to S(\mathbb{A}_{F,f}^{\mathfrak{p}_0})$. On the other the equation (6.43) tells us that λ is a unit in $S(F_{\mathfrak{p}_0})$ because by this equation there exists an element $\alpha \in F_{\mathfrak{p}_0}^\times$ such that $\alpha\lambda$ is a unit in $K_{\mathfrak{p}_0}^\times$. Therefore the image of λ in $S(F)$ is a unit. If we choose C_K^p sufficiently small, the Theorem of Chevalley implies that the image is 1. We conclude that $\lambda \in F^\times$. Going back to (6.44) we obtain that

$$c \in C_K^{\mathfrak{p}_0} \cap (\mathbb{A}_{F,f}^{\mathfrak{p}_0})^\times \subset \mathbf{K}^{\mathfrak{p}_0}.$$

This shows that the right hand side of (6.41) represents the same element on the left hand side of (6.39) as $[\omega_2, h_2]$. □

We can now prove our main result, the Cherednik uniformization of quaternionic Shimura curves.

Theorem 6.7. *Let $\mathbf{K} \subset D^\times(\mathbb{A}_f)$ be of the form $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p$, where $\mathbf{K}_p$ is chosen as in (6.31). Let $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)$ be the model over $\mathrm{Spec}\, O_{E_\nu}$ of the Shimura curve associated to D, cf. Definition 6.5. Then there is a uniformization isomorphism of formal schemes*

$$\Theta\colon \check{D}^\times\backslash((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}) \xrightarrow{\sim} \widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)^\wedge_{/\mathrm{Spf}\, O_{\breve{E}_\nu}} \tag{6.45}$$

For varying $\mathbf{K}$ this uniformization isomorphism is compatible with the action of the Hecke operators in $D^\times(\mathbb{A}_f)$ on both sides.

Let $\Pi \in D_{\mathfrak{p}_0}$ be a prime element in this division algebra over $F_{\mathfrak{p}_0}$. We denote also by Π the image by the canonical embedding $D_{\mathfrak{p}_0} \subset (D \otimes \mathbb{A}_f)^\times$. Let $\tau \in \mathrm{Gal}(\breve{E}_\nu/E_\nu)$ be the Frobenius automorphism and $\tau_c = \mathrm{Spf}\, \tau^{-1}\colon \mathrm{Spf}\, O_{\breve{E}_\nu} \to \mathrm{Spf}\, O_{\breve{E}_\nu}$. The natural Weil descent datum with respect to $O_{\breve{E}_\nu}/O_{E_\nu}$ on the right hand side of (6.45) induces on the left hand side the Weil descent datum given by the following commutative diagram

$$\begin{array}{ccc}
\check{D}^\times\backslash((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}) & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)^\wedge_{/\mathrm{Spf}\, O_{\breve{E}_\nu}} \\
\Big\downarrow {\scriptstyle \mathrm{id}\times|_{\Pi^{-1}}\times\tau_c} & & \Big\downarrow {\scriptstyle \mathrm{id}\times\tau_c} \\
\check{D}^\times\backslash((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}},\varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}) & \longrightarrow & \widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)^\wedge_{/\mathrm{Spf}\, O_{\breve{E}_\nu}}
\end{array}$$

The following Corollary provides us with an intrinsic characterization of the integral model $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)$ of $\mathrm{Sh}_{\mathbf{K}}(H, h_D)$.

Corollary 6.8. *If $\mathbf{K}^p$ is sufficiently small, the integral model $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)$ is a stable relative curve over $\mathrm{Spec}\, O_{E_\nu}$, in the sense of* [DM]. *In addition, it has semi-stable reduction, i.e., it is regular and the special fiber is a reduced divisor with normal crossings.*

Proof. By the Theorem the formal scheme $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H,h_D)_{/\operatorname{Spf} O_{\breve{E}_\nu}}$ a union of connected components which are isomorphism to $\bar{\Gamma}\backslash(\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0} \operatorname{Spf} O_{\breve{E}_\nu})$ where $\bar{\Gamma} \subset \mathrm{PGL}_2(F_{\mathfrak{p}_0})$ is a disrete subroup. It is known [Mum] that $\mathbf{K}^p$ can be chosen such that $\bar{\Gamma}$ acts without fixed point and such that

$$\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0} \operatorname{Spf} O_{\breve{E}_\nu} \to \bar{\Gamma}\backslash(\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\operatorname{Spf} O_{F_{\mathfrak{p}_0}},\varphi_0} \operatorname{Spf} O_{\breve{E}_\nu}) \tag{6.46}$$

is a local isomorphisms for the Zariski topology. We denote by $\Omega_{\bar{\kappa}_\nu}$ the special fibre over $\operatorname{Spf} O_{\breve{E}_\nu}$ of the left hand side. All components of $\bar{\Gamma}\backslash\Omega_{\bar{\kappa}_\nu}$ are rational curves. We show that each of these components $\bar{C}$ is met by other components in at least 3 different points. This proves that the right hand side of (6.46) is a stable curve.

$\bar{C}$ is the image of a component $C \subset \Omega_{\bar{\kappa}_\nu}$ Let $E_1,\ldots,E_t$ all different components of $\Omega_{\bar{\kappa}_\nu}$ which meet C properly. Each E_i meets C in a single point z_i. The points $z_1,\ldots,z_t$ are all different. We know that $t \geq 3$. We denote by $\bar{E}_1,\ldots,\bar{E}_t$ the images in $\bar{\Gamma}\backslash\Omega_{\bar{\kappa}_\nu}$. We note the that $\bar{E}_i \neq \bar{C}$ for $i = 1,\ldots t$. Indeed, let $\bar{z}_i$ be the image of z_i in $\bar{\Gamma}\backslash\Omega_{\bar{\kappa}_\nu}$. The inequality follows because a neighbourhood of z_i is isomorphically mapped to a neighbourhood of $\bar{z}_i$. We will show that the points $\bar{z}_1,\ldots,\bar{z}_t$ are all different. If not we find an element $\gamma \in \bar{\Gamma}$ such that for example $\gamma z_1 = z_2$. Then all three components C, γC, γE_1 contain the point z_2. Therefore two of these components must be equal. By what we said above $C = \gamma C$ follows. But this implies that γ has a fixpoint on C which is excluded by assumption. We conclude that $\gamma z_1 = z_2$ is impossible. This proves that the points $\bar{z}_1,\ldots,\bar{z}_t$ are different. □

Proof of Theorem 6.7. We consider only open compact subgroups $\mathbf{K} \subset H(\mathbb{A}_f) = D^\times(\mathbb{A}_f)$ of the type as in the statement of the theorem. For the proof it will suffice to consider those $\mathbf{K}$ where $\mathbf{K}^p$ is small enough. We choose a chain of open compact subgroups of this type

$$\mathbf{K}_1 \supset \mathbf{K}_2 \supset \ldots \supset \mathbf{K}_t \supset \ldots, \tag{6.47}$$

which is cofinal to all subgroups of this type.

We consider open and compact subgroups $\mathbf{K}^\bullet \subset G^\bullet(\mathbb{A}_f)$ as in Proposition 4.21 with the following properties:

(a) $\mathbf{K}^\bullet \cap H(\mathbb{A}_f) = \mathbf{K}_t$ for some $t \in \mathbb{N}$.

(b) The groups $\mathbf{K}^\bullet_{\mathfrak{p}_i}$ and $\mathbf{K}_{t,\mathfrak{p}_i}$ are related as in (6.31) for $i = 0,\ldots,s$.

(c) The natural morphism

$$\mathrm{Sh}_{\mathbf{K}_t}(H,h_D) \times_{\operatorname{Spec} E(H,h_D)} \operatorname{Spec} E \to \mathrm{Sh}_{\mathbf{K}^\bullet}(G^\bullet,h^\bullet_D)$$

is an open and closed immersion. Here $\mathbf{K}^\bullet$ is chosen such that (a) is satisfied.

We find a chain of open and compact subgroups $\mathbf{K}^\bullet$ with the properties (*abc*)

$$\mathbf{K}^\bullet_1 \supset \mathbf{K}^\bullet_2 \supset \ldots \supset \mathbf{K}^\bullet_s \supset \ldots, \tag{6.48}$$

which has the following properties. For each $\mathbf{K}_t$ there is a group $\mathbf{K}^\bullet_s$ which satifies (*abc*) with respect to $\mathbf{K}_t$. Moreover, for an arbitrary $\mathbf{K}^\bullet$ satisfying (*abc*), there is a group $\mathbf{K}^\bullet_s$ such that $\mathbf{K}^\bullet_s \subset \mathbf{K}^\bullet$ and such that $\mathbf{K}^\bullet_s \cap H(\mathbb{A}_f) = \mathbf{K}_{t'}$ for some $t' > t$. We set

$$\mathrm{Sh}^{pro}(H,h_D)_{\breve{E}_\nu} = \varprojlim_{\mathbf{K}_t} \mathrm{Sh}_{\mathbf{K}_t}(H,h_D)_{\breve{E}_\nu}. \tag{6.49}$$

We remark that the connected components of $\mathrm{Sh}_{\mathbf{K}_t}(H,h_D)_{\breve{E}_\nu}$ are geometrically connected. This follows from [De, (2.7.1) and (3.9.1)] because $\mathbf{K}_{\mathfrak{p}_0} \in D^\times_{\mathfrak{p}_0}$ is maximal. We choose a connected component Z of the left hand side of (6.49). This induces a connected component $Z_{\mathbf{K}_t}$ of $\mathrm{Sh}_{\mathbf{K}_t}(H,h_D)_{\breve{E}_\nu}$ for each t. The closure $\tilde{Z}_{\mathbf{K}_t}$ of $Z_{\mathbf{K}_t}$ in $\widetilde{\mathrm{Sh}}_{\mathbf{K}_t}(H,h_D)$ is a connected component there. Since the last schemes are proper over $\operatorname{Spec} O_{\breve{E}_\nu}$, the natural restriction morphisms $\tilde{Z}_{\mathbf{K}_{t+1}} \to \tilde{Z}_{\mathbf{K}_t}$ are surjective. We choose points $z_{\mathbf{K}_t} \in \tilde{Z}_{\mathbf{K}_t}(\bar{\kappa}_{E_\nu})$ such that $z_{\mathbf{K}_{t+1}}$ is mapped to $z_{\mathbf{K}_t}$ for all t. Let $\mathbf{K}^\bullet_s$ be a subgroup such that $\mathbf{K}^\bullet_s$ induces $\mathbf{K}_t$ as in (*abc*). Then by the open and closed immersion of Definition 6.5, $\tilde{Z}_{\mathbf{K}_t}$ is also a connected component of $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet_s}(G^\bullet,h^\bullet_D)$. We consider $z_{\mathbf{K}_t}$ as a point of $\tilde{\mathcal{A}}^{\bullet t}_{\mathbf{K}^\bullet_s}(\bar{\kappa}_{E_\nu})$. We denote this point by $z^\bullet_{\mathbf{K}^\bullet_s}$. It is represented by the isomorphism class of a tuple

$$z^\bullet_{\mathbf{K}^\bullet_s} = (A(\mathbf{K}^\bullet_s), \iota(\mathbf{K}^\bullet_s), \bar{\lambda}(\mathbf{K}^\bullet_s), \bar{\eta}^p(\mathbf{K}^\bullet_s), (\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_s))_j). \tag{6.50}$$

We note that no datum $(\xi_{\mathfrak{p}_i})_i$ appears because of our choice (6.31). By construction $z_{\mathbf{K}^\bullet_{s+1}}$ is mapped to $z_{\mathbf{K}^\bullet_s}$ for all s. The triples

$$(A(\mathbf{K}^\bullet_s), \iota(\mathbf{K}^\bullet_s), \bar{\lambda}(\mathbf{K}^\bullet_s))$$

are all isomorphic. Therefore we may choose them independent of s. The classes $\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})$ and $\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})$ generate classes modulo $\mathbf{K}^\bullet_s$. We denote these classes by $\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})_{|s}$, resp. $\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})_{|s}$. Since $z_{\mathbf{K}^\bullet_{s+1}}$ is mapped to $z_{\mathbf{K}^\bullet_s}$, we obtain an isomorphism of tuples

$$(A, \iota, \bar{\lambda}, \bar{\eta}^p(\mathbf{K}^\bullet_{s+1})_{|s}, (\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})_{|s})_j) \xrightarrow{\sim} (A, \iota, \bar{\lambda}, \bar{\eta}^p(\mathbf{K}^\bullet_s), (\bar{\eta}_{\mathfrak{q}_i}(\mathbf{K}^\bullet_s))_j).$$

By this isomorphism the data $\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})$ and $\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})$ on the left hand side induce on the right hand side data $\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})'$ and $\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})'$ such that

$$\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})'_{|s} = \bar{\eta}^p(\mathbf{K}^\bullet_s), \quad \bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})'_{|s} = \bar{\eta}_{\mathfrak{q}_i}(\mathbf{K}^\bullet_s).$$

Therefore we may assume that

$$\bar{\eta}^p(\mathbf{K}^\bullet_{s+1})_{|s} = \bar{\eta}^p(\mathbf{K}^\bullet_s), \quad \bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_{s+1})_{|s} = \bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_s). \tag{6.51}$$

Now $\bar{\eta}^p(\mathbf{K}^\bullet_s) \subset \mathrm{Isom}_{B\otimes\mathbb{A}^p_f}(V \otimes \mathbb{A}^p_f, \mathrm{V}^p(A))$ is a compact subset. Therefore the intersection $\cap_s \bar{\eta}^p(\mathbf{K}^\bullet_s)$ is not empty. We choose an element η^p in this intersection. It generates the class $\bar{\eta}^p(\mathbf{K}^\bullet_s)$ for each s. Similiarly, we find for each $j = 1, \ldots, s$ an isomorphism $\eta_{\mathfrak{q}_i} : \Lambda_{\mathfrak{q}_j} \xrightarrow{\sim} T_{\mathfrak{q}_i}$ which induces all classes $\bar{\eta}_{\mathfrak{q}_j}(\mathbf{K}^\bullet_s)$.

The tuple

$$(A, \iota, \bar{\lambda}, \eta^p, (\eta_{\mathfrak{q}_j})_j) \tag{6.52}$$

makes it possible to define the uniformization morphism of the theorem. For this we consider the morphism (6.2) defined by substituting $(A, \iota, \bar{\lambda}, \eta^p, (\eta_{\mathfrak{q}_j})_j)$ for the choice of (6.1) used there. Let $\mathbb{X} = \prod_{i=0}^s \mathbb{X}_{\mathfrak{p}_i}$ be the p-divisible group of A. In the Definition 5.6 we take $\mathbb{X}_0 = \mathbb{X}_{\mathfrak{p}_0}$ as the framing object. If we take for ρ the identity, we obtain a point of $\mathrm{RZ}_{\mathfrak{p}_0}(0,0) \subset \mathrm{RZ}_{\mathfrak{p}_0}$ and by Lemma 5.10 a point

$$\tilde{z} \in \hat{\Omega}^2_{F_{\mathfrak{p}_0}}(\bar{\kappa}_{E_\nu}) = \hat{\Omega}^2_{E_\nu}(\bar{\kappa}_{E_\nu}).$$

From $\mathbb{X}_j = \mathbb{X}_{\mathfrak{p}_j}$, with the $O^\times_{F_{\mathfrak{p}_i}}$-homogeneous polarization induced from $\bar{\lambda}$, the rigidification $\bar{\eta}_{\mathfrak{q}_j}$, and the datum $\rho = \mathrm{id}_{\mathbb{X}_i}$, we obtain a point $\tilde{z}_j(\mathbf{K}^\bullet_{s,\mathfrak{p}_j})$ of $\mathrm{RZ}_{\mathfrak{p}_j, \mathbf{K}^\bullet_{s,\mathfrak{p}_j}}$. If we use for (5.7) the isomorphism given by $\eta_{\mathfrak{q}_j}$, the point $\tilde{z}_j(\mathbf{K}^\bullet_{s,\mathfrak{p}_j})$ corresponds to $1 \in G^\bullet_{\mathfrak{p}_j}/\mathbf{K}^\bullet_{s,\mathfrak{p}_j}$ under the isomorphism of Proposition 5.4. By construction of the uniformization morphism (6.18), the point

$$\tilde{z} \times 1 \in (\hat{\Omega}^2_{E_\nu} \times G^\bullet(\mathbb{A}_f)/\mathbf{K}^\bullet_s)(\bar{\kappa}_{E_\nu})$$

is mapped to the point

$$z_{\mathbf{K}^\bullet_s} \in \tilde{Z}_{\mathbf{K}_t}(\bar{\kappa}_{E_\nu}) \subset \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet_s}(G^\bullet, h^\bullet_D)(\bar{\kappa}_{E_\nu}).$$

This implies that $(\hat{\Omega}^2_{E_\nu} \times_{\mathrm{Spf}\, O_{E_\nu}} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times 1$ is mapped by (6.18) to the formal completion of the connected component $\tilde{Z}_{\mathbf{K}_t}$ of $\widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet_s}(G^\bullet, h^\bullet_D)_{O_{\breve{E}_\nu}}$. Now we restrict (6.18) to

$$\check{D}^\times \backslash ((\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times D^\times(\mathbb{A}_f)/\mathbf{K}_t) \to \widetilde{\mathrm{Sh}}_{\mathbf{K}^\bullet_s}(G^\bullet, h^\bullet_D)^\wedge_{/\,\mathrm{Spf}\, O_{\breve{E}_\nu}}, \tag{6.53}$$

cf. Lemma 6.6. The image of the connected component $(\hat{\Omega}^2_{F_{\mathfrak{p}_0}} \times_{\mathrm{Spf}\, O_{F_{\mathfrak{p}_0}}, \varphi_0} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times 1$ is mapped to a connected component of the open and closed formal subscheme $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)^\wedge_{/\,\mathrm{Spf}\, O_{\breve{E}_\nu}}$. But since the Hecke operators $D^\times(\mathbb{A}_f)$ act transitively on the connected components of the last formal scheme and the morphism (6.18) is compatible with Hecke operators, we conclude that (6.53) is a surjective map onto $\widetilde{\mathrm{Sh}}_{\mathbf{K}}(H, h_D)^\wedge_{/\,\mathrm{Spf}\, O_{\breve{E}_\nu}}$. Since by Theorem 6.3 and Lemma 6.6 the morphism is an open and closed immersion we conclude that (6.45) is an isomorphism for $\mathbf{K} = \mathbf{K}_t$.

Now the tuple $(A, \iota, \bar{\lambda}, \eta^p, (\eta_{\mathfrak{q}_j})_j)$ defines the uniformization morphism for an arbitrary $\mathbf{K}$. By choosing $\mathbf{K}_t \subset \mathbf{K}$, we see that (6.45) is surjective and therefore an isomorphism by Lemma 6.6. The compatibility with the Weil descent data is a consequence of Theorem 6.3. This completes the proof. □

7. Conventions about Galois descent

Let L/E be a Galois extension (possibly infinite) with Galois group $G = \mathrm{Gal}(L/E)$. For $\sigma \in G$ we set

$$\sigma_c = \operatorname{Spec} \sigma^{-1} : \operatorname{Spec} L \to \operatorname{Spec} L. \tag{7.1}$$

If $\tau \in G$ we find $(\sigma \circ \tau)_c = \sigma_c \circ \tau_c$. Let $\pi : X \to \operatorname{Spec} L$ be a scheme over L. We recall that a descent datum on X relative to L/E is a collection of morphisms $\varphi_\sigma : X \to X$ for $\sigma \in G$, making the following diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{\varphi_\sigma} & X \\ \downarrow{\scriptstyle\pi} & & {\scriptstyle\pi}\downarrow \\ \operatorname{Spec} L & \xrightarrow{\sigma_c} & \operatorname{Spec} L, \end{array}$$

such that $\varphi_\sigma \circ \varphi_\tau = \varphi_{\sigma\tau}$ for all $\sigma, \tau \in G$. In other words, a descent datum is a left action of G on X by semi-linear automorphisms. A descent datum (X, φ_σ) defines a left action of G on $X(L) = \mathrm{Hom}_{\operatorname{Spec} L}(\operatorname{Spec} L, X)$,

$$\begin{array}{ccc} G \times X(L) & \longrightarrow & X(L). \\ (\sigma, \alpha) & \longmapsto & \varphi_\sigma \circ \alpha \circ \operatorname{Spec} \sigma \end{array}$$

We denote the right hand side by $\sigma \times_\varphi \alpha$. This is indeed a point of $X(L)$:

$$\pi \circ \varphi_\sigma \circ \alpha \circ \operatorname{Spec} \sigma = \sigma_c \circ \pi \circ \alpha \circ \operatorname{Spec} \sigma = \sigma_c \circ \operatorname{Spec} \sigma = \mathrm{id}_{\operatorname{Spec} L}.$$

Let $u : G \to \mathrm{Aut}_L((X, \varphi))$ be an action of G on this descent datum. This means that for each $\sigma \in G$ an L-morphism $u_\sigma : X \to X$ is given such that for each $\sigma, \tau \in G$

$$u_\sigma \circ u_\tau = u_{\sigma\tau}, \quad u_\sigma \circ \varphi_\tau = \varphi_\tau \circ u_\sigma.$$

Then $\psi_\sigma := u_\sigma \circ \varphi_\sigma$ is another descent datum on X. It defines another action $\sigma \times_\psi \alpha$ of G on $X(L)$. From the definition we obtain

$$\sigma \times_\psi \alpha = u_\sigma \circ (\sigma \times_\phi \alpha). \tag{7.2}$$

If X_0 is a scheme over $\operatorname{Spec} E$, there is the canonical descent datum on $X = X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L$,

$$\kappa_\sigma = \mathrm{id}_{X_0} \times \sigma_c : X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L \to X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L.$$

The action of G induced by κ_σ on $X(L)$ coincides with the action on $X_0(L) = \mathrm{Hom}_{\operatorname{Spec} E}(\operatorname{Spec} L, X_0)$ via L, taking into account the identification $X(L) = X_0(L)$. We denote this action by $\sigma\alpha_0 = \sigma \times_\kappa \alpha_0$, $(\sigma \in G,\ \alpha_0 \in X_0(L))$.

A homomorphism $a : G \to \mathrm{Aut}_E X_0$ defines an action on the canonical descent datum $u : G \to \mathrm{Aut}_L((X, \kappa))$ via $u_\sigma = a_\sigma \times \mathrm{id}_{\operatorname{Spec} L}$. We obtain the new descent datum

$$\psi_\sigma = u_\sigma \circ \kappa_\sigma = a_\sigma \times \sigma_c : X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L \to X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L.$$

The action of G on $X(L) = X_0(L)$ defined by this descent datum is

$$\sigma \times_\psi \alpha_0 = a_\sigma \circ \sigma\alpha_0, \quad \alpha_0 \in X_0(L).$$

Assume that $X_0 = \coprod_{X_0(E)} \operatorname{Spec} E$ is the constant scheme. Then the action of G on $X_0(L)$ is trivial. Let $a : G \to \mathrm{Aut}\, X_0(E)$ be an action. It induces an action on X_0 which we denote by the same letter a. It defines on $X = X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L$ the descent datum $a_\sigma \times \sigma_c$. The action on $X(L) = X_0(L)$ induced by this descent datum is the operation on $X_0(L)$ by a acting on X_0. (Note that $X_0(L) = X_0(E)$).

A descent datum (X, φ_σ) is effective if there is scheme X_0 over E such that there is an isomorphism $X_0 \times_{\operatorname{Spec} E} \operatorname{Spec} L \to X$ which respects the descent data κ_σ resp. φ_σ. Assume that L/K is a finite field extension and that X is quasiprojective over $\operatorname{Spec} L$. Then any decent datum φ_σ is effective (cf. SGA1 Exp. VIII).

8. Conventions about Shimura varieties

Let (G, h) a Shimura datum. We denote by X the $G(\mathbb{R})$-conjugacy class of h. We consider the operation of $G(\mathbb{R})$ on X from the left

$$G(\mathbb{R}) \times X \to X, \quad (g, x) \longmapsto gxg^{-1}.$$

Let $\mathbf{K}_\infty \subset G(\mathbb{R})$ be the stabilizer of h. Then we have a $G(\mathbb{R})$-equivariant map

$$G(\mathbb{R})/\mathbf{K}_\infty \xrightarrow{\sim} X, \quad g \longmapsto ghg^{-1}.$$

Let $\mathbf{K} \subset G(\mathbb{A}_f)$ be an open and compact subgroup. Then we define the complex Shimura variety

$$\begin{aligned} \mathrm{Sh}_{\mathbf{K}}(G,h)_{\mathbb{C}} &= G(\mathbb{Q})\backslash(X \times (G(\mathbb{A}_f)/\mathbf{K})) \\ &= G(\mathbb{Q})\backslash((G(\mathbb{R})/\mathbf{K}_\infty) \times (G(\mathbb{A}_f)/\mathbf{K})). \end{aligned} \tag{8.1}$$

The group $G(\mathbb{Q})$ acts from the left via the homomorphisms $G(\mathbb{Q}) \to G(\mathbb{R})$ and $G(\mathbb{Q}) \to G(\mathbb{A}_f)$.

The group $G(\mathbb{A}_f)$ acts from the right on the tower $\{\mathrm{Sh}_{\mathbf{K}}\}_{\mathbf{K}}$ for varying $\mathbf{K}$. For $a \in G(\mathbb{A}_f)$ this action is given by

$$\begin{array}{rccc} |_a\colon & \mathrm{Sh}_{\mathbf{K}} & \to & \mathrm{Sh}_{a^{-1}\mathbf{K}a}, \\ & (x,u) & \longmapsto & (x, ua) \end{array} \tag{8.2}$$

where $x \in X$ and $u \in G(\mathbb{A}_f)$. We call this a Hecke operator.

Remark 8.1. In [De], the action of $G(\mathbb{R})$ from the right on X is considered,

$$h \times g := g^{-1}hg, \quad h \in X,\ g \in G(\mathbb{R}).$$

The complex Shimura variety is defined as

$$\begin{aligned} \mathrm{Sh}^D_{\mathbf{K}}(G,h)_{\mathbb{C}} &= (X \times (\mathbf{K}\backslash G(\mathbb{A}_f))/G(\mathbb{Q}) \\ &= ((\mathbf{K}_\infty\backslash G(\mathbb{R})) \times (\mathbf{K}\backslash G(\mathbb{A}_f))/G(\mathbb{Q}) \end{aligned}$$

Let X_- be the conjugacy class of h^{-1}. Then there is a natural isomorphism

$$\mathrm{Sh}_{\mathbf{K}}(G,h)_{\mathbb{C}} \xrightarrow{\sim} \mathrm{Sh}^D_{\mathbf{K}}(G,h^{-1})_{\mathbb{C}}, \tag{8.3}$$

given by

$$\begin{array}{ccc} G(\mathbb{Q})\backslash(X \times (G(\mathbb{A}_f)/\mathbf{K})) & \xrightarrow{\sim} & (X \times (\mathbf{K}\backslash G(\mathbb{A}_f))/G(\mathbb{Q}) \\ (x,g) & \longmapsto & (x^{-1}, g^{-1}) \end{array}$$

Let H be a torus over $\mathbb{Q}$ and let

$$h : \mathbb{S} \to H_{\mathbb{R}}$$

be a morphism of algebraic groups over $\mathbb{R}$. It induces a morphism of algebraic groups over $\mathbb{C}$

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to H_{\mathbb{C}}.$$

The field of definition E of μ is the reflex field of (H, h). We consider the composite

$$\mathfrak{r} : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \xrightarrow{\mathrm{Res}\,\mu} \mathrm{Res}_{E/\mathbb{Q}}H_E \xrightarrow{\mathrm{Nm}_{E/\mathbb{Q}}} H.$$

The homomorphism

$$r(H,h) = \mathfrak{r}^{-1} : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to H \tag{8.4}$$

is called the reciprocity law of (H, h), cf. [De, (3.9.1)]. Let $\mathbf{K} \subset H(\mathbb{A}_f)$ be an open compact subgroup. There is an open and compact subgroup $C \subset (E \otimes \mathbb{A}_f)^\times$ such that $r(H,h)(\mathbb{A}_f)(C) \subset \mathbf{K}$. Therefore $r(H, h)$ induces a map

$$E^\times\backslash(E \otimes \mathbb{A}_f)^\times/C \to H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K} \tag{8.5}$$

By class field theory

$$E^\times\backslash(E \otimes \mathbb{A}_f)^\times/C = E^\times\backslash(E \otimes \mathbb{A})^\times/C(E \otimes \mathbb{R})^\times$$

corresponds to a finite abelian extension L of E. We consider the homomorphism

$$\mathrm{Gal}(E^{ab}/E) \to \mathrm{Gal}(L/E) = E^\times\backslash(E \otimes \mathbb{A}_f)^\times/C.$$

If we compose this with (8.5) we obtain the class field version of the reciprocity map,

$$r^{\mathrm{cft}}(H,h) : \mathrm{Gal}(\bar{E}/E) \to \mathrm{Gal}(E^{ab}/E) \to H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}. \tag{8.6}$$

This Galois action on $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}$ defines a finite étale scheme over E which we denote by $\mathrm{Sh}_{\mathbf{K}}(H,h)$. This is called the canonical model of $\mathrm{Sh}_{\mathbf{K}}(H,h)$ over E. By definition

$$\mathrm{Sh}_{\mathbf{K}}(H,h)_{\mathbb{C}} = H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}.$$

In other words we can say that the E-scheme $\mathrm{Sh}_{\mathbf{K}}(H,h)$ is obtained from the constant scheme $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}$ over E by the descent datum

$$\begin{aligned} r^{\mathrm{cft}}(H,h)(\sigma)\times\sigma_c : \quad & (H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} \longrightarrow \\ & \qquad (H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E}, \end{aligned} \tag{8.7}$$

for $\sigma \in \mathrm{Gal}(\bar{E}/E)$ and $\sigma_c := \mathrm{Spec}\,\sigma^{-1}$. We can also express the last statement by a commutative diagram. There is an isomorphism of schemes over $\bar{E}$

$$H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} \to \mathrm{Sh}_{\mathbf{K}}(H,h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} \tag{8.8}$$

such that for each $\sigma \in \mathrm{Gal}(\bar{E}/E)$ the following diagram is commutative,

$$\begin{array}{ccc} (H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(H,h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} \\ {\scriptstyle r^{\mathrm{cft}}(H,h)(\sigma)\times\sigma_c}\downarrow & & \downarrow{\scriptstyle id\times\sigma_c} \\ (H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(H,h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\,\bar{E}. \end{array}$$

For varying $\mathbf{K}$ the isomorphism is compatible with the action of the Hecke operators $H(\mathbb{A}_f)$. In the considerations above we can replace the field of definition E of μ by any finite extension E', $E \subset E' \subset \mathbb{C}$. The definition (8.4) gives

$$r_{E'}(H,h) = \mathfrak{r}^{-1} : \mathrm{Res}_{E'/\mathbb{Q}}\mathbb{G}_{m,E'} \to H.$$

This reciprocity law gives the action of $\mathrm{Gal}(\bar{E}/E')$ on $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K}$ obtained by restriction from (8.6).

Remark 8.2. The map

$$\begin{array}{ccc} H(\mathbb{Q})\backslash H(\mathbb{A}_f)/\mathbf{K} & \xrightarrow{\sim} & \mathbf{K}\backslash H(\mathbb{A}_f)/H(\mathbb{Q}) \\ h & \longmapsto & h^{-1} \end{array}$$

is equivariant with respect to the action of $E^{\times}(\mathbb{A}_f)$ on the left hand side by $r(H,h)$ and the action of $E^{\times}(\mathbb{A}_f)$ on the right hand side by $r(H,h^{-1})$. Therefore we obtain an isomorphism of canonical models over E,

$$\mathrm{Sh}_{\mathbf{K}}(H,h) \xrightarrow{\sim} \mathrm{Sh}^{D}_{\mathbf{K}}(H,h^{-1}).$$

We go back to a general Shimura datum (G,h). We denote by $E(G,h)$ the Shimura field. We call a model $\mathrm{Sh}(G,h)$ over E of $\mathrm{Sh}(G,h)_{\mathbb{C}}$ canonical if for each maximal torus $H \subset G$ and each $h' : \mathbb{S} \to H_{\mathbb{R}}$ which is conjugate to h in $G(\mathbb{R})$, the induced morphism $\mathrm{Sh}(H,h')_{\mathbb{C}} \to \mathrm{Sh}(G,h)_{\mathbb{C}}$ is defined over the compositum of $E(H,h')$ and $E(G,h)$. With this definition (8.3) induces an isomorphism of canonical models, cf. [De, 3.13].

$$\mathrm{Sh}_{\mathbf{K}}(G,h) \xrightarrow{\sim} \mathrm{Sh}^{D}_{\mathbf{K}}(G,h^{-1}). \tag{8.9}$$

We now consider the situation of [De, 4.9–4.11]. Let L be a semisimple algebra over $\mathbb{Q}$ with a positive involution $* : L \to L$. Let $F \subset L$ be a subfield in the center of L which is invariant by the involution $*$. Let V be a faithful L-module which is finite-dimensional over $\mathbb{Q}$. Let $\psi : V \times V \to \mathbb{Q}$ be an alternating $\mathbb{Q}$-bilinear form such that

$$\psi(\ell x, y) = \psi(x, \ell^* y), \quad \ell \in L,\ x,y \in V.$$

We consider the algebraic group G over $\mathbb{Q}$ given by

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}_L(V) \mid \psi(gx,gy) = \psi(\mu(g)x,y),\ \mu(g) \in F^{\times}\}.$$

There is up to conjugation by an element of $G(\mathbb{R})$ a unique complex stucture $J : V\otimes\mathbb{R} \to V\otimes\mathbb{R}$, $J^2 = -\mathrm{id}$ which commutes with the action of L and such that

$$\psi(Jx,y)$$

is a symmetric and positive definite. Let $h : \mathbb{S} \to G_{\mathbb{R}}$ such that $h(z)$ acts on $V \otimes \mathbb{R}$ by multiplication by z with respect to the complex structure just introduced. Then (G, h) is a Shimura datum. We set

$$\mathbf{t}(\ell) = \mathrm{Tr}_{\mathbb{C}}(\ell \mid V \otimes \mathbb{R}), \quad \ell \in L.$$

The numbers $\mathbf{t}(\ell)$ generate over $\mathbb{Q}$ the Shimura field $E(G, h)$. The canonical model $\mathrm{Sh}_{\mathbf{K}}(G, h)$ is the coarse moduli scheme of the following functor $\mathcal{M}(L, V, \psi)$ on the category of E-schemes S.

Definition 8.3. A point of $\mathcal{M}(L, V, \psi)(S)$ is given by the following data:

(a) an abelian scheme A over S up to isogeny with an action $\iota : L \to \mathrm{End}^o A$,

(b) an $F^{\times}$-homogeneous polarization $\bar{\lambda}$ of A,

(c) a class $\bar{\eta}$ modulo $\mathbf{K}$ of $L \otimes \mathbb{A}_f$-module isomorphisms

$$\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A).$$

such that for each $\lambda \in \bar{\lambda}$ there is locally for the Zariski topology a constant $\xi(\lambda) \in (F \otimes \mathbb{A}_f)^{\times}(1)$ with

$$\psi(\xi(\lambda)v_1, v_2) = E^{\lambda}(\eta(v_1), \eta(v_2)).$$

(d) The L-module $H_1(A, \mathbb{Q})$ with its Riemann form defined by λ is isomorphic to (V, ψ), up to a factor in $F^{\times}$.

We require that the following condition holds

$$\mathrm{Tr}(\iota(\ell) \mid \mathrm{Lie}\, A) = \mathbf{t}(\ell), \quad \ell \in L.$$

We reformulate [De, 5.11] with our conventions.

Proposition 8.4. *Let (G, h) be a Shimura datum. Let $Z \subset G$ be the connected center of G. Let $\delta : \mathbb{S} \to Z_{\mathbb{R}}$ be a homomorphism. Let $E \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$ which contains the Shimura fields $E(G, h)$ and $E(Z, \delta)$. Let $\mathbf{K} \subset G(\mathbb{A}_f)$ be an open and compact subgroup. We denote by $\mathrm{Sh}_{\mathbf{K}}(G, h)$ and $\mathrm{Sh}_{\mathbf{K}}(G, h\delta)$ the quasi-canonical models over E. Let*

$$r_E^{\mathrm{cft}}(Z, \delta) : \mathrm{Gal}(\bar{E}/E) \to Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/(\mathbf{K} \cap Z(\mathbb{A}_f))$$

be the reciprocity law. There is an isomorphism of schemes over $\bar{E}$

$$\mathrm{Sh}_{\mathbf{K}}(G, h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E} \to \mathrm{Sh}_{\mathbf{K}}(G, h\delta) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E} \tag{8.10}$$

such that for each $\sigma \in \mathrm{Gal}(\bar{E}/E)$ the following diagram is commutative,

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{K}}(G, h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(G, h\delta) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E} \\ \Big\downarrow {\scriptstyle r_E^{\mathrm{cft}}(Z,\delta)(\sigma) \times \sigma_c} & & \Big\downarrow {\scriptstyle id \times \sigma_c} \\ \mathrm{Sh}_{\mathbf{K}}(G, h) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E} & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(G, h\delta) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \bar{E}. \end{array}$$

For varying $\mathbf{K}$ the morphism (8.10) is compatible with the Hecke operators induced by elements $g \in G(\mathbb{A}_f)$.

Proof. By the definition of a canonical model one can reduce the question to the case when G is an algebraic torus (cf. [De, 5.11]). Then the proposition is a consequence of (8.7). □

We formulate a "local" version of the last proposition, keeping the notations there. We fix a diagram as in (2.13)

$$\mathbb{C} \leftarrow \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p.$$

It determines a p-adic place ν of E. Let $\mu(\delta) : \mathbb{G}_{m,\mathbb{C}} \to Z_{\mathbb{C}}$ the homomorphism associated to δ as usual. It is defined over E_{ν},

$$\mu_{\nu} : \mathbb{G}_{m,E_{\nu}} \to Z_{E_{\nu}}.$$

We consider the homomorphism

$$\mathfrak{r}_{\nu} : \mathbb{G}_{m,E_{\nu}} \xrightarrow{\mu_{\nu}} Z_{E_{\nu}} \xrightarrow{\mathrm{Nm}_{E_{\nu}/\mathbb{Q}_p}} Z_{\mathbb{Q}_p}.$$

We define $r_{\nu}(Z, \delta)$ as the composite

$$r_{\nu}(Z, \delta) : E_{\nu}^{\times} \xrightarrow{\mathfrak{r}_{\nu}^{-1}} Z(\mathbb{Q}_p) \to Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/(\mathbf{K} \cap Z(\mathbb{A}_f)), \tag{8.11}$$

where the last arrow is induced by the inclusion $Z(\mathbb{Q}_p) \subset Z(\mathbb{A}_f)$. By local class field theory, this induces a homomorphism

$$r_\nu^{\mathrm{cft}}(Z,\delta) : \mathrm{Gal}(\bar{E}_\nu/E_\nu) \to Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/(\mathbf{K} \cap Z(\mathbb{A}_f)).$$

Corollary 8.5. *We denote by* $\mathrm{Sh}_{\mathbf{K}}(G,h)_{E_\nu}$ *and* $\mathrm{Sh}_{\mathbf{K}}(G,h\delta)_{E_\nu}$ *the schemes over* E_ν *obtained by base change from the canonical models. There is an isomorphism of schemes over* $\bar{E}_\nu$

$$\mathrm{Sh}_{\mathbf{K}}(G,h)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu \to \mathrm{Sh}_{\mathbf{K}}(G,h\delta)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu$$

such that for any $\sigma \in \mathrm{Gal}(\bar{E}_\nu/E_\nu)$ *the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{K}}(G,h)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(G,h\delta)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu \\ {\scriptstyle r_\nu^{\mathrm{cft}}(Z,\delta)(\sigma)\times\sigma_c}\downarrow & & \downarrow{\scriptstyle id\times\sigma_c} \\ \mathrm{Sh}_{\mathbf{K}}(G,h)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu & \longrightarrow & \mathrm{Sh}_{\mathbf{K}}(G,h\delta)_{E_\nu} \times_{\mathrm{Spec}\, E_\nu} \mathrm{Spec}\, \bar{E}_\nu. \end{array}$$

Proof. This follows from the compatibilities of local and global class field theory. □

Our final topic is the following variant of [De, Prop. 1.15]. A similar variant appears in Kisin [K, Lem. 2.1.2].

Proposition 8.6. *Let* S *be a finite set of prime numbers. Let* $M \subset G$ *be closed immersion of reductive subgroups over* $\mathbb{Q}$*. We assume that* M *is the kernel of a homomorphism* $G \to T$ *to a torus over* $\mathbb{Q}$*. Let* $h : \mathbb{S} \to M_{\mathbb{R}} \to G_{\mathbb{R}}$ *be a homomorphism of algebraic groups such that* (M,h) *and* (G,h) *are Shimura data. Let* $\mathbf{I} = \mathbf{I}_S\mathbf{I}^S$ *be an open and compact subgroup* $M(\mathbb{A}_f)$*, where* $\mathbf{I}_S \subset \prod_{p\in S} M(\mathbb{Q}_p)$ *and* $\mathbf{I}^S \subset M(\mathbb{A}_f^S)$*. Let* $\mathbf{K}_S \subset \prod_{p\in S} G(\mathbb{Q}_p)$ *be a compact open subgroup such that*

$$\mathbf{K}_S \cap (\textstyle\prod_{p\in S} M(\mathbb{Q}_p)) = \mathbf{I}_S.$$

Then there exists an open compact subgroup $\mathbf{K}^S \subset G(\mathbb{A}_f^S)$ *which contains* $\mathbf{I}^S$ *such that the induced morphism of schemes over* $\mathbb{C}$

$$\mathrm{Sh}_{\mathbf{I}}(M,h)_{\mathbb{C}} \to \mathrm{Sh}_{\mathbf{K}_S\mathbf{K}^S}(G,h)_{\mathbb{C}} \tag{8.12}$$

is an open and closed immersion.

Proof. Let Z_G be the center of G and let G^{der} be the derived group. The map $Z_G \times G^{\mathrm{der}} \to G$ is an isogeny. Since G^{der} is mapped to $\{1\} \in T$ we see that G and Z_G have the same image in T. We obtain that the homomorphism $Z_G \times M \to G$ is surjective. Therefore $Z_M \subset Z_G$. We obtain a morphism $M^{\mathrm{ad}} \to G^{\mathrm{ad}}$ which is an isomorphism. Let X_G be the set of conjugates of h by elements of $G(\mathbb{R})$ and define X_M in the same way. Since the adjoint groups are the same, the induced map $X_M \to X_G$ is an isomorphism onto a union of connected components of X_G. This implies that the Shimura varieties $\mathrm{Sh}_{\mathbf{I}}(M,h)_{\mathbb{C}}$ and $\mathrm{Sh}_{\mathbf{K}_S\mathbf{K}^S}(G,h)_{\mathbb{C}}$ have the same dimension. By [K] we may choose $\mathbf{K}^S$ in such a way that (8.12) is a closed immersion. But since both varieties are normal of the same dimension, the induced morphisms on the local rings must be isomorphisms. Therefore (8.12) is also open. □

The main ingredient of the proof in [K] is the following theorem. Because it is needed for other purposes in this paper, we state it here.

Proposition 8.7. *(Theorem of Chevalley) Let* T *be an algebraic torus over* $\mathbb{Q}$*. Let* $\mathcal{E} \subset T(\mathbb{Q})$ *be a finitely generated subgroup. Let* S *be a finite set of rational primes. We denote by* $\mathbb{A}_f^S$ *the restricted product over all* $\mathbb{Q}_\ell$ *where* ℓ *runs over all prime numbers* $\ell \notin S$*. Let* m *be an integer. Then there exists a compact open subgroup* $C \subset T(\mathbb{A}_f^S)$ *such that* $C \cap \mathcal{E} \subset \mathcal{E}^m$*.*

In the case where L is a number field and $T = \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{m,L}$, this is the first theorem in [Che]. The general case is easily reduced to this.

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