A Dieudonné Theory for p-Divisible Groups

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1 Introduction

Let k be a perfect field of characteristic $p > 0$. We denote by $W(k)$ the ring of Witt vectors. Let us denote by $\xi \to {}^F\xi$, $\xi \in W(k)$ the Frobenius automorphism of the ring $W(k)$. A Dieudonné module over k is a finitely generated free $W(k)$-module M equipped with an ^{F}-linear map $F : M \to M$, such that $pM \subset FM$. By a classical theorem of Dieudonné (compare Grothendieck [G]) the category of p-divisible formal groups over k is equivalent to the category of Dieudonné modules over k.

In this paper we will prove a totally similiar result for p-divisible groups over a complete noetherian local ring R with residue field k if either $p > 2$, or if $pR = 0$. For formal p-divisible groups (i.e. without étale part) this is done in [Z2].

We will now give a description of our result. Let R be as above but assume firstly that R is artinian. The maximal ideal of R will be denoted by $\mathfrak{m}$. The most important point is that we do not work with the Witt ring $W(R)$ but with a subring $\hat{W}(R) \subset W(R)$. This subring is characterized by the following properties: It is functorial in R. It is stable by the Frobenius endomorphism ^{F} and by the Verschiebung ^{V} of $W(R)$. We have $\hat{W}(k) = W(k)$. The canonical homomorphism $\hat{W}(R) \to W(k)$ is surjective, and its kernel consists exactly of the Witt vectors in $W(\mathfrak{m})$ with only finitely many non-zero components. The ring $\hat{W}(R)$ is a non-noetherian local ring with residue class field k. It is separated and complete as a local ring.

If R is an arbitrary complete local ring as above we set $\hat{W}(R) = \varprojlim \hat{W}(R/\mathfrak{m}^n)$.

Let us denote by $\hat{I}_R \subset \hat{W}(R)$ the ideal, which consists of all Witt vectors whose first component is zero.

Definition 1 *A Dieudonné display over R is a quadruple (P, Q, F, V^{-1}), where P is a finitely generated free $\hat{W}(R)$-module, $Q \subset P$ is a submodule and F and V^{-1} are ^{F}-linear maps $F : P \to P$, $V^{-1} : Q \to P$.*

The following properties are satisfied:

(i) $\hat{I}_R P \subset Q \subset P$ *and* P/Q *is a free* R*-module.*

(ii) $V^{-1} : Q \longrightarrow P$ *is an* ^{F}*-linear epimorphism.*

(iii) For $x \in P$ *and* $w \in \hat{W}(R)$*, we have*

$$V^{-1}({}^V wx) = wFx. \tag{1}$$

In contrast with Cartier theory there is no operator V in our theory. The strange notation V^{-1} is explained below by the relationship to Cartier's V. But there is a $\hat{W}(R)$-linear map:

$$V^{\sharp} : P \longrightarrow \hat{W}(R) \otimes_{F, \hat{W}(R)} P, \tag{2}$$

which is uniquely determined by the relation $V^{\sharp}(wV^{-1}y) = w \otimes y$ for $w \in \hat{W}(R)$ and $y \in Q$ (see [Z2] Lemma 1.5).

If $\mathcal{P}$ is a Dieudonné display over k, then the pair (P, F) is a Dieudonné module, and this defines an equivalence of categories.

Theorem: *There is a functor $\mathbb{D}$ from category of p-divisible groups over R to the category of Dieudonné displays over R, which is an equivalence of categories.*

Let X be a p-divisible group over R and let $\mathcal{P} = \mathbb{D}(X)$ be the associated Dieudonné display. Then $\text{height} X = \text{rank}_{\hat{W}(R)}(P)$. Moreover the tangent space of X is canonically identified with the R-module P/Q.

I stated this theorem as a conjecture during the p-adic Semester in Paris 1997. Faltings told me that I should prove it using proposition 19 below. We follow here his suggestion. In the proof we will restrict to an artinian ring R, because the general case is then obtained by a standard limit argument.

Other generalizations of Dieudonné theory are Cartier theory, and the crystalline Dieudonné theory, which was developed by Grothendieck, Messing, Berthelot, de Jong and others (compare de Jong [J]). Dieudonné displays are explicitly related to both of these theories. More precisely we construct functors from the category of Dieudonné displays to the category of crystals respectively to the category of Cartier modules. In particular this explains the relationship between Cartier theory and crystalline theory completely. So far this relationship was only understood in special cases (compare the introduction of Mazur and Messing [MM], and [Z3]).

We note that our theory works over rings with nilpotent elements, while the crystalline Dieudonné functor is not fully faithful in this case. From our point of view the reason for this failure of crystalline Dieudonné theory is that we can recover from the crystal associated to a Dieudonné display $\mathcal{P}$ the data P, Q, F but not the operator V^{-1}. On the other hand for a reduced ring the prime number p is a non-zero divisor in $\hat{W}(R)$ and therefore V^{-1} may be recovered from the relation $pV^{-1} = F$.

Let us explain the relationship to Cartier theory. Like a Cartier module a Dieudonné display may be defined by structural equations. Take any invertible matrix $(\alpha_{ij}) \in Gl_h(\hat{W}(R))$, and fix any number $0 \leq d \leq h$. We define a Dieudonné display $\mathcal{P} = (P, Q, F, V^{-1})$ as follows. We take for P the free $\hat{W}(R)$-module with the basis $e_1, \dots, e_h$. We set :

$$Q = \hat{I}_R e_1 \oplus \dots \oplus \hat{I}_R e_d \oplus \hat{W}(R) e_{d+1} \oplus \dots \oplus \hat{W}(R) e_h$$

The operators F and V^{-1} are uniquely determined by (1) and by the following relations:

$$\begin{aligned} F e_j &= \sum_{i=1}^{h} \alpha_{ij} e_i, \quad \text{for} \quad j = 1, \dots, d \\ V^{-1} e_j &= \sum_{i=1}^{h} \alpha_{ij} e_i \quad \text{for} \quad j = d+1, \dots, h \end{aligned} \tag{3}$$

Assume now for simplicity that R is an artinian ring. Let $\mathbb{E}_R$ be the local Cartier ring with respect to p (see [Z1]). Then we may consider in the free $\mathbb{E}_R$-module with basis $e_1, \dots, e_h$ the submodule generated by the elements:

$$\begin{aligned} F e_j - \sum_{i=1}^{h} \alpha_{ij} e_i, \quad \text{for} \quad j = 1, \dots, d \\ e_j - V(\sum_{i=1}^{h} \alpha_{ij} e_i) \quad \text{for} \quad j = d+1, \dots, h \end{aligned} \tag{4}$$

where F and V are now considered as elements of $\mathbb{E}_R$. The quotient by this submodule is the $\mathbb{E}_R$-module which Cartier associates to the connected component of the p-divisible group X with the Dieudonné display $\mathbb{D}(X) = \mathcal{P}$.

Finally we point out two questions, which we hope to answer in another paper.

Let X be a p-divisible group over R, and let $\mathcal{P}$ be the associated Dieudonné display. Then we cannot verify in general that the crystal we associate to X coincides with the crystal Messing [M] associates to X. By [Z2] this is true, if X is connected.

The other probably easier question is, whether our functor respects duality. In [Z2] we proved the following: If X is a connected p-divisible group, whose dual group X^t is also connected, the displays $\mathbb{D}(X)$ and $\mathbb{D}(X^t)$ are dual to each other. The same is then automatically true for the Dieudonné displays. It is not difficult to see that a positive answer to the first question gives also a positive answer to the second question.

2 Dieudonné Displays

Let R be an artinian local ring with perfect residue field k. There is a unique ring homomorphism $W(k) \to R$, which for any element $a \in k$, maps the Teichmüller representative $[a]$ of a in $W(k)$ to the Teichmüller representative of a in R. Let $\mathfrak{m} \subset R$ be the maximal ideal. Then we have the exact sequence

$$(5) \qquad 0 \longrightarrow W(\mathfrak{m}) \longrightarrow W(R) \xrightarrow{\pi} W(k) \longrightarrow 0.$$

It admits a canonical section $\delta : W(k) \to W(R)$, which is a ring homomorphism commuting with ^{F}. It may be deduced from the Cartier morphism [Z2] (2.39), but it has also the following explicit Teichmüller description: Let $x \in W(k)$. Then for any number n there is a unique solution of the equation ${}^{F^n}y_n = x$. Let $\tilde{y_n} \in W(R)$ be any lifting of y_n. Then for big n the element ${}^{F^n}\tilde{y_n}$ is independent of n and the lifting chosen, and is the desired $\delta(x)$.

Since $\mathfrak{m}$ is a nilpotent algebra we have a subalgebra of $W(\mathfrak{m})$:

$$\hat{W}(\mathfrak{m}) = \{(x_0, x_1, \dots) \in W(\mathfrak{m}) \quad | \quad x_i = 0 \quad \text{for almost all} \quad i\}$$

$\hat{W}(\mathfrak{m})$ is stable by ^{F} and ^{V}. Moreover $\hat{W}(\mathfrak{m})$ is an ideal in $W(R)$. Indeed, since any element in $\hat{W}(\mathfrak{m})$ may be represented as a finite sum $\sum_{i=1}^{N} {}^{V^i}[x_i]$, it is enough to show that $[x_0]\xi \in \hat{W}(\mathfrak{m})$, for $x_0 \in \mathfrak{m}, \quad \xi \in W(R)$. But this is obvious from the formula:

$$[x_0]\,(\xi_0, \xi_1, \ldots, \xi_i, \ldots) \quad = \quad \left(x_0\xi_0, x_0^p\xi_1, \ldots, x_0^{p^i}\xi_i, \ldots\right).$$

We may now define a subring $\hat{W}(R) \subset W(R)$:

$$\hat{W}(R) = \{\xi \in W(R) \quad | \quad \xi - \delta\pi(\xi) \in \hat{W}(\mathfrak{m})\}.$$

Again we have a split exact sequence

$$0 \longrightarrow \hat{W}(\mathfrak{m}) \longrightarrow \hat{W}(R) \xrightarrow{\pi} W(k) \longrightarrow 0,$$

with a canonical section δ of π.

Lemma 2 *Assume that the characteristic p of k is not 2, or that $2R = 0$. Then the subring $\hat{W}(R)$ of $W(R)$ is stable under ^{F} and ^{V}.*

Proof: Since δ commutes with ^{F}, the stability under ^{F} is obvious. For the stability under ^{V} one has to show that

$$(6) \qquad \delta({}^V x) - {}^V\delta(x) \in \hat{W}(\mathfrak{m}) \quad \text{for} \quad x \in W(k).$$

If we write $x = {}^F y$ and use that $\hat{W}(\mathfrak{m})$ is an ideal in $W(R)$, we see that it suffices to verify (6) for $x = 1$. For the proof we may replace R by $W_N(k)$ for a big number N. In $W(W(k))$ we have the following formula using logarithmic coordinates (compare (7) below, and [Z2] 2.11):

$$\delta({}^V 1) - {}^V\delta(1) = [{}^V 1, 0, \ldots, 0, \ldots] = [p, 0, \ldots].$$

Our assertion is, that the Witt components of this Witt vector in $W(W(k))$ converge to zero in the p-adic topology of $W(k)$ for $p \neq 2$ respectively that they become divisible by 2 in the case $p = 2$. We write

$$[p, 0, 0 \ldots] = (u_0, u_1, \ldots u_i, \ldots) \quad , \quad u_i \in {}^V W(k).$$

The u_i are determined by the equations

$$\begin{aligned} p &= u_0 \\ 0 &= \frac{u_0^p}{p} + u_1 \\ 0 &= \frac{u_0^{p^2}}{p^2} + \frac{u_1^p}{p} + u_2 \\ &\vdots \end{aligned}$$

An elementary induction shows $\operatorname{ord}_p u_n = p^n - p^{n-1} - \cdots - 1.$ *Q.E.D.*

We remark that in the case $pR = 0$ the section δ also commutes with ^{V} . Indeed, in this case taking the Teichmüller representative is a ring homomorphism $k \to R$. We obtain δ , if we apply the functor W to this homomorphism.

Since the ring $\hat{W}(R)$ has obviously all the properties mentioned in the introduction,i.e. the definition 1 has now a precise meaning.

We consider now a surjection $S \to R$ of artinian local rings with residue class field k as in the lemma. We assume that the kernel $\mathfrak{a}$ of the surjection is equipped with divided powers $\gamma_i : \mathfrak{a} \to \mathfrak{a}$. Then we have an exact sequence

$$0 \longrightarrow \hat{W}(\mathfrak{a}) \longrightarrow \hat{W}(S) \longrightarrow \hat{W}(R) \longrightarrow 0,$$

and the divided Witt polynomials define an injective homomorphism:

$$\hat{W}(\mathfrak{a}) \longrightarrow \mathfrak{a}^{(\mathbb{N})}, \tag{7}$$

If the divided powers are nilpotent in the sense that for a given element $a \in \mathfrak{a}$ the divided powers $\gamma_{p^k}(a)$ become zero for big k the homomorphism (7) becomes an isomorphism (compare [Z2] (3.4)). In this paper a pd-thickening is a triple (S, R, γ_i) , which satisfies this nilpotence condition. We write an element from the right hand side of (7) as $[a_0, ..., a_i, ...]$, where $a_i \in \mathfrak{a}$ are almost all zero. We call it a Witt vector in logarithmic coordinates.

The ideal $\mathfrak{a} \subset \hat{W}(S)$ is by definition the set of all elements of the form $[a, 0, \ldots, 0, \ldots]$, where $a \in \mathfrak{a}$. Let $\mathcal{P}$ be a Dieudonné display over S and $\overline{\mathcal{P}} = \mathcal{P}_R$ be its reduction over R . Let us denote by $\hat{Q}$ the inverse image of $\overline{Q}$ by the homomorphism

$$P \longrightarrow \overline{P} = \hat{W}(R) \otimes_{\hat{W}(S)} P.$$

Then $V^{-1}: Q \to P$ extends uniquely to $V^{-1}: \hat{Q} \to P$ such that $V^{-1}\mathfrak{a}P = 0$.

Theorem 3 *Let us consider a pd -thickening $S \to R$ as above. Let $\mathcal{P}_i = (P_i, Q_i, F, V^{-1})$ for $i = 1, 2$ be Dieudonné displays over S . Let $\overline{\mathcal{P}}_i = (\overline{P}_i, \overline{Q}_i, F, V^{-1}) = \mathcal{P}_{i,R}$ be the reductions over R . Assume we are given a morphism of Dieudonné displays $\overline{u}: \overline{\mathcal{P}}_1 \to \overline{\mathcal{P}}_2$. Then there exists a unique morphism of quadruples*

$$u: (P_1, \hat{Q}_1, F, V^{-1}) \longrightarrow (P_2, \hat{Q}_2, F, V^{-1}),$$

which lifts the morphism $\overline{u}$.

Proof: For the uniqueness it is enough to consider the case $\overline{u} = 0$. As in the proof of [Z2] lemma 1.34 one obtains a commutative diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{u} & \hat{W}(\mathfrak{a})P_2 \\ {\scriptstyle (V^N)^\#}\downarrow & & \uparrow{\scriptstyle (V^{-N})^\#} \\ \hat{W}(S) \otimes_{F^N, \hat{W}(S)} P_1 & \xrightarrow{1\otimes u} & \hat{W}(S) \otimes_{F^N, \hat{W}(S)} \hat{W}(\mathfrak{a})P_2 \end{array}$$

Since $V^{-N}[a_0, a_1, \ldots]x = [a_N, a_{N+1} \ldots]F^N x$, for $[a_0, \ldots] \in \hat{W}(\mathfrak{a})$ and $x \in P_2$, any given element of $\hat{W}(\mathfrak{a})P_2$ is annihilated by V^{-N} for big N . Since P_1 is finitely generated it follows that $V^{-N}u = 0$ for big N . Then the diagram shows $u = 0$ which proves the uniqueness.

As in the proof of [Z2] theorem 2.5 it is enough to consider the case where $\overline{\mathcal{P}}_1 = \overline{\mathcal{P}}_2 = \overline{\mathcal{P}}$ and $\overline{u}$ is the identity, if one wants to prove the existence of u . One simply repeats the proof of [Z2] theorem 2.3 with $\hat{W}$ instead of W . The proof goes through without changing a word up to the last argument showing the nilpotency of the operator U defined by loc.cit. (2.16).

To complete the proof we have to show that for any ^{F} -linear map $\widetilde{\omega}: L_1 \to p^N\hat{W}(\mathfrak{a})/p^{N+1}\hat{W}(\mathfrak{a}) \otimes_{W(S)} P_2$ there exists a number m , such that $U^m\widetilde{\omega} = 0$.

To see this we consider the following F^{m+1} -linear map

(8)

$$\tau_m: L_1 \xrightarrow{\widetilde{\omega}} p^N\hat{W}(\mathfrak{a})/p^{N+1}\hat{W}(\mathfrak{a}) \otimes_{\hat{W}(S)} P_2 \xrightarrow{V^{-m}} p^N\hat{W}(\mathfrak{a})/p^{N+1}\hat{W}(\mathfrak{a}) \otimes_{\hat{W}(S)} P_2$$

By definition $U^m\widetilde{\omega}$ factors through the ^{F}-linear map obtained from τ_m by partial linearization to an ^{F}-linear map:

$$\hat{W}(S) \otimes_{F^m, \hat{W}(S)} L_1 \xrightarrow{1 \otimes \tau_m} p^N \hat{W}(\mathfrak{a})/p^{N+1}\hat{W}(\mathfrak{a}) \otimes_{\hat{W}(S)} P_2.$$

But as in the proof of the uniqueness any given element of $p^N\hat{W}(\mathfrak{a})/p^{N+1}\hat{W}(\mathfrak{a}) \otimes_{\hat{W}(S)} P_2$ is annihilated by some power of V^{-1}. Since L_1 is a finitely generated $\hat{W}(S)$-module, it follows that τ_m is zero for big m. This proves $U^m\widetilde{\omega} = 0$ for big m. *Q.E.D.*

Theorem 3 gives the possibility to associate a crystal to a Dieudonné display: Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a Dieudonné display over R. Let $S \to R$ be a pd-thickening. Then we define a functor on the category of pd-thickenings:

$$\mathcal{K}_{\mathcal{P}}(S) = \tilde{P},$$

where $\tilde{\mathcal{P}} = (\tilde{P}, \tilde{Q}, F, V^{-1})$ is any lifting of $\mathcal{P}$ to S. The theorem 3 assures, that $\tilde{P}$ is unique up to a canonical isomorphism. This functor is called the Witt crystal. We also define the Dieudonné crystal:

$$\mathcal{D}_{\mathcal{P}}(S) = S \otimes_{\mathbf{w}_0, \hat{W}(S)} \mathcal{K}_{\mathcal{P}}(S)$$

The filtration:

$$Q/\hat{I}_R P \subset P/\hat{I}_R P = \mathcal{D}_{\mathcal{P}}(R)$$

is called the Hodge filtration. The following statement is similiar to a result of Grothendieck and Messing in crystalline Dieudonné theory.

Theorem 4 *Let $\mathcal{C}$ be the category of all pairs $(\mathcal{P}, Fil)$, where $\mathcal{P}$ is a Dieudonné display over R and $Fil \subset \mathcal{D}_{\mathcal{P}}(S)$ is a direct summand, which lifts the Hodge filtration of $\mathcal{D}_{\mathcal{P}}(R)$. Then the category $\mathcal{C}$ is canonically isomorphic to the category of Dieudonné displays over S.*

This follows immediately from theorem 3 (compare [Z2] 2.2).

To a Dieudonné display $\mathcal{P} = (P, Q, F, V^{-1})$ we may associate a 3n-display $\mathcal{F}(\mathcal{P}) = (P', Q', F, V^{-1})$, where we set $P' = W(R) \otimes_{\hat{W}(R)} P$. The submodule Q' is defined to be the kernel of the natural map $W(R) \otimes_{\hat{W}(R)}$

$P \to P/Q$. The operators F and V^{-1} for $\mathcal{F}(\mathcal{P})$ are uniquely determined by the relations:

$$\begin{array}{rcll} F(\xi \otimes x) & = & {}^F\xi \otimes Fx & \xi \in W(R),\ x \in P \\ V^{-1}(\xi \otimes y) & = & {}^F\xi \otimes V^{-1}y & y \in Q \\ V^{-1}({}^V\xi \otimes x) & = & \xi \otimes Fx & \end{array}$$

We call a Dieudonné display $\mathcal{P}$ over R V -nilpotent, if $\mathcal{F}(\mathcal{P})$ is a display in the sense of [Z2] 1.6. We recall that this is also equivalent to the following condition. Let $\mathcal{P}_k = (P_k, Q_k, F, V^{-1})$ be the Dieudonné display obtained by base change to k . Then the operator $V = pF^{-1} : P_k \to P_k$ is topologically nilpotent for the p -adic topology.

If $\mathcal{P}$ is V -nilpotent a Dieudonné crystal $\mathcal{D}_{\mathcal{F}(\mathcal{P})}(S)$ was defined in [Z2] 2.6. The trivial statement that the functor $\mathcal{F}$ respects liftings leads to a canonical isomorphism:

$$\mathcal{D}_{\mathcal{P}}(S) \cong \mathcal{D}_{\mathcal{F}(\mathcal{P})}(S) \tag{9}$$

Theorem 5 *The functor $\mathcal{F}$ is an equivalence of the category of V -nilpotent Dieudonné displays over R with the category of displays over R .*

Proof: If $R = k$ the functor $\mathcal{F}$ is the identical functor. By induction it suffices to prove the following. Let $S \to R$ be a pd-thickening and assume that the theorem holds for R . Then the theorem holds for S . But the category of Dieudonné displays over S is decribed from the category of Dieudonné displays over R and the Dieudonné crystal. Since the same description holds for displays by [Z2] 2.7 , we can do by (9) the induction step. *Q.E.D.*

Corollary 6 *The category of p-divisible formal groups over R is equivalent to the category of V -nilpotent Dieudonné displays over R*

This is clear because the corresponding theorem holds for displays by [Z2] theorem 3.21. The equivalence of the corollary is given by the functor which associates to a V -nilpotent Dieudonné display the p -divisible group $BT(\mathcal{F}(\mathcal{P}))$. Let us describe this functor which will be simply denoted by $BT(\mathcal{P})$.

Let X be a p -divisible group over R . It is an inductive limit of finite schemes over $\operatorname{Spec} R$. Hence we have a fully faithful embedding of the category of p-divisible groups to the category of functors from the category of finite R-algebras to the category of abelian groups. We describe $BT(\mathcal{P})$ by giving this functor.

Proposition 7 *Let $\mathcal{P}$ be a V-nilpotent Dieudonné display over R, and let S be a finite R-algebra. Let $\mathcal{P}_S = (P_S, Q_S, F, V^{-1})$ be the Dieudonné display obtained by base change. Then we have an exact sequence:*

$$0 \to Q_S \xrightarrow{V^{-1}-id} P_S \longrightarrow BT(\mathcal{P})(S) \to 0$$

Proof: First of all we note that S is a direct product of local artinian algebras satifying the same assumptions as R. Therefore the notion of a Dieudonné display makes sense over S. Moreover we may assume that S is local with maximal ideal $\mathfrak{m}_S$. In [Z2] we have considered $BT(\mathcal{P})$ as a functor on the category Nil_R of nilpotent R-algebras. In this sense we have:

$$BT(\mathcal{P})(S) = BT(\mathcal{P})(\mathfrak{m}_S)$$

Let $P_{\mathfrak{m}_S} = \hat{W}(\mathfrak{m}_S) \otimes_{\hat{W}(R)} P \subset P_S$. We set $Q_{\mathfrak{m}_S} = P_{\mathfrak{m}_S} \cap Q_S$. Then [Z2] theorem 3.2 tells us that there is an exact sequence:

$$0 \to Q_{\mathfrak{m}_S} \xrightarrow{V^{-1}-\mathrm{id}} P_{\mathfrak{m}_S} \longrightarrow BT(\mathcal{P})(S) \to 0$$

Let k_S be the residue class field of S, and $\mathcal{P}_{k_S}$ the display obtained by base change. Then we have $P_S/P_{\mathfrak{m}_S} = P_{k_S}$ and $Q_S/Q_{\mathfrak{m}_S} = Q_{k_S}$. Hence the proposition follows, if we show that the map $V^{-1} - \mathrm{id} : Q_{k_S} \to P_{k_S}$ is bijective. Indeed, because V is topologically nilpotent on P_{k_S} for the p-adic topology, the operator $-V - V^2 - V^3 - \ldots$ is an inverse. *Q.E.D.*

3 The Multiplicative Part and the Étale Part

For a p-divisible group G over an artinian ring there is an exact sequence:

$$0 \to G^c \longrightarrow G \longrightarrow G^{et} \to 0$$

Here G^c is a connected p-divisible group and G^{et} is an étale p-divisible group. The aim of this section is to show that the same result holds for Dieudonné displays.

Let us first recall a well–known lemma of Fitting (Lazard [L] VI 5.7):

Lemma 8 *Let A be a commutative ring and $\tau : A \to A$ a ring automorphism. Let M be an A-module of finite length and $\varphi : M \to M$ be a τ-linear endomorphism. Then M admits a unique decomposition*

$$M = M^{\mathrm{bij}} \oplus M^{\mathrm{nil}},$$

such that φ leaves the submodules M^{bij} and M^{nil} stable, and such that φ is a bijection on M^{bij} and operates nilpotently on M^{nil}.

We omit the proof, but we remark that M^{bij} and M^{nil} are given by the following formulas:

$$M^{\mathrm{bij}} = \bigcap_{n \in \mathbb{N}} \operatorname{Image} \varphi^n, \qquad M^{\mathrm{nil}} = \bigcup_{n \in \mathbb{N}} \operatorname{Ker} \varphi^n. \tag{10}$$

Here $\operatorname{Image} \varphi^n$ is an A-module because τ is surjective. In order to deal with a more general situation we add two complements to this lemma.

Let A be a commutative ring and $\mathfrak{a} \subset A$ an ideal, which consists of nilpotent elements. We set $A_0 = A/\mathfrak{a}$ and more generally we denote for an A-module M the A_0-module $M/\mathfrak{a}M$ by M_0. Let $\tau : A \to A$ be a ring homomorphism, such that $\tau(\mathfrak{a}) \subset \mathfrak{a}$, and such that there exists a natural number r with $\tau^r(\mathfrak{a}) = 0$. We denote by $\tau_0 : A_0 \to A_0$ the ring homomorphism induced by τ.

Lemma 9 *Let P be a finitely generated projective A-module and $\varphi : P \to P$ be a τ-linear endomorphism. Then φ induces a τ_0-linear endomorphism $\varphi_0 : P_0 \to P_0$ of the A_0-module P_0.*

Let E_0 be a direct summand of P_0, such that φ_0 induces a τ_0-linear isomorphism.

$$\varphi_0 : E_0 \longrightarrow E_0.$$

Then there exists a direct summand $E \subset P$, which is uniquely determined by the following properties:

(i) $\varphi(E) \subseteq E$.

(ii) *E lifts E_0.*

(iii) *$\varphi : E \to E$ is a τ-linear isomorphism.*

(*iv*) *Let* C *be an* A *-module, which is equipped with a* τ *-linear isomorphism* $\psi : C \to C$. *Let* $\alpha : (C, \psi) \to (P, \varphi)$ *be an* A *-module homomorphism such that* $\alpha \circ \psi = \varphi \circ \alpha$. *Let us assume that* $\alpha_0(C_0) \subset E_0$. *Then we have* $\alpha(C) \subset E$.

Proof: By our assumption on r we have an isomorphism

$$A \otimes_{\tau^r, A} P = A \otimes_{\tau^r, A_0} P_0.$$

We define E to be the image of the A -module homomorphism

$$(\varphi^r)^{\#} : A \otimes_{\tau^r, A_0} E_0 \longrightarrow P. \tag{11}$$

It follows immediately that $\varphi(E) \subset E$.

Let us prove that E is a direct summand of P . We choose a A_0 -submodule $F_0 \subset P_0$, which is complementary to E_0 :

$$P_0 = E_0 \oplus F_0.$$

Then we lift F_0 to a direct summand F of P . We consider the map induced by (11)

$$(\varphi^r)^{\#} : A \otimes_{\tau^r, A_0} E_0 \longrightarrow P/F. \tag{12}$$

By assumption the last map becomes an isomorphism, when tensored with $A_0 \otimes_A$. Hence we conclude by the lemma of Nakayama that (12) is an isomorphism. We see that E is a direct summand:

$$P = E \oplus F$$

Applying Nakayama's lemma to the projective and finitely generated module E , we obtain that:

$$\varphi^{\#} : A \otimes_{\tau, A} E \longrightarrow E$$

is an isomorphim.

Therefore we have checked the properties $(i) - (iii)$. The last property follows from the commutative diagram:

$$\begin{array}{ccccc} A \otimes_{\tau^r, A_0} E_0 & \longrightarrow & E & \longrightarrow & P \\ \uparrow {\scriptstyle 1 \otimes \alpha_0} & & & & \uparrow {\scriptstyle \alpha} \\ A \otimes_{\tau^r, A_0} C_0 & & \xrightarrow{\quad \sim \quad} & & C \end{array} \qquad Q.E.D.$$

We have also a dual form of the last lemma.

Lemma 10 *Let* A, A_0, τ, τ_0 *be as before. Let* P *be a finitely generated projective* A *-module and*

$$\varphi : P \longrightarrow A \otimes_{\tau, A} P$$

be a homomorphism of A *-modules. Let* $E_0 \subset P_0$ *be a direct summand of the* A_0 *-module* P_0 *, such that* φ_0 *induces an isomorphism*

$$P_0/E_0 \longrightarrow A_0 \otimes_{\tau_0, A_0} P_0/E_0.$$

Then there exists a direct summand $E \subset P$ *of the* A *-module* P *, which is uniquely determined by the following properties:*

(i) $\varphi(E) \subset A \otimes_{\tau, A} E.$

(ii) E *lifts* $E_0.$

(iii) $\varphi : P/E \longrightarrow A \otimes_{\tau, A} P/E$ *is an isomorphism.*

(iv) Let C *be any* A *-module, which is equipped with an isomorphism* $\psi : C \longrightarrow A \otimes_{\tau, A} C$ *. Let* $\alpha : P \longrightarrow C$ *be an* A *-module homomorphism, such that* E_0 *is in the kernel of* α_0 *. Then* E *is in the kernel of* α *.*

Proof: The proof is obtained by dualizing the last lemma with the functor $\mathrm{Hom}_A(-, A)$, except for the property (iv) . We omit the details, but we write down explicitly the definition of E . Let r be such that $\tau^r(\mathfrak{a}) = 0$. From the isomorphism $A \otimes_{\tau^r, A} P = A \otimes_{\tau^r, A_0} P_0$ we obtain a map

$$\varphi^r : P \longrightarrow A \otimes_{\tau^r, A_0} P_0/E_0.$$

Then E is the kernel of this map. *Q.E.D.*

We will apply these lemmas in the situation where $A = \hat{W}(R), \quad A_0 = W(k)$ and τ is the Frobenius endomorphism of $\hat{W}(R)$, i.e. $\tau w = {}^F w$. For this we have to convince ourself that the kernel $\hat{W}(\mathfrak{m})$ of the map $\hat{W}(R) \rightarrow W(k)$ is nilpotent and that ${}^{F^r}\hat{W}(\mathfrak{m}) = 0$ for a sufficiently big number r . By induction it is enough to prove that for a surjection of artinian rings $S \rightarrow R$ with kernel $\mathfrak{b}$, such that $p\mathfrak{b} = \mathfrak{b}^2 = 0$, we have $\hat{W}(\mathfrak{b})^2 =^F \hat{W}(\mathfrak{b}) = 0$. This we know. Hence the lemmas are applicable and give the following:

Proposition 11 *Let* P *be a finitely generated projective* $\hat{W}(R)$ *-module and* $\varphi : P \rightarrow P$ *be an* ^{F} *-linear homomorphism.*

Then there exists a uniquely determined direct summand $P^{\mathrm{mult}} \subset P$ *with the following properties* *(i)* φ *induces an* ^{F} *-linear isomorphism:*

$$\varphi : P^{\mathrm{mult}} \longrightarrow P^{\mathrm{mult}}$$

(ii) Let M *be any* $\hat{W}(R)$ *-module and* $\psi : M \to M$ *be an* ^{F} *-linear isomorphism. Let* $\alpha : M \to P$ *be a homomorphism of* $\hat{W}(R)$ *-modules, such that* $\alpha \circ \psi = \varphi \circ \alpha$ *. Then* α *factors through* P^{mult} *.*

Proof: Let us begin with the case $R = k$. For any natural number n the Frobenius induces an isomorphism ${}^F : W_n(k) \to W_n(k)$. Therefore Fitting's lemma is applicable to $P_n = W_n(k) \otimes_{W(k)} P$ and $\varphi_n = W_n(k) \otimes_{W(k)} \varphi$. In the notation of that lemma we set $P^{\mathrm{mult}} = \varprojlim_n P_n^{\mathrm{bij}}$. From the definition of P_n^{bij} (see (10)) it follows that $\alpha_n : M_n \to P_n$ factors through P_n^{bij} . Hence $\alpha(M) \subset P^{\mathrm{mult}} + p^n P$ for any n , which proves (ii) .

Let us now consider the general case. We set $P_0 = W(k) \otimes_{\hat{W}(R)} P$. Then we have already proved the existence of P_0^{mult} . We lift P_0^{mult} by the lemma (9) to a direct summand P^{mult} of P . Then that lemma states that P^{mult} has the desired properties. *Q.E.D.*

Proposition 12 *Let* R *be as in the last proposition. Let* P *be a finitely generated projective* $\hat{W}(R)$ *-module and let* $\varphi : P \to \hat{W}(R) \otimes_{F,\hat{W}(R)} P$ *be a* $\hat{W}(R)$ *-module homomorphism. Then there exists a projective factor module* P^{et} *of* P *, which is uniquely determined by the following properties.*

(i) φ *induces an isomorphism of* $\hat{W}(R)$ *-modules:*

$$\varphi : P^{\mathrm{et}} \longrightarrow \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}}$$

(ii) Let M *be a* $\hat{W}(R)$ *-module and* $\psi : M \to \hat{W}(R) \otimes_{F,\hat{W}(R)} M$ *be an isomorphism. Let* $\alpha : P \to M$ *be a homomorphism of* $\hat{W}(R)$ *-modules, such that* $(id \otimes \alpha) \circ \varphi = \psi \circ \alpha$ *. Then* α *factors through* P^{et} *.*

Proof: Again we begin with the case $R = k$. Then ${}^F : W(k) \to W(k)$ is bijective. We denote its inverse by τ . Then we have a τ -linear isomorphism:

$$\begin{array}{ccc} W(k) \otimes_{F,W(k)} P & \longrightarrow & P \\ w \otimes x & \longmapsto & \tau(w)x \end{array}$$

Hence we consider φ as a τ-linear map:

$$\varphi : P \longrightarrow P.$$

To this map we apply Fitting's lemma as in the last proposition. We obtain the decomposition $P = P^{\mathrm{bij}} \oplus P^{\mathrm{nil}}$. Then we set $P^{\mathrm{et}} = P^{\mathrm{bij}}$, and we obtain the lemma for $W(k)$.

The general case is obtained, if we apply the lemma 10 to the situation $A = W(R), A_0 = W(k)$ and $\tau w = {}^F w$ for $w \in W(R)$. *Q.E.D.*

We will now define the étale part and the multiplicative part of a Dieudonné display over R.

Definition 13 *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a Dieudonné display over R. We say that $\mathcal{P}$ is étale, if one of the following equivalent conditions is satisfied:*

(i) $P = Q$.

(ii) $V^{\#} : P \longrightarrow \hat{W}(R) \otimes_{F,\hat{W}(R)} P$ *is an isomorphism.*

Proof: Assume (i) is fulfilled. Then we have for any $x \in P$ the formula $V^{\#}(\xi V^{-1} x) = \xi \otimes x$, where $\xi \in \hat{W}(R)$. This implies that $V^{\#}$ is surjective, and hence an isomorphism. Conversly if $V^{\#}$ is surjective, we consider the composite of the following surjections:

$$\hat{W}(R) \otimes_{F,\hat{W}(R)} Q \overset{(V^{-1})^{\#}}{\longrightarrow} P \overset{V^{\#}}{\longrightarrow} \hat{W}(R) \otimes_{F,\hat{W}(R)} P.$$

Since the composite is by (2) induced by the inclusion $Q \subset P$, we conclude

$$\hat{W}(R) \otimes_{F,\hat{W}(R)} P/Q = (\hat{W}(R)/\ {}^F \hat{I}_R) \otimes_{F,R} P/Q = 0.$$

But since P/Q is a projective R-module this implies $P/Q = 0$. Indeed ${}^F I_R = p \cdot \hat{W}(R)$ and p is not a unit in $\hat{W}(R)$. *Q.E.D.*

Definition 14 *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a Dieudonné display over R. We say that $\mathcal{P}$ is of multiplicative type, if one of the following equivalent conditions is satisfied:*

(i) $Q = I_R P$.

(ii) $F^{\#} : \hat{W}(R) \otimes_{F,\hat{W}(R)} P \to P$ *is an isomorphism.*

Proof: The first condition implies that P is generated by elements of the form $V^{-1}({}^V\xi x) = \xi Fx, \xi \in \hat{W}(R), x \in P$. This implies the second condition.

Assume that the second condition holds. The image of a normal decomposition $P = L \oplus T$ by $F^{\#}$ gives a direct decomposition

$$P = \hat{W}(R)pV^{-1}L \oplus \hat{W}(R)FT.$$

Comparing this with the standard decomposition

$$P = \hat{W}(R)V^{-1}L \oplus \hat{W}(R)FT,$$

we obtain $p \cdot \hat{W}(R)V^{-1}L = \hat{W}(R)V^{-1}L$. Hence again since p is not a unit, we have $\hat{W}(R)V^{-1}L = 0$. This implies $L = 0$. *Q.E.D.*

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a Dieudonné display. Recall that $\mathcal{P}$ is called V-nilpotent, if the following map is zero for big numbers N:

$$(V^N)^{\#} : P \longrightarrow R \otimes_{F^N, \hat{W}(R)} P$$

The Dieudonné display is called F-nilpotent if the following map is zero for big numbers N:

$$(F^N)^{\#} : \hat{W}(R) \otimes_{F^N, \hat{W}(R)} P_2 \longrightarrow R \otimes_{\hat{W}(R)} P_2$$

Proposition 15 *Let $\alpha : \mathcal{P}_1 = (P_1, Q_1, F, V^{-1}) \to \mathcal{P}_2 = (P_2, Q_2, F, V^{-1})$ be a homomorphism of Dieudonné displays. Then α is zero, if one of the following conditions is satisfied.*

(i) One of the Dieudonné displays $\mathcal{P}_1$ and $\mathcal{P}_2$ is étale and the other is V-nilpotent.

(ii) One of the Dieudonné displays $\mathcal{P}_1$ and $\mathcal{P}_2$ is of multiplicative type and the other is F-nilpotent.

Proof: By rigidity (i.e. the uniqueness assertion of theorem 3) it is easy to reduce this proposition to the case, where $R = k$ is a perfect field. In this case the proposition is well known. *Q.E.D.*

Proposition 16 *Let $\mathcal{P}$ be a Dieudonné display over R. Then there is a morphism $\mathcal{P} \rightarrow \mathcal{P}^{\mathrm{et}}$ to an étale Dieudonné display over R, such that any other morphism to an étale Dieudonné display $\mathcal{P} \rightarrow \mathcal{P}_1$ factors uniquely through $\mathcal{P}^{\mathrm{et}}$. Moreover $\mathcal{P}^{\mathrm{et}}$ has the following properties:*

1) *The induced map $P \rightarrow P^{\mathrm{et}}$ is surjective.*

2) *Let P^{nil} be the kernel of $P \rightarrow P^{\mathrm{et}}$. Then $(P^{\mathrm{nil}}, P^{\mathrm{nil}} \cap Q, F, V^{-1})$ is a V-nilpotent Dieudonné display, which we will denote by $\mathcal{P}^{\mathrm{nil}}$.*

Proof: The map $V^{\#} : P \rightarrow \hat{W}(R) \otimes_{F,\hat{W}(R)} P$ determines by proposition 12 a projective factor module $P \xrightarrow{\alpha} P^{\mathrm{et}}$, such that $V^{\#}$ induces an isomorphism $P^{\mathrm{et}} \rightarrow \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}}$. We consider the inverse map

$$V^{-1\#} : \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}} \longrightarrow P^{\mathrm{et}}.$$

It is induced by an ^{F}-linear map $V^{-1} : P^{\mathrm{et}} \rightarrow P^{\mathrm{et}}$. We set $Q^{\mathrm{et}} = P^{\mathrm{et}}$ and $F = pV^{-1} : P^{\mathrm{et}} \rightarrow P^{\mathrm{et}}$. Then we obtain a Dieudonné display $\mathcal{P}^{\mathrm{et}} = (P^{\mathrm{et}}, Q^{\mathrm{et}}, F, V^{-1})$. We will now check that the map $\alpha : P \rightarrow P^{\mathrm{et}}$ induces a homomorphism of displays $\mathcal{P} \rightarrow \mathcal{P}^{\mathrm{et}}$. To see that α commutes with F we consider the following diagram:

$$\begin{array}{ccccc} \hat{W}(R) \otimes_{F,\hat{W}(R)} P & \xrightarrow{F^{\#}} & P & \xrightarrow{V^{\#}} & \hat{W}(R) \otimes_{F,\hat{W}(R)} P \\ \downarrow{\scriptstyle 1\otimes_F \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1\otimes_F \alpha} \\ \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}} & \xrightarrow{F^{\#}} & P^{\mathrm{et}} & \xrightarrow{V^{\#}} & \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}}. \end{array}$$

the right hand square is commutative by definition. Our assertion is that the left hand square is commutative. Since $V^{\#}$ for P^{et} is an isomorphism it is enough to show that the diagram becomes commutative, if we delete the vertical arrow in the middle. But this is trivial because $V^{\#} \circ F^{\#} = p$. Since we have trivially $\alpha(Q) \subset P^{\mathrm{et}}$ it only remains to be checked that α commutes with V^{-1}. For this we consider the diagram

$$\begin{array}{ccccc} \hat{W}(R) \otimes_{F,\hat{W}(R)} Q & \xrightarrow{(V^{-1})^{\#}} & P & \xrightarrow{V^{\#}} & \hat{W}(R) \otimes_{F,\hat{W}(R)} P \\ \downarrow{\scriptstyle 1\otimes_F \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1\otimes_F \alpha} \\ \hat{W}(R) \otimes_{F,\hat{W}(R)} P^{\mathrm{et}} & \xrightarrow{(V^{-1})^{\#}} & P^{\mathrm{et}} & \xrightarrow{V^{\#}} & \hat{W}(R) \otimes_{F,\hat{W}(R)} P \end{array}$$

By (2) the composition of the arrows in the first horizontal row is induced by the inclusion $Q \subset P$, while the composition in the lower horizontal row is the identity. We deduce the commutativity of the first square as before.

Hence we have a morphism of Dieudonné displays $\alpha : \mathcal{P} \to \mathcal{P}^{\mathrm{et}}$. the proposition is known for $R = k$ and in fact easily deduced from Fitting's lemma. In this case V exists and $Q = VP$. It follows that the map $Q \to P^{\mathrm{et}}$ is surjective. In general we conclude the same by Nakayama's lemma. To show that $(P^{\mathrm{nil}}, P^{\mathrm{nil}} \cap Q, F, V^{-1})$ defined in the proposition is a Dieudonné display, it remains to be shown that $P^{\mathrm{nil}}/P^{\mathrm{nil}} \cap Q$ is a projective R-module. But because of the sujective map $Q \to P^{\mathrm{et}}$ we have an isomorphism $P^{\mathrm{nil}}/P^{\mathrm{nil}} \cap Q \simeq P/Q$. The universality of $\mathcal{P} \to \mathcal{P}^{\mathrm{et}}$ is an immediate consequence of proposition 12. *Q.E.D.*

Dually to the last proposition we have:

Proposition 17 *Let $\mathcal{P}$ be a Dieudonné display over R. Then there is a morphism from a multiplicative Dieudonné display $\mathcal{P}^{\mathrm{mult}} \to \mathcal{P}$, such that any other morphism $\mathcal{P}_1 \to \mathcal{P}$ from a multiplicative Dieudonné display $\mathcal{P}_1$ factors uniquely as $\mathcal{P}_1 \to \mathcal{P}^{\mathrm{mult}} \to \mathcal{P}$. Moreover $\mathcal{P}^{\mathrm{mult}}$ has the following properties:*

1) *The map $P^{\mathrm{mult}} \to P$ is injective and $P^{\mathrm{mult}} \cap Q = I_R P^{\mathrm{mult}}$.*

2) *$(P/P^{\mathrm{mult}}, Q/I_R P^{\mathrm{mult}}, F, V^{-1})$ is an F-nilpotent Dieudonné display.*

Proof: We consider the map $F : P \to P$, and we define the direct summand $P^{\mathrm{mult}} \subset P$ according to proposition 11. We define $Q^{\mathrm{mult}} = I_R P^{\mathrm{mult}}$, and obtain a Dieudonné display $\mathcal{P}^{\mathrm{mult}} = (P^{\mathrm{mult}}, Q^{\mathrm{mult}} F, V^{-1})$, which has the required universal property. To prove 1) we consider a normal decomposition $P = L \oplus T$. Let $\overline{P} = W(k) \otimes_{\hat{W}(R)} P$ and let $\overline{L}$ and $\overline{P}^{\mathrm{mult}}$ be the images of L and P^{mult}. Since the proposition is known (and easy to prove by Fitting's lemma) for $R = k$, it follows that $\overline{L} \oplus \overline{P}^{\mathrm{mult}}$ is a direct summand of $\overline{P}$. By Nakayama's lemma one verifies that $L \oplus P^{\mathrm{mult}}$ is a direct summand of P. Hence we may assume without loss of generality that P^{mult} is a direct summand of T. From this 1) and 2) follow immediately, except for the F-nilpotence, which may be reduced to the case $k = R$. *Q.E.D.*

4 The p-Divisible Group of a Dieudonné Display

In this section we will extend the functor BT of proposition 7 to the category of all Dieudonné displays, and show that this defines an equivalence of categories.

Let R be an artinian local ring with perfect residue class field k, satisfying the assumptions in the introduction. We will denote by $\overline{R}$ the unramified extension of R, such that $\overline{R}$ is local and has residue class field $\overline{k}$ the algebraic closure of k. We will write $\Gamma = \mathrm{Gal}(\overline{k}/k)$ for the Galois group. Then Γ acts continuously on the discrete module $\overline{R}$.

Let H be a finitely generated free $\mathbb{Z}_p$-module. Assume we are given an action of Γ on H, which is continuous with respect to the p-adic topology on H. The actions of Γ on $\hat{W}(\overline{R})$ and H induce an action on $\hat{W}(\overline{R}) \otimes_{\mathbb{Z}_p} H$. We set:

$$P(H) = (\hat{W}(\overline{R}) \otimes_{\mathbb{Z}_p} H)^{\Gamma}$$

One can show by reduction to the case $R = k$ that $P(H)$ is a finitely generated free $\hat{W}(R)$-module and that the natural map:

$$\hat{W}(\overline{R}) \otimes_{\hat{W}(R)} P(H) \to \hat{W}(\overline{R}) \otimes_{\mathbb{Z}_p} H$$

is an isomorphism. We define an étale Dieudonné display over R:

$$\mathcal{P}(H) = (P(H), Q(H), F, V^{-1})$$

Here $P(H) = Q(H)$ and V^{-1} is induced by the map:

$$\begin{gathered} \hat{W}(\overline{R}) \otimes_{\mathbb{Z}_p} H \to \hat{W}(\overline{R}) \otimes_{\mathbb{Z}_p} H \\ w \otimes h \mapsto {}^{F}w \otimes h \end{gathered}$$

Conversely if $\mathcal{P}$ is an étale Dieudonné display over R we define $H(\mathcal{P})$ to be the kernel of the homomorphism of $\mathbb{Z}_p$-modules

$$V^{-1} - id : \hat{W}(\overline{R}) \otimes_{\hat{W}(R)} P \to \hat{W}(\overline{R}) \otimes_{\hat{W}(R)} P.$$

Hence the category of étale Dieudonné displays over R is equivalent to the category of continuous $\mathbb{Z}_p[\Gamma]$-modules, which are free and finitely generated over $\mathbb{Z}_p$.

On the category of Dieudonné displays over R we have the structure of an exact category: A morphism $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ is called a strict monomorphism if $\varphi : P_1 \to P_2$ is injective and $Q_1 = \varphi^{-1}(Q_2)$, and it is called a strict epimorphism if $\varphi : P_1 \to P_2$ is an epimorphism and $\varphi(Q_1) = Q_2$.

Proposition 18 *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a V -nilpotent Dieudonné display over R . Let us denote by $C_{\overline{R}}$ the cokernel of the map $V^{-1} - \mathrm{id} : Q_{\overline{R}} \to P_{\overline{R}}$ with its natural structure of a Γ -module. Then we have a natural equivalence of categories:*

$$\mathrm{Hom}_{\Gamma}(H, C_{\overline{R}}) \cong \mathrm{Ext}^1(\mathcal{P}(H), \mathcal{P}) \tag{13}$$

Proof: Let us start with a remark on Galois cohomology. Let $\overline{P}$ be any free and finitely generated $\hat{W}(\overline{R})$ -module with a semilinear Γ -action, which is continuous with respect to the topology induced by the ideals ${}^{V^n}\hat{W}(\overline{R})$. Then we have

$$H^1(\Gamma, \mathrm{Hom}_{\mathbb{Z}_p}(H, \overline{P})) = 0. \tag{14}$$

Indeed we reduce this to the assertion, that for a finite dimensional vector space $\overline{U}$ over $\overline{k}$ with a semilinear continuous action of Γ , we have $H^1(\Gamma, \overline{U}) = 0$, because this is an induced Galois module by usual descent theory. To make the reduction we consider first the case $\overline{R} = \overline{k}$. Then we have a filtration with graded pieces $\mathrm{Hom}_{\mathbb{Z}_p}(H, p^n\overline{P}/p^{n+1}\overline{P})$. Since the cohomology of these graded pieces vanishes and our group is complete and separated for this filtration, we are done for $\overline{R} = \overline{k}$. In the general case we consider a surjection $\overline{R} \to \overline{S}$ with kernel $\mathfrak{a}$ and argue by induction. We may assume that $\mathfrak{m} \cdot \mathfrak{a} = p \cdot \mathfrak{a} = 0$. It is enough to show that $H^1(\Gamma, \mathrm{Hom}_{\mathbb{Z}_p}(H, \hat{W}(\mathfrak{a})\overline{P}) = 0$. Because $\hat{W}(\mathfrak{a})P \simeq \bigoplus_n \mathfrak{a} \otimes_{Frob^n, \overline{k}} \overline{P}_{\overline{k}}/I_{\overline{k}}\overline{P}_{\overline{k}}$ the vanishing (14) follows.

We will use a bar to denote base change to $\overline{R}$, i.e. $Q_{\overline{R}} = \overline{Q}$ etc. Let us start with an extension from the right hand side of (13) :

$$0 \to \mathcal{P} \to \mathcal{P}_1 \to \mathcal{P}(H) \to 0 \tag{15}$$

It induces an exact sequence

$$0 \to \overline{Q} \to \overline{Q}_1 \to \hat{W}(\overline{R}) \otimes H \to 0 \tag{16}$$

of Γ-modules. The same argument as above shows $H^1(\Gamma, \mathrm{Hom}_{\mathbb{Z}_p}(H, \overline{Q})) = 0$, if we use a normal composition for $\overline{Q}$. Hence the sequence (16) admits a Γ-equivariant section over H:

$$s : H \to \overline{Q}_1$$

We consider the function $u : H \to \overline{P}$ given by:

$$u(h) = V^{-1}s(h) - s(h) \tag{17}$$

Since we may change s exactly by a homomorphism of Γ-modules $H \to \overline{Q}$, we obtain that the class of u in the cokernel of the map:

$$\mathrm{Hom}_\Gamma(H, \overline{Q}) \stackrel{V^{-1}-id}{\longrightarrow} \mathrm{Hom}_\Gamma(H, \overline{P}),$$

is well-defined by the extension (15). Since the group cohomology vanishes this cokernel is exactly $\mathrm{Hom}_\Gamma(H, C_{\overline{R}})$. This provides an injective group homomorphism:

$$\mathrm{Ext}^1(\mathcal{P}(H), \mathcal{P}) \to \mathrm{Hom}_\Gamma(H, C_{\overline{R}}) \tag{18}$$

Conversely it is easy to construct from a homomorphism $u \in \mathrm{Hom}_\Gamma(H, \overline{P})$ an extension of Dieudonné displays over $\overline{R}$:

$$0 \to \overline{\mathcal{P}} \to \overline{\mathcal{P}}_u \to \overline{\mathcal{P}}(H) \to 0, \tag{19}$$

by taking (17) as a definition for the operator V^{-1} of $\overline{\mathcal{P}}_u$. Then one has an action of Γ on $\overline{\mathcal{P}}_u$, for which the sequence (19) becomes Γ-equivariant. Taking the invariants by Γ we obtain an element in $\mathrm{Ext}^1(\mathcal{P}(H), \mathcal{P})$, whose image by (18) is u. Hence (18) is an isomorphism. *Q.E.D.*

Remark: Our construction is functorial in the following sense. Let $\mathcal{P}'$ be a second V-nilpotent Dieudonné display, and H' be a second $\mathbb{Z}_p[\Gamma]$-module, which is free and finitely generated as a $\mathbb{Z}_p$-module. Let $u' \in \mathrm{Hom}_\Gamma(H, C'_{\overline{R}})$ be a homomorphism to the cokernel of $V^{-1} - \mathrm{id} : \overline{Q}' \to \overline{P}'$. A morphism of data $(\mathcal{P}, H, u) \to (\mathcal{P}', H', u')$ has the obvious meaning. Then it is clear that such a morphism induces a morphism of the corresponding extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}_u & \longrightarrow & \mathcal{P}(H) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{P}' & \longrightarrow & \mathcal{P}'_{u'} & \longrightarrow & \mathcal{P}(H') & \longrightarrow & 0 \end{array}$$

and conversely. Moreover, since there are no nontrivial homomorphims $\mathcal{P} \to \mathcal{P}(H')$, we conclude

$$\mathrm{Hom}((\mathcal{P}, H, u), (\mathcal{P}', H', u')) = \mathrm{Hom}(\mathcal{P}_u, \mathcal{P}'_{u'})$$

We associate to a continuous Γ-module H, which is free and finitely generated as a $\mathbb{Z}_p$-module a Barsotti-Tate group as usual. The finite Γ-module $p^{-n}H/H$ corresponds to a finite étale group scheme G_n. We set

$$BT(H) = \varinjlim_{n} G_n.$$

The following analogue of proposition 18 seems to be well-known.

Proposition 19 *Let H be as above and let G be a formal p-divisible group over R. Then there is a canonical isomorphism of categories*

$$\mathrm{Hom}_{\Gamma}(H, G(\overline{R})) \simeq \mathrm{Ext}^1(BT(H), G)$$

Moreover this isomorphism is functorial in the sense of the last remark.

Before we proof this, we remark that it implies the main theorem of this paper:

Theorem 20 *There is a functor BT from the category of Dieudonné displays over R to the category of p-divisible groups over R which is an equivalence of categories. On the subcategory of V-nilpotent Dieudonné displays this is the functor BT of proposition 7.*

Proof: By the last proposition the category of p-divisible groups over R is equivalent to the category of data $(G, H, u : H \to G(\overline{R}))$. But since we already know that the category of formal p-divisible groups is equivalent to the category of V-nilpotent Dieudonné displays, such that $G(\overline{R})$ is identified with $C_{\overline{R}}$ we conclude by the remark after proposition 18. *Q.E.D.*

Proof of proposition 19: Let us start with an extension

$$0 \to G \to G_1 \to BT(H) \to 0 \tag{20}$$

Let S be a local R-algebra, such that the residue class field l of S is contained in a fixed algebraic closure $\overline{k}$ of k. Then we obtain an exact sequence of $\Gamma_l = \mathrm{Gal}(\overline{k}/l)$-modules

$$0 \to G(\overline{S}) \to G_1(\overline{S}) \to H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0 \tag{21}$$

In fact this sequence is exact because the flat Čech-cohomology of a formal group vanishes (use [Z2] 4.6 or more directly [Z3] 5.5).

Conversely, if we are given for any S an extension of Γ_l -modules (21), which depends functorially on S we obtain an extension (20).

If we pull back the extension (20) by the morphism $H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p/\mathbb{Z}_p$ it splits uniquely as a sequence of abelian groups because $G(S)$ is annihilated by some power of p . By the uniqueness it splits also as a sequence of Γ_l - modules. Hence to give an extension (20) is the same thing as to give a homomorphism of Γ_l -modules $H \to G(\overline{S})$. The functoriality in S means in particular that we have a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & G(\overline{R}) \\ \| & & \downarrow \\ H & \longrightarrow & G(\overline{S}), \end{array}$$

which is equivariant with respect to $\Gamma_l \subset \Gamma_k$. Hence to give functorially extensions (21) is the same thing as a Γ_k -equivariant homomorphism $H \to G(\overline{R})$. *Q.E.D.*

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