Comparison between overconvergent de Rham-Witt and crystalline cohomology for projective and smooth varieties

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Let X be a smooth variety over a perfect field k of characteristic $p > 0$. In [DLZ] we constructed an overconvergent de Rham-Witt complex $W^\dagger\Omega_{X/k}$ as a suitable sub-complex of the completed de Rham-Witt complex $W\Omega_{X/k}$ of Deligne-Illusie. It is a Zariski sheaf of differential graded algebras. We proved that the hypercohomology $\mathbb{H}^{\cdot}(X, W^\dagger\Omega_{X/k})$ tensored with $\mathbb{Q}$ is canonically isomorphic to the rigid cohomology of X. It is an open question whether $\mathbb{H}^{\cdot}(X, W^\dagger\Omega_{X/k})$ is modulo torsion a finitely generated $W(k)$-module.

The main result of this note answers this question if X is projective and smooth.

Theorem *Let X be smooth and projective over k. Then the canonical map*

$$H^i(X, W^\dagger\Omega_{X/k}) \to H^i(X, W\Omega_{X/k}) = H^i_{cris}(X/W(k)$$

is an isomorphism. These modules are of finite type over $W(k)$ for all $i \geq 0$.

1 Proof of the Theorem

Lemma 1. *Let X be smooth and projective over k. Then we have a commutative diagram*

$$\begin{array}{ccc} H^i(X, W^\dagger\Omega_{X/k}) & \xrightarrow{\gamma} & H^i(X, W\Omega_{X/k}) \\ \downarrow & & \downarrow \\ H^i(X, W^\dagger\Omega_{X/k} \otimes \mathbb{Q}) & \longrightarrow & H^i(X, W\Omega_{X/k} \otimes \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ H^i_{\mathrm{rig}}(X/W(k)[\frac{1}{p}]) & \xrightarrow{\cong} & H^i_{\mathrm{cris}}(X/W(k)) \otimes \mathbb{Q} \end{array} \tag{1}$$

where all maps in the lower square are isomorphisms.

Proof. The isomorphisms of the lower square follow from [I] II Théorème 1.4, [DLZ] Theorem 4.40 and [B] Théorème 1.9.

We only need to show that the lower square commutes. All maps in the diagram are defined for any quasi-projective smooth scheme. The cohomology can be computed by simplicial methods. Therefore it is enough to check the commutativity if X is affine, $X = \operatorname{Spec} A$.

Let $\tilde{A}^\dagger$ be an overconvergent (Monsky-Washnitzer) Witt-lift and let $\hat{\tilde{A}}$ be its p-adic completion.

We have a commutative diagram of complexes

$$\begin{array}{ccc} \Omega_{\tilde{A}^\dagger/W(k)} & \longrightarrow & W^\dagger\Omega_{A/k} \\ \downarrow & & \downarrow \\ \Omega_{\hat{\tilde{A}}/W(k)} & \longrightarrow & W\Omega_{A/k} \end{array}$$

where the vertical maps are the canonical inclusion maps. The lower horizontal map is a quasi-isomorphism and its cohomology is crystalline cohomology.

The upper horizontal map becomes a quasi-isomorphism after tensoring with $\mathbb{Q}$ (by [DLZ], Corollary 3.25) and then induces an isomorphism between Monsky-Washnitzer (resp. rigid) cohomology and rational overconvergent de Rham-Witt cohomology. This proves the lemma. □

Lemma 2. *For X/k smooth we have a quasi-isomorphism*

$$W^\dagger\Omega_{X/k}/p^nW^\dagger\Omega_{X/k} \cong W\Omega_{X/k}/p^nW\Omega_{X/k} \cong W_n\Omega_{X/k}.$$

Proof. The quasi-isomorphism on the right is shown in [I], I, 3.17.3. For the left quasi-isomorphism it is enough to prove this locally, that is for a finite étale monogenic extension B of a localised polynomial algebra. In this case we have shown [DLZ] (proof of Theorem 3.19) that there are direct decompositions

$$W^\dagger\Omega_{B/k} = W^\dagger\Omega^{\text{int}}_{B/k} \oplus W^\dagger\Omega^{\text{frac}}_{B/k}, \quad W\Omega_{B/k} = W\Omega^{\text{int}}_{B/k} \oplus W\Omega^{\text{frac}}_{B/k}$$

into integral and fractional parts.

Moreover, the fractional parts are acyclic sub-complexes. Surely multiplication with p^n is an injection and respects this decomposition. It follows that $W^\dagger\Omega^{\text{frac}}_{B/k} \otimes \mathbb{Z}/(p^n)$ and $W\Omega^{\text{frac}}_{B/k} \otimes \mathbb{Z}/(p^n)$ are acyclic. On the other hand it is easy to see that $W^\dagger\Omega^{\text{int}}_{B/k} \otimes \mathbb{Z}/(p^n)$ is isomorphic to the de Rham complex $\Omega_{\tilde{B}/W_n(k)}$ where $\tilde{B}$ is a Witt-lift of B over $W_n(k)$, but this complex also coincides with $W\Omega^{\text{int}}_{B/k} \otimes \mathbb{Z}/(p^n)$. □

Proposition 3. *Let X be smooth and projective over k. Then the canonical map*

$$H^i(X, W^\dagger\Omega_{X/k}) \to H^i(X, W\Omega_{X/k}) = H^i_{cris}(X/W(k))$$

is an isomorphism. In particular these groups are finitely generated $W(k)$-modules for all $i \geq 0$.

We begin with some general remarks. Let A be a $W(k)$-module. We denote the kernel of the multiplication by $p^n : A \to A$ by $A[p^n]$ and the cokernel by $A[/p^n]$. We denote by $A_{tors} \subset A$ the subset of all elements which are annihilated by a power of p. We write $\hat{A} = \varprojlim_n A[/p^n]$ for the p-adic completion of A.

Lemma 4. *Let A be a $W(k)$-module. Assume that the following properties hold:*

(i) $\hat{A}$ *is a finitely generated* $W(k)$*-module.*

(ii) The kernel I *of the canonical map* $\iota : A \to \hat{A}$ *is torsion, i.e.* $I = I_{tors}$.

(iii) There is no injection $W(k)_{\mathbb{Q}}/W(k) \to A$.

Then $\iota : A \to \hat{A}$ *is an isomorphism.*

Proof. Let $y \in I$. We show that there is $x \in I$, such that $px = y$. We find a number m such that $p^m \hat{A}_{tors} = 0$. Since $y = 0 \mod p^{m+1}A$ we find $z \in A$ such that $p^{m+1}z = y$. By the choice of p we have $x := p^m z \in I$. Therefore any element of I is divisible by p. The condition (iii) implies that $I = 0$. Then A is finitely generated and $A = \hat{A}$. □

We turn now to the proof of the Proposition. By [I] II 2.7.2 and Lemma 2 we have for a proper and smooth scheme X/k isomorphisms

$$\varprojlim_n H^i(X, W^\dagger\Omega_{X/k}[/p^n]) = \varprojlim_n H^i(X, W_n\Omega_{X/k}) = H^i(X, W\Omega_{X/k}). \qquad (2)$$

We have the exact sequence

$$0 \to W^\dagger\Omega_{X/k} \xrightarrow{p^n} W^\dagger\Omega_{X/k} \to W^\dagger\Omega_{X/k}[/p^n] \to 0.$$

Taking the projective limit in the obvious sense with respect to n, we obtain from (2) the exact sequence:

$$\varprojlim_n H^i(X, W^\dagger\Omega_{X/k})[/p^n]) \rightarrowtail H^i(X, W\Omega_{X/k}) \twoheadrightarrow \varprojlim_n H^{i+1}(X, W^\dagger\Omega_{X/k})[p^n]) \qquad (3)$$

We note that the morphism γ (1) factors over the second arrow in this sequence. Since the cokernel of γ has finite length, we see that the last limit of this sequence is a module of finite length. Since X is proper the modules $A[p^n] := H^{i+1}(X, W^\dagger\Omega_{X/k})[p^n])$ appearing in the projective system have finite length too. Let $A_n \subset A[p^n]$ be the universal images of the projective system. We see that for each n there is a number $n' > n$, such that A_n is the image of $p^{n'-n} : A[p^{n'}] \to A[p^n]$. It follows that the natural map $A_{n+1} \to A_n$ is surjective for each n. Let $A = H^{i+1}(X, W^\dagger\Omega_{X/k})$. Then A_n consists of all elements $x \in A[p^n]$ such that for each number m there is $y_m \in A$ such that $p^m y_m = x$. We see that for $\ell \leq n$ we have $A_\ell = A[p^\ell] \cap A_n$. Therefore we have for each n an exact sequence

$$0 \to A_1 \to A_{n+1} \to A_n \to 0.$$

Since

$$\lim_{\substack{\longleftarrow \\ n}} A_n = \lim_{\substack{\longleftarrow \\ n}} A[p^n]$$

is a module of finite length we conclude that $A_1 = 0$. But then all A_n are zero. Therefore the last projective system of (3) is essentially zero.

Now we set $A = H^i(X, W^\dagger\Omega_{X/k})$. We claim that A satisfies the assumptions of Lemma 4. Indeed by what we have shown the inclusion

$$\hat{A} \subset H^i(X, W\Omega_{X/k}).$$

is an isomorphism. It follows that $\hat{A}$ is a $W(k)$-module of finite type. Therefore the kernel I of $\iota : A \to \hat{A}$ coincides with the kernel of γ in diagram (1). This is a torsion module.

Finally assume that there is an injection $W(k)_{\mathbb{Q}}/W(k) \to A$. Then $i \geq 1$. But this implies that $\lim_{\substack{\longleftarrow \\ n}} A[p^n]$ contains a submodule isomorphic to $W(k)$. We have already shown that the last projective limit is zero. This contradiction shows that the last assumption (iii) of Lemma 4 is fulfilled for A. Therefore $A \to \hat{A}$ is an isomorphism. This proves the Proposition and the Theorem.

References

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