ON THE p-ADIC UNIFORMIZATION OF UNITARY SHIMURA CURVES

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ABSTRACT. We prove p-adic uniformization for Shimura curves attached to the group of unitary similitudes of certain binary skew hermitian spaces V with respect to an arbitrary CM field K with maximal totally real subfield F. For a place $v|p$ of F that is not split in K and for which V_v is anisotropic, let ν be an extension of v to the reflex field E. We define an integral model of the corresponding Shimura curve over $\operatorname{Spec} O_{E,(\nu)}$ by means of a moduli problem for abelian schemes with suitable polarization and level structure prime to p. The formulation of the moduli problem involves a *Kottwitz condition*, an *Eisenstein condition*, and an *adjusted invariant*. The first two conditions are conditions on the Lie algebra of the abelian varieties; the last condition is a condition on the Riemann form of the polarization. The uniformization of the formal completion of this model along its special fiber is given in terms of the formal Drinfeld upper half plane $\widehat{\Omega}_{F_v}$ for F_v. The proof relies on the construction of the *contracting functor* which relates a relative Rapoport-Zink space for strict formal O_{F_v}-modules with a Rapoport-Zink space of p-divisible groups which arise from the moduli problem, where the O_{F_v}-action is usually not strict when $F_v \neq \mathbb{Q}_p$. Our main tool is the theory of displays, in particular the *Ahsendorf functor*.

CONTENTS

1. INTRODUCTION

1.1. **History of uniformization.** One of the major results of the Mathematics of the 19th century is the *uniformization theorem*. It states that any non-singular projective algebraic curve X of genus $g(X) \geq 2$ can be uniformized, i.e., can be written as

$$X \simeq \Gamma \backslash \Omega_{\mathbb{R}}, \tag{1.1.1}$$

where $\Omega_{\mathbb{R}} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is the union of the upper and the lower half plane and Γ denotes a discrete cocompact subgroup of $\mathrm{PGL}_2(\mathbb{R})$. This notation reinforces the analogy with the p-adic uniformization discussed below. The history of this theorem is very complicated, and involves the names of many mathematicians, among them Poincaré, Hilbert and Koebe, comp. [12]. Inspired by the uniformization theorem, Poincaré gave a systematic construction of cocompact discrete subgroups of $\mathrm{PGL}_2(\mathbb{R})$. For this he used the exceptional isomorphism between inner forms of PGL_2 and special orthogonal groups of ternary quadratic forms. In fact, for his construction, he used arithmetic subgroups of the special orthogonal group of an indefinite anisotropic ternary quadratic form over $\mathbb{Q}$, cf. [12].

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Now let p be a prime number. The history of the p-adic uniformization of algebraic curves starts with Tate's uniformization theory of elliptic curves. It turns out that not all elliptic curves over p-adic fields admit a p-adic uniformization, but only those with (split) multiplicative reduction [30, §6].

The next step was Mumford's p-adic uniformization theory of algebraic curves of higher genus, [23]. Again, it turns out that not all such algebraic curves over p-adic fields admit a p-adic uniformization by an admissible open subset of $\mathbb{P}^1$, but only those with totally degenerate reduction [23]. In view of Mumford's results, it becomes interesting to single out classes of algebraic curves with totally degenerate reduction. Such classes are exhibited by Cherednik's theorem [7].

Cherednik's theorem states that Shimura curves associated to certain quaternion algebras over a totally real field F admit p-adic uniformization. The quaternion algebra is required to be split at precisely one archimedean place w of F (and ramified at all other archimedean places), and to be ramified at a non-archimedean place v of residue characteristic p. In this case, the reflex field can be identified with F. Then one obtains p-adic uniformization by the Drinfeld halfplane associated to F_v, provided that the level structure is prime to v. Cherednik's theorem implies that if X is a connected component of the Shimura tower for such a level, considered as an algebraic curve over $\bar{F}$, then there is an isomorphism of algebraic curves over $\bar{F}_v$,

$$X \otimes_{\bar{F}} \bar{F}_v \simeq (\bar{\Gamma}\backslash\Omega_{F_v}) \otimes_{F_v} \bar{F}_v. \tag{1.1.2}$$

Here $\Omega_{F_v} = \mathbb{P}^1_{F_v} \setminus \mathbb{P}^1(F_v)$ denotes the Drinfeld upper halfplane for the local field F_v, and $\bar{\Gamma}$ denotes a discrete cocompact subgroup of $\mathrm{PGL}_2(F_v)$. Recall that Ω_{F_v} is a rigid-analytic space over F_v. The isomorphism (1.1.2) is to be interpreted as follows: the rigid-analytic space $\bar{\Gamma}\backslash\Omega_{F_v}$ is (uniquely) algebraizable by a projective algebraic curve over F_v. After extension of scalars $F_v \longrightarrow \bar{F}_v$, there exists an isomorphism as in (1.1.2). We thus see that (1.1.2) allows us to pass from the original complex uniformization $X \otimes_{\bar{F}} \mathbb{C} \simeq \Gamma\backslash\Omega_{\mathbb{R}}$, where Γ is a congruence subgroup maximal at v, to p-adic uniformization.

Let us comment on the proof of Cherednik's theorem. When $F = \mathbb{Q}$, these Shimura curves are moduli spaces of abelian varieties with additional structure, and Drinfeld [9] gave a moduli-theoretic proof of Cherednik's theorem in this special case. Furthermore, he proved an 'integral version' of this theorem (which has the original version as a corollary). This integral version describes a concrete model of the Shimura variety over $\operatorname{Spec} O_{F,(\nu)}$ and a description of the formal completion along its special fiber in terms of a formal scheme version of Ω_{F_ν}. It relies on a theorem on formal moduli spaces of p-divisible groups, which is in fact the deepest part of Drinfeld's paper. When $F \neq \mathbb{Q}$, Cherednik's Shimura curves do not represent a moduli problem of abelian varieties, and Cherednik's method of proof is indirect and apparently uses Ihara's theory of elliptic elements in congruence monodromy problems.

There are also higher-dimensional versions of p-adic uniformization. Drinfeld's method has been generalized by Rapoport and Zink [27] to Shimura varieties associated to certain *fake unitary groups*. These are associated to central division algebras over a CM-field equipped with an involution of the second kind; for Rapoport-Zink uniformization, one has to assume that the p-adic place of the totally real subfield splits in the CM-field. This higher-dimensional generalization also includes integral uniformization theorems. In [27], these integral uniformization theorems appear as a special instance of a general *non-archimedean uniformization theorem*, which describes the formal completion of PEL-type Shimura varieties along a fixed isogeny class. In the case of p-adic uniformization, the whole special fiber forms a single isogeny class.

The method of [27] has been applied by Boutot and Zink [5] to prove variants of Cherednik's original theorem by embedding Cherednik's Shimura curves into Shimura curves obtained by the Rapoport-Zink method; however, the integral uniformization theorems in [5] are rather weak in that they only show that there exists some integral model of the Shimura curve for which one has integral uniformization.

A variant of Cherednik's method has been developed by Varshavsky [31, 32] to obtain p-adic uniformization of certain higher-dimensional Shimura varieties associated to fake unitary groups, again at a split place. We refer to Boutot's Bourbaki talk [3] for an account of all these developments.

In the present paper, we deal with Shimura curves attached to unitary similitude groups associated to anti-hermitian[1] vector spaces V of dimension 2 over a CM-field K with totally real subfield F of arbitrary degree. Our results generalize those in [18], where the case $F = \mathbb{Q}$ is considered. Like Cherednik, we assume that V is split at precisely one archimedean place w of F (and ramified at all other archimedean places). We also assume that V is ramified at a non-archimedean place v of residue characteristic p of F. However, in contrast to the cases of p-adic uniformization mentioned above, we assume that v does *not* split in K. Of course, these Shimura curves are closely related to the Shimura curves considered by Cherednik (we refer to [18] for a general discussion of the relation between quaternion algebras and two-dimensional hermitian vector spaces). However, they are different. In particular, they have the enormous advantage that they always represent a moduli problem of abelian varieties. Our uniformization theorem is optimal when the level structure imposed is prime to p, in the sense that it extends to an integral uniformization that allows an explicit interpretation of the points in the reduction modulo p.

As in Drinfeld's approach, our uniformization theorem relies on a theorem on formal moduli spaces of p-divisible groups. In fact, the main work in proving our theorems is to establish an isomorphism of our formal moduli spaces with the moduli space of Drinfeld. Such an isomorphism is also constructed by Scholze and Weinstein [29]. Their construction relies on Scholze's theory of *local Shimura varieties* and his *integral p-adic Hodge theory*, as well as on results in a preliminary (unpublished) version of the present paper on *local models*. Our construction here is more direct and more elementary; it relies on the *theory of displays*, cf. [33].

Drinfeld's version of Cherednik's theorem for $F = \mathbb{Q}$ has found numerous arithmetic applications, to *level raising, level lowering* and *bounding Selmer groups*, at the hands of Ribet, Bertolini, Darmon, Nekovar and many others, comp. also the references in the introduction of [18]. It is to be hoped that our direct construction is the basis of similar such applications for general totally real fields F.

Our results are an expression of the exceptional isomorphism between an inner twist of the adjoint group of GL_2 and an inner twist of the adjoint group of U_2. Just as for Poincaré's exceptional isomorphism of inner forms of PGL_2 and special orthogonal groups of ternary quadratic forms, there is no higher rank analogue.

1.2. **Global results.** Now let us state our global results. Let K be a CM-field, with totally real subfield F. We denote the non-trivial F-automorphism of K by $a \mapsto \bar{a}$. Let V be a two-dimensional K-vector space, equipped with an alternating $\mathbb{Q}$-bilinear form $\psi\colon V \times V \longrightarrow \mathbb{Q}$ such that

$$\psi(ax, y) = \psi(x, \bar{a}y), \quad x, y \in V,\, a \in K. \tag{1.2.1}$$

There is a unique anti-hermitian form $\varkappa$ on V such that

$$\mathrm{Tr}_{K/\mathbb{Q}_p}\, a\varkappa(x, y) = \psi(ax, y), \quad x, y \in V,\, a \in K. \tag{1.2.2}$$

Conversely, the anti-hermitian form $\varkappa$ determines the alternating bilinear form ψ with (1.2.1). We say that $\varkappa$ arises from ψ by contraction. Recall that anti-hermitian spaces V are determined up to isomorphism by their signature at the archimedean places of F and their local invariants $\mathrm{inv}_v(V)$ at the non-archimedean places v of F. Let w be an archimedean place such that V_w has signature $(1, 1)$ and such that $V_{w'}$ is definite for all archimedean places $w' \neq w$. Let us be more precise. Let $\Phi = \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(K, \mathbb{C})$. Let r be a *generalized CM-type of rank 2, special w.r.t. w*, i.e., a function

$$r : \Phi \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \longmapsto r_\varphi, \tag{1.2.3}$$

such that $r_\varphi + r_{\bar{\varphi}} = 2$ for all $\varphi \in \Phi$, and such that for the extensions $\{\varphi_0, \bar{\varphi}_0\}$ of w we have $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$ and with $r_\varphi \in \{0, 2\}$ for $\varphi \notin \{\varphi_0, \bar{\varphi}_0\}$, comp. [18]. Then we demand that the signature of $V_\varphi = V \otimes_{K,\varphi} \mathbb{C}$ be equal to $(r_\varphi, 2 - r_\varphi)$.

We denote the reflex field of r by $E = E_r$. It is a subfield of $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Note that F embeds via φ_0 into E, and that the archimedean place of F induced by

$$F \xrightarrow{\varphi_0} E \longrightarrow \mathbb{C} \tag{1.2.4}$$

[1]It turns out to be more natural to consider anti-hermitian forms, rather than hermitian forms, cf. below.

is equal to w. If $F = \mathbb{Q}$, then $E = F$.

Associated to these data, there is a *Shimura pair* $(G, \{h\})$. Here G denotes the group of unitary similitudes of V, with similitude factor in $\mathbb{G}_m$, an algebraic subgroup of $\mathrm{GSp}(V, \psi)$ over $\mathbb{Q}$. For an open compact subgroup $\mathbf{K} \subset G(\mathbb{A}_f)$, there is a Shimura variety $\mathrm{Sh}_{\mathbf{K}}$, with canonical model over the reflex field E, whose complex points are given by

$$\mathrm{Sh}_{\mathbf{K}}(\mathbb{C}) \simeq G(\mathbb{Q})\backslash[\Omega_{\mathbb{R}} \times G(\mathbb{A}_f)/\mathbf{K}].$$

Here $\Omega_{\mathbb{R}}$ is acted on by $G(\mathbb{R})$ via the projection to $\mathrm{GU}(V_w)_{\mathrm{ad}}$ and a fixed isomorphism $\mathrm{GU}(V_w)_{\mathrm{ad}} \simeq \mathrm{PGL}_2(\mathbb{R})$.

Consider the following moduli problem on (Sch/E). It associates to an E-scheme S the set of isomorphism classes of tuples $(A, \iota, \bar{\lambda}, \bar{\eta})$. Here

- A is an abelian scheme up to isogeny of dimension $2[F : \mathbb{Q}]$ over S.
- $\iota\colon O_K \longrightarrow \mathrm{End}(A)$ is an action of O_K on A such that

$$\mathrm{Tr}(\iota(a)|\,\mathrm{Lie}\,A) = \sum\nolimits_{\varphi\in\Phi} r_\varphi \varphi(a), \qquad \text{for all } a \in O_K.$$

- $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous polarization of A such that its Rosati involution induces the conjugation on K/F.
- a $\mathbf{K}$-orbit of K-linear similitudes $\bar{\eta}\colon V \otimes \mathbb{A}_f \xrightarrow{\sim} \widehat{V}(A)$.

Here the rational Tate module is equipped with its natural anti-hermitian form arising by contraction from its polarization form. In the case where S is connected, a more precise formulation of the last datum is as follows. Fix a geometric point $\omega \to S$. Then $\bar{\eta}$ is a $\mathbf{K}$-orbit of K-linear similitudes $\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A_\omega)$ which is invariant by the action of $\pi_1(S, \omega)$ on $\hat{V}(A_\omega)$.

This moduli problem is represented by a quasi-projective scheme $\mathcal{A}_{\mathbf{K},E}$ which is the *canonical model* of $\mathrm{Sh}_{\mathbf{K}}$ over E. It is a projective scheme when the existence of v as below is imposed.

Let p be a prime number and let v be a p-adic place of F which is non-split in K and such that V_v is a non-split K_v/F_v-anti-hermitian space, i.e., $\mathrm{inv}_v(V) = -1$. We take the open compact subgroup of the form $\mathbf{K}^\star = \mathbf{K}_p^\star \cdot \mathbf{K}^p$, where $\mathbf{K}^p$ is an arbitrary open compact subgroup of $G(\mathbb{A}_f^p)$, and where $\mathbf{K}_p^\star$ has the following shape. Let

$$V \otimes \mathbb{Q}_p = \bigoplus\nolimits_{\mathfrak{p}|p} V_{\mathfrak{p}}$$

be the orthogonal decomposition according to the prime ideals of F over p. Note that the prime ideal $\mathfrak{p}_v$ corresponding to v occurs as an index here. Then

$$G(\mathbb{Q}_p) \subset \prod\nolimits_{\mathfrak{p}} G_{\mathfrak{p}}(\mathbb{Q}_p)\,,$$

where $G_{\mathfrak{p}}$ denotes the group of unitary similitudes of $V_{\mathfrak{p}}$ with similitude factor in $\mathbb{G}_m$. We take $\mathbf{K}_p^\star$ of the form

$$\mathbf{K}_p^\star = G(\mathbb{Q}_p) \cap \mathbf{K}_v \mathbf{K}_p^{\star,v}\,, \tag{1.2.5}$$

where $\mathbf{K}_v$ is the unique maximal compact subgroup of $G_{\mathfrak{p}_v}(\mathbb{Q}_p)$, and where $\mathbf{K}_p^{\star,v} \subset \prod_{\mathfrak{p}\neq\mathfrak{p}_v} G_{\mathfrak{p}}(\mathbb{Q}_p)$ is an arbitrary open compact subgroup.

Let J be the inner form of G which is anisotropic at w and quasi-split at v, and which locally coincides with G at all places $\neq v, w$ of F. Then there exists an identification of the adjoint group $J_{v,\mathrm{ad}}(\mathbb{Q}_p)$ with $\mathrm{PGL}_2(F_v)$ and an action of $J(\mathbb{Q})$ on $G(\mathbb{A}_f)/\mathbf{K}^\star$ (which is, however, not induced by an action of $J(\mathbb{Q})$ on $G(\mathbb{A}_f)$).

We now formulate our main theorem in the version over a p-adic field, cf. Corollary 7.5.2. Recall the embedding (1.2.4) of F into E. We choose a place ν of E over v. Throughout the paper, we always assume[2] $p \neq 2$ if v is ramified in K.

Theorem 1.2.1. *Let $\mathbf{K}^\star = \mathbf{K}_p^\star \mathbf{K}^p$, where $\mathbf{K}_p^\star$ is of the form (1.2.5). Let $\breve{E}_\nu$ be the completion of the maximal unramified extension of E_ν in $\overline{\mathbb{Q}}_p$. There is a finite abelian extension $\breve{E}_\nu^\star$ of $\breve{E}_\nu$ and an isomorphism of algebraic curves over $\breve{E}_\nu^\star$,*

$$\mathcal{A}_{\mathbf{K}^\star,E} \times_{\mathrm{Spec}\ E} \mathrm{Spec}\ \breve{E}_\nu^\star \simeq \big(J(\mathbb{Q})\backslash[\Omega_{F_v} \times G(\mathbb{A}_f)/\mathbf{K}^\star]\big) \times_{\mathrm{Spec}\ F_v} \mathrm{Spec}\ \breve{E}_\nu^\star$$

[2]In the light of the results of Kirch [14], it should be possible to remove this blanket assumption.

Here, as before, Ω_{F_v} denotes the Drinfeld halfplane relative to the local field F_v, and the interpretation is as before that the scheme on the LHS is the algebraization of the rigid-analytic variety on the RHS. If $\mathbf{K}_p$ is of the form (1.2.7) below, then we may take $\breve{E}_\nu^\star = \breve{E}_\nu$, cf. Theorem 1.2.3 below; but in general, one needs a non-trivial extension, comp. Theorem 1.2.4.

From this theorem we deduce an analogue of Cherednik's isomorphism (1.1.2), noting that any geometric connected component X of $\mathrm{Sh}_{\mathbf{K}^\star}$ is defined over the maximal abelian extension E^{ab} of E,

$$X \otimes_{E^{\mathrm{ab}}} \breve{E}_\nu^{\mathrm{ab}} \simeq (\bar{\Gamma}\backslash\Omega_{F_v}) \otimes_{F_v} \breve{E}_\nu^{\mathrm{ab}}. \tag{1.2.6}$$

Here $\breve{E}_\nu^{\mathrm{ab}}$ denotes the maximal abelian extension of $\breve{E}_\nu$, and $\bar{\Gamma}$ is a cocompact discrete subgroup of $\mathrm{PGL}_2(F_v)$. Since the Cherednik Shimura datum is a central twist of $(G, \{h\})$, the geometric connected components of $\mathrm{Sh}_{\mathbf{K}}$ can be identified with those appearing in Cherednik's theorem, so that in fact (1.1.2) follows from Theorem 1.2.1.

By extending the moduli problem for $\mathrm{Sh}_{\mathbf{K}}$ integrally over $\operatorname{Spec} O_{E,(\mathfrak{p}_\nu)}$, we obtain semi-global integral models of these Shimura varieties. This gives us the possibility of formulating an 'integral' version of this theorem. Let us explain the moduli problem in question.

We first explain the level structure. For every $\mathfrak{p}|p$, we fix a lattice $\Lambda_\mathfrak{p}$ in $V_\mathfrak{p}$. We assume that $\Lambda_\mathfrak{p}$ is a self-dual lattice when $\mathfrak{p}$ is either split in K or ramified. When $\mathfrak{p}$ is unramified in K, we assume that $\Lambda_\mathfrak{p}$ is selfdual when $\mathrm{inv}_\mathfrak{p}(V) = 1$, and almost selfdual when $\mathrm{inv}_\mathfrak{p}(V) = -1$. Let

$$\mathbf{K}_p = \{g \in G(\mathbb{Q}_p) \mid g\Lambda_\mathfrak{p} = \Lambda_\mathfrak{p}, \text{ for all } \mathfrak{p}|p\}. \tag{1.2.7}$$

We also fix an open compact subgroup $\mathbf{K}^p \subset G(\mathbb{A}_f^p)$ and set $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p$. We continue to assume that for the distinguished p-adic place v we have $\mathrm{inv}_{\mathfrak{p}_v}(V) = -1$.

We now define a functor $\mathcal{A}_{\mathbf{K}}$ on the category of $O_{E,(\mathfrak{p}_\nu)}$-schemes. Let $S \in (\mathrm{Sch}/O_{E,(\mathfrak{p}_\nu)})$. Then a point of $\mathcal{A}_{\mathbf{K}}(S)$ consists of an equivalence class of quadruples $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$. Here

- A is an abelian scheme over S and $\iota\colon O_K \longrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is an action of O_K on A.
- $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous polarization of A such that its Rosati involution induces the conjugation on K/F.
- $\bar{\eta}^p\colon V \otimes \mathbb{A}_f^p \xrightarrow{\sim} \widehat{V}^p(A)$ is $\mathbf{K}^p$-class of K-linear similitudes.

Here the prime-to-p-rational Tate module $\widehat{V}^p(A)$ of A is equipped with its natural anti-hermitian form arising by contraction from its polarization form. Two quadruples $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ and $(A', \iota', \bar{\lambda}', \bar{\eta}^{p,\prime})$ are equivalent, if there exists an isogeny $A \longrightarrow A'$ of degree prime to p compatible with the remaining data.

We impose the following conditions on the quadruples $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$. First, for the action of O_K on $\operatorname{Lie} A$ induced by ι, we impose the *Kottwitz condition* (KC_r) relative to r, comp. [18]. In addition, we demand that this action also satisfies the *Eisenstein condition* (EC_r) relative to r. This condition is defined in section 2, and is a key novelty of this paper. The condition (EC_r) follows from the Kottwitz condition (KC_r) when S is an E-scheme but is quite subtle when p is nilpotent in $\mathcal{O}_S$. Imposing this condition ensures the flatness of the moduli scheme.

Secondly, we demand that there exists a polarization $\lambda \in \bar{\lambda}$ such that, for every $\mathfrak{p}|p$, the localization of the kernel of the polarization λ at the place $\mathfrak{p}$ satisfies

$$|(\mathrm{Ker}\ \lambda)_\mathfrak{p}| = [\Lambda_\mathfrak{p}^\vee : \Lambda_\mathfrak{p}]. \tag{1.2.8}$$

Thirdly, we impose that for each $\mathfrak{p}|p$, the *r-adjusted invariant* $\mathrm{inv}_\mathfrak{p}^r(A, \iota, \lambda)$ coincides with the invariant $\mathrm{inv}_\mathfrak{p}(V)$ of the hermitian space V. Here the r-adjusted invariant of the triple (A, ι, λ), defined in Appendix A, is another key novelty of this paper. This condition is automatically satisfied when $\mathfrak{p}_v$ is the only prime ideal of F over p. In general, this condition cuts out the open and closed part of the moduli scheme defined by the Shimura variety. The reason for the name *r-adjusted* is that this adjusts the definition of the invariant in [18], where it was erroneously asserted that the invariant is locally constant in families. We prove here that this local constancy indeed holds for the r-adjusted invariant, cf. Proposition 8.2.1.

Proposition 1.2.2. *Let r be a generalized CM-type of* even *rank n, with associated reflex field E. Let S be an O_E-scheme. Let (A, ι, λ) be a CM-triple over S which satisfies the Kottwitz* *condition* (KC_r). *Let* $c \in \{\pm 1\}$. *Then for every non-archimedean place* v *of* F, *the set of points* $s \in S$ *such that*

$$\mathrm{inv}_v^r(A_s, \iota_s, \lambda_s) = c$$

is open and closed in S.

We refer to the body of the text for the terminology used.

Here, now, is our main theorem in the context of schemes over p-adic integer rings, cf. Theorem 7.3.3.

Theorem 1.2.3. *Let* $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p$, *where* $\mathbf{K}_p$ *is of the form* (1.2.7).
(i) The functor $\mathcal{A}_{\mathbf{K}}$ *is representable by a projective flat* $O_{E,(\mathfrak{p}_\nu)}$*-scheme, with semi-stable reduction. Its generic fiber* $\mathcal{A}_{\mathbf{K}} \otimes_{O_{E,(\mathfrak{p}_\nu)}} E$ *is identified with* $\mathcal{A}_{\mathbf{K},E}$, *which is the canonical model of* $\mathrm{Sh}_{\mathbf{K}}$.
(ii) Let $\hat{\mathcal{A}}_{\mathbf{K}}$ *be the formal completion of* $\mathcal{A}_{\mathbf{K}}$ *along its special fiber, which is a formal scheme over* $\mathrm{Spf}\, O_{E_\nu}$. *Then there exists an isomorphism of formal schemes over* $\mathrm{Spf}\, O_{\breve{E}_\nu}$,

$$\hat{\mathcal{A}}_{\mathbf{K}} \times_{\mathrm{Spf}\, O_{E_\nu}} \mathrm{Spf}\, O_{\breve{E}_\nu} \simeq J(\mathbb{Q})\backslash\big[\big(\widehat{\Omega}_{F_v} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{\breve{E}_\nu}\big) \times G(\mathbb{A}_f)/\mathbf{K}\big]\,.$$

For varying $\mathbf{K}^p$, *this isomorphism is compatible with the action of* $G(\mathbb{A}_f^p)$ *through Hecke correspondences on both sides.*

Here $\widehat{\Omega}_{F_v}$ denotes the formal scheme version of Ω_{F_v} over $\mathrm{Spf}\, O_{F_v}$ due to Deligne, Drinfeld and Mumford, cf. [9]. In section 7 we give a variant of the RHS, which allows us to express the descent datum from $O_{\breve{E}_\nu}$ to O_{E_ν} of the LHS. Theorem 1.2.3 is optimal in the sense that it describes explicitly the scheme $\mathcal{A}_{\mathbf{K}}$ over O_{E_ν} and its p-adic uniformization.

If we assume that there are prime ideals $\mathfrak{p}|p$ different from $\mathfrak{p}_v$, we may pass to deeper level structures and still prove an integral version of p-adic uniformization. Let $\mathbf{K}_p^\star \subset G(\mathbb{Q}_p)$ be of the form

$$\mathbf{K}_p^\star = G(\mathbb{Q}_p) \cap \mathbf{K}_v\mathbf{K}_p^{\star,v}, \tag{1.2.9}$$

where $\mathbf{K}_v$ is the stabilizer of $\Lambda_{\mathfrak{p}_v}$, and where $\mathbf{K}_p^{\star,v}$ is an arbitrary open compact subgroup of $G^v(\mathbb{Q}_p) = \prod_{\mathfrak{p}\neq\mathfrak{p}_v} G_{\mathfrak{p}}(\mathbb{Q}_p)$. The system of such subgroups is stable under conjugation with elements of $G(\mathbb{Q}_p)$. For such subgroups, we have the following version of our main theorem, cf. Corollary 7.4.14.

Theorem 1.2.4. *Let* $\mathbf{K}^\star = \mathbf{K}_p^\star\mathbf{K}^p$, *for a choice of* $\mathbf{K}^p \subset G(\mathbb{A}_f^p)$, *where* $\mathbf{K}_p^\star$ *is of the form* (1.2.9). *There exists a normal scheme* $\mathcal{A}^\star_{\mathbf{K}^\star}$ *over* $\mathrm{Spec}\, O_{\breve{E}_\nu}$ *such that for the p-adic completion of this scheme there is an isomorphism*

$$\hat{\mathcal{A}}^\star_{\mathbf{K}^\star} \simeq J(\mathbb{Q})\backslash[(\hat{\Omega}_{F_v} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times G^v(\mathbb{Q}_p)/\mathbf{K}_p^{\star,v} \times G(\mathbb{A}_f^p)/\mathbf{K}^p].$$

For varying $\mathbf{K}^\star$, *these schemes form a tower with an action of the group* $G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, *where the action of* $G(\mathbb{Q}_p)$ *factors through* $G(\mathbb{Q}_p) \to G^v(\mathbb{Q}_p)$. *The isomorphism of formal schemes is compatible with these actions.*

The general fibre of $\mathcal{A}^\star_{\mathbf{K}^\star}$ *is a Galois twist of* $\mathcal{A}_{\mathbf{K}^\star,E} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E}_\nu$ *as follows. There is a character* $\chi_0 : \mathrm{Gal}(\breve{E}_\nu^{ab}/\breve{E}_\nu) \to G(\mathbb{Q}_p)$ *with values in the center of* $G(\mathbb{Q}_p)$ *and an isomorphism*

$$\mathcal{A}_{\mathbf{K}^\star,E} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E}_\nu^{ab} \to \mathcal{A}^\star_{\mathbf{K}^\star} \times_{\mathrm{Spec}\, O_{\breve{E}_\nu}} \breve{E}_\nu^{ab}$$

such that, for $\sigma \in \mathrm{Gal}(\breve{E}_\nu^{ab}/\breve{E}_\nu)$, *the action of* $\mathrm{id}_{\mathcal{A}^\star_{\mathbf{K}^\star}} \times \mathrm{Spec}\, \sigma$ *on the right hand side induces on the left hand side* $\chi_0(\sigma) \times \mathrm{Spec}\, \sigma$, *where* $\chi_0(\sigma)$ *acts as a Hecke operator. The Galois twist respects the Hecke operators.*

The scheme $\mathcal{A}^\star_{\mathbf{K}^\star}$ represents a moduli problem of abelian varieties with additional structure over $O_{\breve{E}_\nu}$, cf. section 7.4. We refer to section 7.6 for the explicit determination of χ_0.

It should be pointed out that Theorem 1.2.4 is not optimal since we cannot describe the descent to E_ν. Also, when v is ramified in K, we can only give the character χ_0 explicitly after restricting to a subgroup of index 2. This is in contrast to Theorem 1.2.3. The deeper reason for this deficiency lies in the fact that the natural context for Theorem 1.2.4 is the class of Shimura varieties appearing in [24]. Let $\Psi \subset \Phi$ be a CM-type for K/F such that

$$\Psi \cap (\Phi \setminus \{\varphi_0, \bar{\varphi}_0\}) = \{\varphi \in \Phi \setminus \{\varphi_0, \bar{\varphi}_0\} \mid r_\varphi = 2\}. \tag{1.2.10}$$

There are two possibilities for Ψ. Let E_Ψ be the reflex field of Ψ and let $\tilde{E}$ be the composite of E_Ψ and $E = E_r$. Then $\tilde{E}$ is an extension of degree one or two of E_r. Associated to (V, ψ, Ψ), there is a finite number of Shimura varieties $\mathrm{Sh}_{\tilde{\mathbf{K}}}(\tilde{G}, \{\tilde{h}\})$ with reflex field $\tilde{E}$, cf. [24, §3]. Here $\tilde{G}$ maps surjectively to G with kernel a central torus, hence the Shimura varieties $\mathrm{Sh}_{\tilde{\mathbf{K}}}(\tilde{G}, \{\tilde{h}\})$ are central twists of the Shimura variety $\mathrm{Sh}_{\mathbf{K}}(G, \{h\})$. Each one represents a moduli problem on $(\mathrm{Sch}/\tilde{E})$. In a sequel to this paper, we will construct semi-global integral models of these Shimura varieties over $\mathrm{Spec}\, O_{\tilde{E},(\tilde{\nu})}$. These are described by moduli problems of abelian varieties on $(\mathrm{Sch}/O_{\tilde{E},(\tilde{\nu})})$ and admit p-adic uniformization in the strong sense of Theorem 1.2.3, when the congruence condition on the open compact subgroup $\tilde{\mathbf{K}}$ is prime to the chosen place v. The trade-off in comparison with our Shimura variety is that the corresponding reflex field $\tilde{E}$ is larger than the reflex field E of our Shimura variety (which, in turn, is larger than the reflex field F of Cherednik's Shimura variety).

Both Theorems 1.2.1 and 1.2.3 are proved in [18] when $F_v = \mathbb{Q}_p$. Most of the work in [18] was local, and an essential ingredient was the *alternative moduli interpretation* of the Drinfeld halfplane in [17]. Once this is accomplished, the proof of the global theorems follows in a relatively straightforward way from the general non-archimedean uniformization theory of [27, Chap. 6]. The same is true here. In [18], we expressed the hope that it might be possible to eliminate the strong limitation $F_v = \mathbb{Q}_p$ made there, and this hope is achieved in the present paper. As explained in [18], the main issue is the contrast between the condition on the action of O_{F_v} on the Lie algebras of the p-divisible groups in the local moduli problems. On the one hand, for the moduli problem represented by the Drinfeld half-plane $\widehat{\Omega}_{F_v}$, the action of O_{F_v} on the Lie algebra is required to be strict, i.e., to factor through the structure morphism of the base scheme S. On the other hand, in the global moduli problem, the Lie algebras of the relevant abelian schemes are often free $O_F \otimes_{\mathbb{Z}} \mathcal{O}_S$-modules. The main results of the present paper, and in particular the contracting functor defined in section 4, provide the bridge between the two types of moduli problems.

1.3. **Local results.** Let us now formulate our local results, referring to section 2 for more details and more explanations of some terms used here. Let p be a prime number, and let F be a finite extension of degree $d = [F : \mathbb{Q}_p]$ of $\mathbb{Q}_p$ and let K/F be a quadratic extension. Let $\Phi = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K, \bar{\mathbb{Q}}_p)$, and fix a pair $\{\varphi_0, \bar{\varphi}_0\}$ of conjugate elements in Φ. Here $\bar{\varphi}_0(a) = \varphi_0(\bar{a})$. Let r be a *local CM-type of rank 2* which is special w.r.t $\{\varphi_0, \bar{\varphi}_0\}$, i.e., a function

$$r : \Phi \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \longmapsto r_\varphi, \tag{1.3.1}$$

such that $r_\varphi + r_{\bar{\varphi}} = 2$ for all $\varphi \in \Phi$, and $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$ and $r_\varphi \in \{0, 2\}$ for $\varphi \notin \{\varphi_0, \bar{\varphi}_0\}$, comp. [18]. We denote the reflex field of r by E. It is a subfield of $\bar{\mathbb{Q}}_p$.

For an O_E-scheme S, we consider triples (X, ι, λ), where X is a p-divisible group of height $4d$ and dimension $2d$ over S, where $\iota\colon O_K \longrightarrow \mathrm{End}(X)$ is an action of O_K on X, and where $\lambda\colon X \longrightarrow X^\vee$ is a polarization of X such that its Rosati involution induces on O_K the conjugation involution over O_F. We impose the Kottwitz condition (KC_r) and the Eisenstein conditions (EC_r) on the action of O_K on $\mathrm{Lie}\, X$. Furthermore, we assume that λ is a *principal* polarization if K/F is ramified, and that λ is an *almost principal* polarization if K/F is unramified.

We fix such a triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over the algebraic closure $\bar{k}$ of the residue field κ_E of E, and refer to it as a *framing object.* When K/F is unramified, then any two such triples are isogenous by an O_K-isogeny of height zero which preserves the polarizations. The same is true when K/F is ramified, provided we impose that the r-adjusted invariant $\mathrm{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is -1 (this last condition is automatic when K/F is unramified). In either case, the group $J(\mathbb{Q}_p)$ of O_K-self-quasi-isogenies of $(\mathbb{X}, \iota_{\mathbb{X}})$ preserving the polarization $\lambda_{\mathbb{X}}$ can be identified with the unitary group of a *split* K/F-hermitian space of dimension 2.

We consider the Rapoport-Zink space $\mathcal{M}_r$ over $\mathrm{Spf}\ O_{\breve{E}}$ representing the functor on $(\mathrm{Sch}/\,\mathrm{Spf}\, O_{\breve{E}})$ which associates to $S \in (\mathrm{Sch}/\,\mathrm{Spf}\, O_{\breve{E}})$ the set of isomorphism classes of 4-tuples $(X, \iota, \lambda, \rho)$, where (X, ι, λ) is as above, and where ρ is a *framing* of height zero, with framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. Our main local result may now be formulated as follows. We fix an isomorphism $J^1(\mathbb{Q}_p) \simeq \mathrm{SL}_2(F)$.

Theorem 1.3.1. *The RZ-space $\mathcal{M}_r$ is isomorphic to $\widehat{\Omega}_F \hat{\otimes}_{O_F,\varphi_0} O_{\breve{E}}$. More precisely, there exists a unique isomorphism of formal schemes*

$$\mathcal{M}_r \simeq \widehat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{E}},$$

which is equivariant with respect to the fixed identification $J^1(\mathbb{Q}_p) \simeq \mathrm{SL}_2(F)$. In particular, $\mathcal{M}_r$ is flat over $\mathrm{Spf}\, O_{\breve{E}}$ with semi-stable reduction.

The proof of this theorem uses the *contracting functor* which is a key technical novelty of this paper and is based on the theory of displays. Let R be a p-adic ring, and let $W(R)$ be its ring of Witt vectors. A display over R is a finitely generated $W(R)$-module with additional structures, namely a Frobenius morphism, a Hodge filtration and a divided Frobenius morphism. A display will be denoted by $\mathcal{P}$ and the underlying $W(R)$-module by P. Under suitable hypotheses, the category of *formal* p-divisible groups over R is equivalent to the category of *nilpotent* displays over R. Let R be an O_E-algebra which we regard as a O_F-algebra via φ_0. The contracting functor is the composition of two functors. The first functor associates to a triple (X, ι, λ) as above over R a new tuple (X', ι', λ'), where X' is a p-divisible group of height $4d$ and dimension 2 and where ι' is an O_K-action such that its restriction to O_F is strict, i.e., the induced action on $\mathrm{Lie}\, X'$ coincides with the action via $O_F \to R$. This functor is defined on the level of displays, i.e., we construct from the display $\mathcal{P}$ of X the display $\mathcal{P}'$ of X'. The third entry λ' is a bilinear form of displays

$$\mathcal{P}' \times \mathcal{P}' \to \mathcal{L}_R, \tag{1.3.2}$$

where $\mathcal{L}$ is the display of a Lubin-Tate group associated to the local field F. The bilinear form (1.3.2) is given by a bilinear form of $W(R)$-modules $P' \times P' \to L_R$ which satisfies additional conditions. We call λ' a polarization *with values in the Lubin-Tate group* or a polarization in the sense of Faltings, comp. [11]. Note that because of the values for the height and the dimension of X', there cannot exist a polarization in the usual sense on X'.

The second functor is the *Ahsendorf functor* $\mathfrak{A}_{O_F/\mathbb{Z}_p,R}$ from [1]. This functor associates to a display with strict O_F-action a display over the ring of *relative* Witt vectors $W_{O_F}(R)$. The Ahsendorf functor is the analogue for displays of the Drinfeld functor which associates to a Cartier module of a p-divisible group with strict O_F-action its *relative* Cartier module, cf. [9, §2]. We give here a new construction of the Ahsendorf functor which is more elementary and uses as an intermediate step the Lubin-Tate frames as in Mihatsch [22]. However, we make a special choice for these frames, cf. Definition 3.3.8. Then the Ahsendorf functor becomes compatible with polarizations, as stated in the following Theorem.

Theorem 1.3.2. *Let R be an O_F-algebra such that p is nilpotent in R. The Ahsendorf functor is a functor*

$$\mathfrak{A}_{O_F/\mathbb{Z}_p,R} : \begin{pmatrix} W(R)\text{-displays} \\ \text{with strict } O_F\text{-action} \end{pmatrix} \longrightarrow \Big(W_{O_F}(R)\text{-displays}\Big).$$

It induces an equivalence of categories

$$\mathfrak{A}_{O_F/\mathbb{Z}_p,R} : \begin{pmatrix} \text{nilpotent } W(R)\text{-displays} \\ \text{with strict } O_F\text{-action} \end{pmatrix} \longrightarrow \Big(\text{nilpotent } W_{O_F}(R)\text{-displays}\Big).$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be $W(R)$-displays over R with a strict O_F-action. We denote by $\mathcal{P}_{1,\mathrm{a}}$ and $\mathcal{P}_{2,\mathrm{a}}$ their images by the Ahsendorf functor $\mathfrak{A}_{O_F/\mathbb{Z}_p,R}$. Then there is a natural homomorphism between groups of bilinear forms of displays,

$$\mathrm{Bil}_{O_F\text{-displays}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{L}_R) \longrightarrow \mathrm{Bil}_{W_{O_F}(R)\text{-displays}}(\mathcal{P}_{1,\mathrm{a}} \times \mathcal{P}_{2,\mathrm{a}}, W_{O_F}(R)(\pi^{ef}/p^f)).$$

If the dual $(\mathcal{P}_{1,\mathrm{a}})^\vee$ of $\mathcal{P}_{1,\mathrm{a}}$ and $\mathcal{P}_{2,\mathrm{a}}$ are nilpotent $W_{O_F}(R)$-displays, then this homomorphism is an isomorphism.

Here the ring $W_{O_F}(R)$ with its Frobenius F' and Verschiebung V' is regarded as a display. For a unit $\varepsilon \in W_{O_F}(R)^\times$ we obtain a *twisted display* $W_{O_F}(R)(\varepsilon)$ by replacing F' by $\varepsilon F'$ and V' by $V'\varepsilon^{-1}$. The underlying module of this display remains $W_{O_F}(R)$. We apply this construction to the image of the unit $\pi^{ef}/p^f \in O_F^\times$ in $W_{O_F}(R)^\times$.

Let us return to Theorem 1.3.1. It is more honest to formulate this theorem as follows. We fix an extension $\breve{\varphi}_0 \colon O_{\breve{F}} \longrightarrow O_{\breve{E}}$ of $\varphi_0 \colon O_F \longrightarrow O_E$. Let $\mathcal{M}$ be the *relative* RZ-space over

$\mathrm{Spf}\, O_{\breve{F}}$ from [17]. It parametrizes tuples $(X', \iota', \lambda', \rho')$, where X' is a *strict* formal O_F-module of *relative* height 4 and dimension 2, and where ι' is an O_K-action on X, and where λ' is a *relative* polarization compatible with ι', which is principal if K/F is ramified and almost principal if K/F is unramified. Also, ρ' is a framing of height zero with the framing object $(\mathbb{X}', \iota'_{\mathbb{X}'}, \lambda'_{\mathbb{X}'})$ which is obtained from $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ by the contracting functor. Then the contracting functor defines a morphism

$$\mathcal{M}_r \longrightarrow \mathcal{M} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathrm{Spf}\, O_{\breve{E}}. \tag{1.3.3}$$

Using Theorem 1.3.2, we prove that this is an isomorphism of formal schemes over $\mathrm{Spf}\, O_{\breve{E}}$. Moreover, we prove that X' is the p-divisible group of a special formal O_D-module in the sense of [9]. Therefore, by Drinfeld's local theorem [9], the right hand side of (1.3.3) has an interpretation in terms of the Drinfeld halfplane $\widehat{\Omega}_F$. This is the main result of [17], for which we give a new proof, after it was already reproved by Kirch [14]. Theorem 1.3.1 follows.

Let us now put the local results of this paper in perspective. We address in our special case the general problem of identifying a *basic* Rapoport-Zink space associated to the pair $(G, \{\mu\})$ with a twist of the basic Rapoport-Zink space associated to the pair $(G, \{\mu'\})$, where μ' differs from μ by a central character, cf. the Introduction of [28]. This problem is also addressed by Scholze in [29, Chap. 23], in both the case considered here and in the *fake Drinfeld case* of [28]. As mentioned above, Scholze's proof uses in an essential way our formulation of the local moduli problem, via the theory of *local models* (and hence implicitly the linear algebra lemma [28, Lem. 4.9]). One of the main reasons that we are successful in constructing the contracting functor in the case treated here is that here we are able to develop a good understanding of the Kottwitz condition (KC_r), even in unequal characteristic. Our failure to do the same in the fake Drinfeld case is the essential reason that in [28] we only succeeded in defining the contracting functor in the special fiber. The contracting functor is an expression of the exceptional isomorphism between the quasi-split special unitary group in two variables and the special linear group in two variables. We restricted ourselves here to the case of curves; it would have been possible to prove a higher-dimensional version where the uniformizing space Ω_{F_v} is replaced by a product of such spaces, comp. [18] and [27, §6].

1.4. **Layout of the paper.** We now explain the lay-out of the paper. The whole paper, with the exception of section 7, is devoted to the local theory. In section 2 we explain in detail the definition of the formal moduli spaces of (polarized) p-divisible formal groups, including the Kottwitz conditions relevant here and the Eisenstein conditions; in particular, Subsection 2.6 contains the detailed statements of our main local results. Section 3 summarizes the relevant facts on relative Dieudonné theory and relative display theory. The most important fact proved in this section is the relation established by the Ahsendorf functor between the *Lubin-Tate display* and the *relative multiplicative display.* In section 4 we first consider the relation between the Kottwitz condition and the Eisenstein condition; this is used in the rest of the section to construct the contracting functor. More precisely, we first consider the first step in its construction which we call the *pre-contracting functor*, cf. above. After this, we complete the second step in the case of a *special* generalized CM-type. In the final subsection of section 4, we consider the second step in the case of a *banal* generalized CM-type. Section 5 is devoted to an alternative proof of the main result of [17], based on the theory of displays. In section 6 we prove the main local results, namely Theorem 1.3.1 and its banal counterpart. In the appendix, section 8, we give the correct version of the sign factor of [18] by defining the *adjusted invariant* of a CM-triple of generalized CM-type r of even rank n, and investigate its behaviour under the contracting functor. Section 7 deduces the global results from the local theory.

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We also acknowledge the hospitality of the MSRI during the fall of 2014 when the work on this paper was begun.

1.6. **Notation.** • If R and R' are $\mathbb{Z}_p$-algebras, we often write $R \otimes R'$ for $R \otimes_{\mathbb{Z}_p} R'$. Also, we often write $X \otimes_A B$ for $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$.
• If F is a finite extension of $\mathbb{Q}_p$, we write $\breve{F}$ for the completion of a maximal unramified extension, and F^t for the maximal subfield unramified over $\mathbb{Q}_p$. We write $d = ef$, where $d = [F : \mathbb{Q}_p]$ and $f = [F^t : \mathbb{Q}_p]$ and $e = [F : F^t]$. We denote by O_F, resp. O_{F^t}, resp. $O_{\breve{F}}$ the rings of integers.
• Let V be an $\mathbb{C}/\mathbb{R}$-anti-hermitian vector space. The signature of V is (a, b) if the anti-hermitian form is equivalent to $\operatorname{diag}(\mathbf{i}^{(a)}, \mathbf{i}^{(b)})$, where $\mathbf{i}$ is the imaginary unit.
• Let F be a finite extension of $\mathbb{Q}_p$ and let K/F be a quadratic extension. Let V be a K-vector space, equipped with an alternating $\mathbb{Q}$-bilinear form $\psi\colon V \times V \longrightarrow \mathbb{Q}$ satisfying (1.2.1). Let Λ be a O_K-lattice in V. Then the dual O_K-lattice is $\Lambda^\vee = \{x \in V \mid \psi(x, y) \in \mathbb{Z}_p \text{ for all } y \in \Lambda\}$. The lattice Λ is called self-dual if $\Lambda = \Lambda^\vee$; it is called almost self-dual if Λ is contained in $\Lambda^\vee$ with colength one.
• If O is a discrete valuation ring with uniformizer π, we write Nilp_O for the category of O-algebras R such that π is locally on $\operatorname{Spec} R$ nilpotent. Similarly, we denote by $(\mathrm{Sch}/\operatorname{Spf} O)$ the category of O-schemes such that $\pi\mathcal{O}_S$ is a locally nilpotent ideal sheaf.
• Given modules M and N over a ring R, we write $M \subset^r N$ to indicate that M is an R-submodule of N of finite colength r.

Warning. It is customary to denote a finite extension of $\mathbb{Q}_p$ and the Frobenius by the same symbol F. This should not lead to confusions.

2. Main local statements

In this section we formulate our main results in the local theory. We fix a prime number p and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let F be a finite field extension of $\mathbb{Q}_p$, with residue class field κ_F. We set $d = [F : \mathbb{Q}_p]$, $f = [\kappa_F : \mathbb{F}_p]$ and define e through $d = ef$. We let K/F be an étale algebra of degree 2. We denote the non-trivial automorphism of $\mathrm{Gal}(K/F)$ by $a \mapsto \bar{a}$.

In the case where K/F is a ramified extension of local fields (ramified case) we choose a prime element $\Pi \in O_K$ such that $\bar{\Pi} = -\Pi$. Then $\pi = -\Pi^2$ is a prime element of F. In the case where K/F is unramified extension of local fields (unramified case) or $K \cong F \times F$ (split case) we choose a prime element $\pi \in F$ and we set $\Pi = \pi$.

Let $\Phi = \Phi_K = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K, \overline{\mathbb{Q}}_p)$ be the set of algebra homomorphisms.

2.1. **Special and banal local CM-types.** Let r be a generalized local CM-type of rank 2 (relative to K/F) in the sense of [18, section 5], i.e., a function

$$r : \Phi \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \longmapsto r_\varphi, \tag{2.1.1}$$

such that $r_\varphi + r_{\overline{\varphi}} = 2$ for all $\varphi \in \Phi$. Here $\overline{\varphi}(a) = \varphi(\overline{a})$, where $a \mapsto \overline{a}$ is the non-trivial automorphism of K over F. The corresponding reflex field $E = E(r)$ is the subfield of $\overline{\mathbb{Q}}_p$ fixed by

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/E) := \{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\tau\varphi} = r_\varphi,\ \forall \varphi\}.$$

Let O_E be the ring of integers of E.

When we fix an embedding $\varphi_0 : F \longrightarrow \overline{\mathbb{Q}}_p$, we denote by $\varphi_0, \overline{\varphi}_0$ the two extensions of φ_0 to K (by abuse of notation).

Definition 2.1.1. A local CM-type r of rank 2 is called *special* relative to the choice of embedding $\varphi_0 : F \longrightarrow \overline{\mathbb{Q}}_p$ if K/F is a field extension and

$$r_{\varphi_0} = r_{\overline{\varphi}_0} = 1, \text{ and } r_\varphi \in \{0, 2\}, \qquad \text{for all } \varphi \in \Phi \setminus \{\varphi_0, \overline{\varphi}_0\}.$$

It is called *banal non-split* if K/F is a field extension and $r_\varphi \in \{0, 2\}$, for all $\varphi \in \Phi$. It is called *banal split* if $K \simeq F \oplus F$ and $r_\varphi \in \{0, 2\}$, for all $\varphi \in \Phi$.

From now on, we will assume r to be either special (relative to a fixed choice of φ_0) or banal (non-split or split). We will consider p-divisible groups X with an action of O_K over O_E-schemes S. We will want to impose certain conditions on the induced action of O_K on $\operatorname{Lie} X$.

2.2. **The Kottwitz and the Eisenstein conditions.** Let S be an O_E-scheme, and let $\mathcal{L}$ be a locally free $\mathcal{O}_S$-module, equipped with an action

$$\iota : O_K \longrightarrow \mathrm{End}_{\mathcal{O}_S} \mathcal{L}.$$

of O_K.

We say that $(\mathcal{L}, \iota)$ satisfies the Kottwitz condition (KC_r) relative to r if the identity of polynomials with coefficients in $\mathcal{O}_S$ holds

$$\mathrm{char}(T, \iota(a)|\mathcal{L}) = i\big(\prod_{\varphi\in\Phi} (T - \varphi(a))^{r_\varphi}\big), \quad \text{for all } a \in O_K , \tag{2.2.1}$$

where $i : O_E \longrightarrow \mathcal{O}_S$ is the structure homomorphism (compare [28]).

We denote by $F^t \subset F$ the maximal subextension which is unramified over $\mathbb{Q}_p$. If K is the field we use the same notation for K, and in the split case $K \cong F \times F$ we set $K^t = F^t \times F^t$. We set $\Psi = \Psi_K = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K^t, \overline{\mathbb{Q}}_p)$. We call $\psi \in \Psi$ *banal* if $r_\varphi \in \{0, 2\}$ for each $\varphi \in \Phi$ such that $\varphi \mid \psi$. If this is not the case we call ψ *special*. We use the notation

$$\Phi_\psi = \{\varphi \in \Phi \mid \varphi_{|K^t} = \psi\}, \quad \psi \in \Psi. \tag{2.2.2}$$

We denote by E' be the compositum of E with the normal closure of the image of K^t in $\bar{\mathbb{Q}}_p$ under any embedding $\psi \in \Psi$. We remark that this normal closure is isomorphic to K^t via any $\psi \in \Psi$. In particular E'/E is an unramified extension of local fields.

Let S be an O_E-scheme. Let $\alpha : S \longrightarrow \mathrm{Spec}\, O_{E'}$ be a morphism of O_E-schemes. Then α gives rise to an isomorphism of $O_{K^t} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ algebras

$$O_{K^t} \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \bigoplus\nolimits_{\psi\in\Psi} \mathcal{O}_S , \tag{2.2.3}$$

where the action of O_{K^t} on the ψ-th factor is via ψ. Hence for a locally free $\mathcal{O}_S$-module $\mathcal{L}$ with action by O_K, we obtain a decomposition into locally free $\mathcal{O}_S$-modules,

$$\mathcal{L} = \bigoplus\nolimits_{\psi\in\Psi} \mathcal{L}_\psi . \tag{2.2.4}$$

If $(\mathcal{L}, \iota)$ satisfies the Kottwitz condition we obtain from (2.2.1) applied to $a \in O_{K^t}$ that

$$\mathrm{rank}\ \mathcal{L}_\psi = \sum_{\varphi\in\Phi_\psi} r_\varphi \tag{2.2.5}$$

We say that $(\mathcal{L}, \iota)$ satisfies the rank condition (RC_r), if (2.2.5) is satisfied for all ψ. The rank condition does not depend on the α chosen above because a second α' differs from α by an automorphism of E' over E if S is connected. If there is no α we use base change $\mathrm{Spec}\, O_{E'} \times_{\mathrm{Spec}\, O_E} S \longrightarrow S$ to define the condition (RC_r). This agrees with the old definition if α exists.

We consider a pair $(\mathcal{L}, \iota)$ that satisfies (RC_r). Then we will define the Eisenstein condition (EC_r) (this definition is analogous to [28, section 2], but different). We introduce the notation

$$\begin{aligned} A_\psi &= \{\varphi : K \longrightarrow \overline{\mathbb{Q}}_p \mid \varphi_{|K^t} = \psi, \text{ and } r_\varphi = 2\} \\ B_\psi &= \{\varphi : K \longrightarrow \overline{\mathbb{Q}}_p \mid \varphi_{|K^t} = \psi, \text{ and } r_\varphi = 0\}. \end{aligned} \tag{2.2.6}$$

We note that under the action of the non-trivial automorphism of K/F,

$$\bar{A}_\psi = B_{\overline{\psi}}. \tag{2.2.7}$$

Also, let $a_\psi = |A_\psi|$ and $b_\psi = |B_\psi|$.

With this notation we may rewrite the rank condition (RC_r)

$$\mathrm{rank}\ \mathcal{L}_\psi = 2a_\psi + \epsilon_\psi , \tag{2.2.8}$$

where

$$\epsilon_\psi = \begin{cases} 0, \text{ if } \psi \text{ is banal} \\ 1, \text{ if } \psi \text{ is special and } K/F \text{ is unramified} \\ 2, \text{ if } \psi \text{ is special and } K/F \text{ is ramified.} \end{cases}$$

In the case where K/F is ramified we have $K^t = F^t$, $[K : K^t] = 2e$, and for each $\psi \in \Psi$

$$\varphi \mid \psi \quad \Rightarrow \bar{\varphi} \mid \psi.$$

Therefore $a_\psi = b_\psi$, and the rank condition reads, in the ramified case,

$$\operatorname{rank} \mathcal{L}_\psi = 2e,$$

regardless of whether r is banal or not.

Consider the Eisenstein polynomial $\mathbf{E}(T)$ of Π in $O_{K^t}[T]$. We consider the image $\mathbf{E}_\psi(T)$ of $\mathbf{E}(T)$ in $\overline{\mathbb{Q}}_p[T]$ under ψ, for $\psi \in \Psi$. In $\overline{\mathbb{Q}}_p[T]$ this has a decomposition into linear factors,

$$\mathbf{E}_\psi(T) = \prod_{\varphi\in\Phi_\psi} (T - \varphi(\Pi)). \tag{2.2.9}$$

We define

$$\mathbf{E}_{A_\psi}(T) = \prod_{\varphi\in A_\psi} (T - \varphi(\Pi)), \quad \mathbf{E}_{B_\psi}(T) = \prod_{\varphi\in B_\psi} (T - \varphi(\Pi)). \tag{2.2.10}$$

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E')$ stabilizes the corresponding subsets in the index set on the right hand sides of (2.2.9) and (2.2.10). Therefore all three polynomials lie in $O_{E'}[T]$.

If r is special we fix an embedding $\varphi_0 : K \longrightarrow \bar{\mathbb{Q}}_p$ such that $r_{\varphi_0} = 1$. We denote by ψ_0 the restriction of φ_0 to K^t. In the ramified case we have $\psi_0 = \bar{\psi}_0$ and in the unramified case $\psi_0 \neq \bar{\psi}_0$.

We define $\mathbf{S}_\psi$ by the following factorization in $O_{E'}[T]$,

$$\mathbf{E}_\psi(T) = \mathbf{S}_\psi(T) \cdot \mathbf{E}_{A_\psi}(T) \cdot \mathbf{E}_{B_\psi}(T). \tag{2.2.11}$$

Hence

$$\mathbf{S}_\psi(T) = \begin{cases} 1, & \text{if } \psi \text{ is banal} \\ (T - \varphi_0(\Pi))(T - \overline{\varphi}_0(\Pi)), & \text{if } \psi = \psi_0 \text{ and } K/F \text{ is ramified} \\ T - \varphi_0(\Pi), & \text{if } \psi = \psi_0 \text{ and } K/F \text{ is unramified} \\ T - \overline{\varphi}_0(\Pi), & \text{if } \psi = \overline{\psi}_0 \text{ and } K/F \text{ is unramified.} \end{cases}$$

Now using the structure morphism $O_{E'} \longrightarrow \mathcal{O}_S$, each of the three factors in (2.2.11), when evaluated on Π, defines an endomorphism of the $\mathcal{O}_S$-module $\mathcal{L}_\psi$. These endomorphisms are denoted by $\mathbf{E}_{A_\psi}(\iota(\Pi)|\mathcal{L}_\psi)$, resp. $\mathbf{E}_{B_\psi}(\iota(\Pi)|\mathcal{L}_\psi)$, resp. $\mathbf{S}_\psi(\iota(\Pi)|\mathcal{L}_\psi)$.

We say that $(\mathcal{L}, \iota)$ satisfies the *Eisenstein conditions* if (RC_r) is fulfilled and if for each ψ

$$\begin{aligned} \big(\mathbf{S}_\psi \cdot \mathbf{E}_{A_\psi}\big)(\iota(\Pi) \mid \mathcal{L}_\psi) &= 0, \\ \bigwedge^{4-[K^t:F^t]} \big(\mathbf{E}_{A_\psi}(\iota(\Pi) \mid \mathcal{L}_\psi)\big) &= 0. \end{aligned} \tag{2.2.12}$$

In the case where ψ is banal the first condition says

$$\mathbf{E}_{A_\psi}(\iota(\Pi)|\mathcal{L}_\psi) = 0, \text{ for all } \psi \in \Psi. \tag{2.2.13}$$

and the second condition follows from the first.

The Eisenstein conditions do not depend on the O_E-morphism $\alpha : S \longrightarrow \operatorname{Spec} O_{E'}$. Indeed, if S is connected, any other choice of α differs by an automorphism $\rho \in \mathrm{Gal}(E'/E)$. In the decomposition (2.2.4) $\mathcal{L}_\psi$ is then replaced by $\mathcal{L}_{\rho\psi}$ and $\mathbf{E}_\psi$ is replaced by $\rho(\mathbf{E}_\psi) = \mathbf{E}_{\rho\psi}$. Here the last identity holds by the definition of the reflex field E. Therefore changing α does not change the Eisenstein conditions (EC_r). If there exists no α, we use base change $\operatorname{Spec} O_{E'} \times_{\operatorname{Spec} O_E} S \longrightarrow S$ to define the condition (EC_r). The same arguments apply to the condition (KC_r).

We first note the following statement.

Proposition 2.2.1. *Let S be an O_E-scheme and $\mathcal{L}$ a locally free $\mathcal{O}_S$-module with an O_K-action $\iota\colon O_K \longrightarrow \mathrm{End}_{\mathcal{O}_S}(\mathcal{L})$.*

(i) *The Eisenstein conditions* (EC_r) *are independent of the uniformizer* Π.

(ii) *When $K/\mathbb{Q}_p$ is unramified, the Eisenstein conditions* (EC_r) *are implied by the Kottwitz condition* (KC_r). *The same conclusion holds if $F = \mathbb{Q}_p$ and K/F is ramified.*

(iii) *When S is an E-scheme, the Eisenstein conditions* (EC_r) *hold automatically.*

Proof. Let us prove (i). Let Π' be another uniformizer. It is enough to show that the elements of $O_K \otimes_{O_{K^t},\psi} O_{E'}$,

$$\begin{aligned} &\mathbf{E}_{A_\psi}(\Pi \otimes 1),\ \mathbf{E}_{A_\psi}(\Pi' \otimes 1), \quad \text{resp.} \\ &\mathbf{S}_\psi(\Pi \otimes 1)\mathbf{E}_{A_\psi}(\Pi \otimes 1),\ \mathbf{S}_\psi(\Pi' \otimes 1)\mathbf{E}_{A_\psi}(\Pi' \otimes 1), \end{aligned}$$

differ by a unit in $O_K \otimes_{O_{K^t},\psi} O_{E'}$. Indeed, let $\tilde{E}'$ be the normalization of E' in $\bar{\mathbb{Q}}_p$. Since $O_K \otimes_{O_{K^t},\psi} O_{E'} \to O_K \otimes_{O_{K^t},\psi} O_{\tilde{E}'}$ is a flat extension of local rings, we can replace E' by $\tilde{E}'$. By the definitions (2.2.10) and (2.2.11), it suffices to show that the elements $\Pi \otimes 1 - 1 \otimes \varphi(\Pi)$ and $\Pi' \otimes 1 - 1 \otimes \varphi(\Pi')$ differ by a unit in $O_K \otimes_{O_{K^t},\psi} O_{E'}$. But by [25, Lem. 6.11] the elements $\Pi \otimes 1 - 1 \otimes \Pi$ and $\Pi' \otimes 1 - 1 \otimes \Pi'$ of $O_K \otimes_{O_{K^t}} O_K$ differ by a unit, whence the assertion.

Now we prove (ii). Let us only treat the case where r is special; the banal case is similar. When $K = K^t$ is unramified over $\mathbb{Q}_p$, then $\mathbf{E}(T) = T - \pi$ is a linear polynomial. Furthermore, A_ψ has at most one element for $\psi \notin \{\psi_0, \overline{\psi}_0\}$, and $A_{\psi_0} = A_{\overline{\psi}_0} = \emptyset$. Let $\psi \notin \{\psi_0, \overline{\psi}_0\}$. If $A_\psi = \emptyset$, then $\mathcal{L}_\psi = (0)$ and the Eisenstein condition relative to the index ψ is empty; if A_ψ has one element, the Eisenstein condition relative to the index ψ is just equivalent to the definition of the ψ-th eigenspace in the decomposition (2.2.4). Something analogous applies to the indices $\psi_0, \overline{\psi}_0$. The case when $F = \mathbb{Q}_p$ is handled in the same way.

Finally we prove (iii). Let $\tilde{K}$ be the normal closure of K in $\overline{\mathbb{Q}}_p$. It suffices to prove the assertion after replacing S by its base change $S \times_{\operatorname{Spec} E} \operatorname{Spec} \tilde{K}$. Then we have a decomposition

$$O_K \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \bigoplus_{\varphi \in \Phi} \mathcal{O}_S.$$

Correspondingly, we have $\mathcal{L} = \oplus \mathcal{L}_\varphi$, and the endomorphism $\iota(\Pi)$ is diagonal with respect to this decomposition, with entries $\varphi(\Pi)\mathrm{id}_{\mathcal{L}_\varphi}$. It is easy to see that (KC_r) is equivalent to the condition

$$\operatorname{rank} \mathcal{L}_\varphi = r_\varphi, \quad \forall \varphi \in \Phi. \tag{2.2.14}$$

The Eisenstein conditions (EC_r) involve endomorphisms of $\mathcal{L}$ which are products of endomorphisms of the form $(\iota(\Pi)_{|\mathcal{L}_\varphi} - \varphi(\Pi)\mathrm{id}_{\mathcal{L}_\varphi}) \oplus_{\varphi' \neq \varphi} \mathrm{id}_{\mathcal{L}_{\varphi'}}$. From this, the conditions follow trivially. □

Let us make the Eisenstein conditions more explicit in the case where r is special. For this, we distinguish between the case when K/F is ramified and the case when K/F is unramified. Let S be an $O_{E'}$-scheme and $\mathcal{L}$ be a locally free $\mathcal{O}_S$-module satisfying (RC_r).

- *K/F ramified.* In this case, we have $K^t = F^t$, and $\psi = \overline{\psi}$ for all $\psi \in \Psi$. Hence, (2.2.7) implies in this case

$$a_\psi = b_\psi = \begin{cases} e, & \text{if } \psi \neq \psi_0 \\ e-1, & \text{if } \psi = \psi_0. \end{cases} \tag{2.2.15}$$

We have

$$\mathbf{S}_{\psi_0}(T) = (T - \varphi_0(\Pi))(T - \overline{\varphi}_0(\Pi)).$$

The Eisenstein conditions become in this case

$$\begin{aligned} &\big(\mathbf{S}_{\psi_0} \cdot \mathbf{E}_{A_{\psi_0}}\big)(\iota(\Pi)|\mathcal{L}_{\psi_0}) = 0, \\ &\bigwedge^3 \big(\mathbf{E}_{A_{\psi_0}}(\iota(\Pi)|\mathcal{L}_{\psi_0})\big) = 0, \\ &\mathbf{E}_{A_\psi}(\iota(\Pi)|\mathcal{L}_\psi) = 0, \ \text{ for all } \psi \neq \psi_0. \end{aligned} \tag{2.2.16}$$

- *K/F unramified.* In this case, $[K^t : F^t] = 2$, and $\psi \neq \overline{\psi}$ for all $\psi \in \Psi$. Furthermore, $a_\psi = b_{\overline{\psi}}$ and

$$\sum_{\psi \in \Psi \setminus \{\psi_0, \overline{\psi}_0\}} a_\psi = e(f-1), \quad a_{\psi_0} + a_{\overline{\psi}_0} = e - 1. \tag{2.2.17}$$

In this case, the Eisenstein conditions become

$$
\begin{aligned}
\big(\mathbf{S}_{\psi_0}\cdot\mathbf{E}_{A_{\psi_0}}\big)(\iota(\pi)|\mathcal{L}_{\psi_0}) &= 0,\\
\bigwedge^2\big(\mathbf{E}_{A_{\psi_0}}(\iota(\pi)|\mathcal{L}_{\psi_0})\big) &= 0,\\
\big(\mathbf{S}_{\overline{\psi}_0}\cdot\mathbf{E}_{A_{\overline{\psi}_0}}\big)(\iota(\pi)|\mathcal{L}_{\overline{\psi}_0}) &= 0,\\
\bigwedge^2\big(\mathbf{E}_{A_{\overline{\psi}_0}}(\iota(\pi)|\mathcal{L}_{\overline{\psi}_0})\big) &= 0,\\
\mathbf{E}_{A_\psi}(\iota(\pi)|\mathcal{L}_\psi) &= 0,\ \text{ for all } \psi\neq\psi_0,\overline{\psi}_0.
\end{aligned}
\tag{2.2.18}
$$

2.3. **Local CM-pairs and CM-triples.** Let S be an O_E-scheme such that p is locally nilpotent i.e. a scheme over $\operatorname{Spf} O_E$. A *local CM-pair of type* r is a pair (X,ι) such that X is a p-divisible group of height $4d$ and dimension $2d$ and ι is an $\mathbb{Z}_p$-algebra homomorphism

$$\iota : O_K \longrightarrow \operatorname{End} X$$

such that the rank condition (RC_r) is satisfied for the induced action of O_K on $\operatorname{Lie} X$. In the split case $O_K = O_F\times O_F$ we require moreover that in the induced decomposition $X = X_1\times X_2$ each factor is a p-divisible group of height $2d$.

Later we will introduce displays $\mathcal{P}$, and these have a Lie algebra $\operatorname{Lie}\mathcal{P}$, cf. Definition 3.1.4. Therefore, we can also speak of local CM-pairs $(\mathcal{P},\iota)$ of type r, where $\mathcal{P}$ is a display over S, cf. section 3.

Let $S = \operatorname{Spec} k$ be a perfect field of characteristic p which is endowed with an $O_{E'}$-algebra structure. In this case, a display in the same thing as a Dieudonné module $\mathcal{P} = (P,F,V)$, where P is a finitely generated free module over the ring of Witt vectors $W(k)$. If $\mathcal{P}$ is the Dieudonné module of X, there is a canonical isomorphism of k-vector spaces $\operatorname{Lie} X\cong P/VP$.

Via ι we regard P as a $O_K\otimes_{\mathbb{Z}_p} W(k)$-module. The homomorphisms

$$\psi : O_{K^t}\longrightarrow O_{E'}\longrightarrow k, \tag{2.3.1}$$

$\psi\in\Psi$ lift uniquely to homomorphisms

$$\tilde{\psi} : O_{K^t}\longrightarrow W(k). \tag{2.3.2}$$

We obtain a ring isomorphism

$$O_K\otimes_{\mathbb{Z}_p} W(k) = \prod_{\psi\in\Psi} O_K\otimes_{O_{K^t},\tilde{\psi}} W(k).$$

This induces a decomposition

$$P = \oplus_{\psi\in\Psi} P_\psi. \tag{2.3.3}$$

More explicitly

$$P_\psi = \{x\in P \mid \iota(a)x = \tilde{\psi}(a)x,\ \text{for } a\in O_{K^t}\}.$$

Let us denote by σ the Frobenius automorphism of $W(k)$. The operators F and V on P induce σ-linear maps

$$F : P_\psi\longrightarrow P_{\sigma\psi},\quad V : P_{\sigma\psi}\longrightarrow P_\psi. \tag{2.3.4}$$

Here $\sigma\psi$ denotes the composite of (2.3.1) with the absolute Frobenius of k.

Lemma 2.3.1. *Let (P,ι) be local CM-pair of type r over a perfect field k. Then P is a free $O_K\otimes_{\mathbb{Z}_p} W(k)$-module of rank 2.*

Proof. Since $FV = p$ it follows that

$$\operatorname{rank}_{O_K\otimes_{O_{K^t},\tilde{\psi}} W(k)} P_\psi = \operatorname{rank}_{O_K\otimes_{O_{K^t},\sigma\tilde{\psi}} W(k)} P_{\sigma\psi}. \tag{2.3.5}$$

Since $\operatorname{rank}_{W(k)} P = 4d$, and by the extra condition in the split case, this implies that the common rank of (2.3.5) is 2. This proves the Lemma. □

To each local CM-pair (X,ι) we define the *conjugate dual* $(X^\vee,\iota^\wedge)$. Here $X^\vee$ is the dual p-divisible group of X but we change the action dual to ι by the conjugation of K/F, i.e., $\iota^\wedge(a) = \iota^\vee(\bar{a})$. We will denote the conjugate dual simply by $X^\wedge$.

Lemma 2.3.2. *The conjugate dual of a CM-pair (X, ι) of type r is again a local CM-pair of type r. If (X, ι) satisfies the Kottwitz condition* (KC_r)*, resp., the Eisenstein conditions* (EC_r)*, then so does its conjugate dual.*

Proof. For the first assertion, we may assume that we are over an algebraically closed field. We use the Dieudonné module $\mathcal{P}$. We set

$$P^\vee = \mathrm{Hom}_{W(k)}(P, W(k)).$$

We use the canonical pairing

$$\langle\ ,\ \rangle : P \times P^\vee \longrightarrow W(k). \tag{2.3.6}$$

The operators F and V on the dual Dieudonné module $\mathcal{P}^\vee$ are defined by the equations

$$\begin{aligned} \langle Vx, Vx^\vee \rangle &= p\sigma^{-1}(\langle x, x^\vee \rangle), \qquad x \in P,\ x^\vee \in P^\vee \\ \sigma(\langle Vx, x^\vee \rangle) &= \langle x, Fx^\vee \rangle. \end{aligned}$$

One of these equations implies the other. It follows that $VP/pP \subset P/pP$ and $VP^\vee/pP^\vee \subset P^\vee/pP^\vee$ are orthogonal complements with respect to the non-degenerate pairing of k-vector spaces,

$$P/pP \times P^\vee/pP^\vee \longrightarrow k.$$

If we use the action $\iota^\wedge$, we write for the decomposition (2.3.3)

$$P^\vee = \oplus_{\psi \in \Psi} P^\wedge_\psi = P^\wedge.$$

Then P_{ψ_1} and $P^\wedge_{\psi_2}$ are for $\psi_1 \neq \bar\psi_2$ orthogonal with respect to (2.3.6) and

$$\langle\ ,\ \rangle : P_\psi \times P^\wedge_{\bar\psi} \longrightarrow W(k). \tag{2.3.7}$$

is a perfect pairing. The k-vector spaces $VP_{\sigma\psi}/pP_\psi$ and $VP^\wedge_{\sigma\bar\psi}/pP^\wedge_{\bar\psi}$ are orthogonal complements with respect to the induced non-degenerate k-bilinear form

$$P_\psi/pP_\psi \times P^\wedge_{\bar\psi}/pP^\wedge_{\bar\psi} \longrightarrow k.$$

Let us assume that K/F is unramified or split. In this case Lemma 2.3.1 implies $\mathrm{rank}_{W(k)} P_\psi = 2e$ and by (2.3.7) $\mathrm{rank}_{W(k)} P^\wedge_{\bar\psi} = 2e$. Since P satisfies (RC_r) we find by the orthogonality above

$$\mathrm{rank}_k P^\wedge_{\bar\psi}/VP^\wedge_{\sigma\bar\psi} = 2e - \mathrm{rank}_k P_\psi/VP_{\sigma\psi} = 2e - \sum_{\varphi|\psi} r_\varphi = \sum_{\varphi|\psi}(2 - r_\varphi) = \sum_{\varphi|\psi} r_{\bar\varphi}.$$

This shows that the conjugate dual satisfies (RC_r). The case K/F ramified is similiar.

For the proof of the assertion concerning (KC_r), we refer to Proposition 4.2.13. For the proof of the assertion concerning (EC_r), we refer to Corollary 4.2.8 in the case when K/F is unramified or split, resp., Corollary 4.2.12 when K/F is ramified. □

The notion of a *local CM-triple of type* r over S was introduced in [18]. This is a triple (X, ι, λ), where (X, ι) is a local CM-pair of type r and $\lambda : X \longrightarrow X^\vee$ is an anti-symmetric isogeny (also called a *polarization*) such that the corresponding Rosati involution induces the non-trivial automorphism on K/F. In particular λ induces a morphism of local CM-pairs

$$\lambda : (X, \iota) \longrightarrow (X^\wedge, \iota^\wedge).$$

In the present paper, we will also say that (X, ι, λ) is a *polarized local CM-pair* (of type r). We call the polarized local CM-pair (X, ι, λ) *principal* if $\mathrm{Ker}\,\lambda = 0$; we call it *almost principal* if $\mathrm{Ker}\,\lambda \subset X[\iota(\pi)]$ and $\mathrm{Ker}\,\lambda$ has order p^{2f}. We will distinguish these two cases by attaching the integer $\mathtt{h} = 0$ to the principal case, and $\mathtt{h} = 1$ to the almost principal case, i.e., height $\lambda = 2f\mathtt{h}$.

2.4. **The invariant of a local CM-triple.** Let K/F be a field extension. We recall from [18] the definition of the invariant of a CM-triple, in a slightly more general context.

Let k be an algebraically closed field of characteristic p and let $W(k)$ be the ring of Witt vectors. We set $W(k)_{\mathbb{Q}} = W(k) \otimes \mathbb{Q}$. Let (M, F, V) be a Dieudonné module of height $4d$ and dimension $2d$ which is endowed with a $\mathbb{Z}_p$-algebra homomorphism

$$\iota : K \longrightarrow \mathrm{End}^0(M, F, V).$$

We set $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. We assume that M is endowed with a non degenerate alternating bilinear form

$$\beta : M_{\mathbb{Q}} \times M_{\mathbb{Q}} \longrightarrow W(k)_{\mathbb{Q}}$$

Let us denote by σ the Frobenius automorphism of $W(k)$. We require the following properties:

$$\beta(Vx, Vy) = p\sigma^{-1}(\beta(x, y)), \quad x, y \in M_{\mathbb{Q}}$$
$$\beta(\iota(a)x, y) = \beta(x, \iota(\bar{a})y), \qquad a \in K.$$

We will associate to such a set of data (M, ι, β) an invariant $\mathrm{inv}(M, \iota, \beta) \in \{\pm 1\}$. We set $\Psi = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K^t, W(k)_{\mathbb{Q}})$.

The ring $O_K \otimes_{\mathbb{Z}_p} W(k)$ decomposes

$$K \otimes_{\mathbb{Z}_p} W(k) = \prod_{\psi} K \otimes_{O_{K^t}, \tilde{\psi}} W(k). \tag{2.4.1}$$

If $\xi = (\xi_\psi)$ is an element of (2.4.1). Then we set

$$\mathrm{ord}_{K \otimes W(k)} \xi = \mathrm{ord}_p \mathrm{Nm}_{K/\mathbb{Q}_p} \xi = \sum_{\psi} \mathrm{ord}_{\Pi} \xi_\psi \in \mathbb{Z}. \tag{2.4.2}$$

The Frobenius homomorphism σ acts via the second factor on $K \otimes_{\mathbb{Z}_p} W(k)$. The σ-conjugacy class of an element $\xi \in (K \otimes W(k))^\times$ is uniquely determined by $\mathrm{ord}_{K \otimes W(k)} \xi$.

We view $M_{\mathbb{Q}}$ as a $K \otimes_{\mathbb{Z}_p} W(k)$-module and suppress the notation ι. This is a free module of rank 2. We define an anti-hermitian form $\varkappa = \varkappa_\beta$,

$$\varkappa : M_{\mathbb{Q}} \times M_{\mathbb{Q}} \longrightarrow K \otimes_{\mathbb{Z}_p} W(k),$$

on the $K \otimes_{\mathbb{Z}_p} W(k)$-module $M_{\mathbb{Q}}$ by the formula

$$\mathrm{Tr}_{K/\mathbb{Q}_p}(a\varkappa(x, y)) = \beta(ax, y), \quad x, y \in M_{\mathbb{Q}},\ a \in K \otimes W(k). \tag{2.4.3}$$

Then $\varkappa$ satisfies

$$\varkappa(Vx, Vy) = p\sigma^{-1}(\varkappa(x, y)). \tag{2.4.4}$$

We write $\wedge^2 M_{\mathbb{Q}} := \bigwedge^2_{K \otimes W(k)} M_{\mathbb{Q}}$ for the exterior product as a $K \otimes_{\mathbb{Z}_p} W(k)$-module. This is a free $K \otimes_{\mathbb{Z}_p} W(k)$-module of rank 1. According to (2.4.1) we have decompositions

$$M_{\mathbb{Q}} = \bigoplus\nolimits_{\psi} M_{\mathbb{Q}, \psi}$$
$$\wedge^2 M_{\mathbb{Q}} = \bigoplus\nolimits_{\psi} \Big(\bigwedge^2_{K \otimes_{K^t, \psi} W_{\mathbb{Q}}(k)} M_{\mathbb{Q}, \psi} \Big).$$

We choose an isomorphism $\wedge^2 M_{\mathbb{Q}} \cong K \otimes_{\mathbb{Z}_p} W(k)$. Then we can write

$$\wedge^2 V(z) = \gamma \sigma^{-1}(z).$$

We have

$$\mathrm{ord}_{K \otimes W(k)} \wedge^2 V = \mathrm{ord}_p \mathrm{Nm}_{K/\mathbb{Q}_p} \det{}_{K \otimes_{\mathbb{Z}_p} W(k)}(V | M_{\mathbb{Q}})$$
$$= \mathrm{ord}_p \det{}_{W(k)}(V | M_{\mathbb{Q}}) = \dim M = 2d.$$

Therefore we find $\mathrm{ord}_{K \otimes W(k)} \gamma = 2d$. Since $\mathrm{ord}_{K \otimes W(k)} p = 2d$, the elements $p, \gamma \in K \otimes_{\mathbb{Z}_p} W(k)$ are in the same σ-conjugacy class by the remark after (2.4.2). We conclude that there is a generator $x \in \wedge^2 M_{\mathbb{Q}}$ such that

$$\wedge^2 V(x) = px. \tag{2.4.5}$$

Note that the last equation is equivalent with $\wedge^2 F(x) = px$. Any other generator with this property has the form ux, where $u \in K^\times$. We consider the hermitian form

$$h = \wedge^2 \varkappa : \wedge^2 M_{\mathbb{Q}} \times \wedge^2 M_{\mathbb{Q}} \longrightarrow K \otimes W(k).$$

One deduces that $h(x,x)$ is an element of $F \subset K \otimes W(k)$ which is $\neq 0$ because β is non-degenerate by assumption. We denote by

$$\mathfrak{d}_{K/F}(M, \iota, \beta) \in F^\times / \operatorname{Nm}_{K/F} K^\times \tag{2.4.6}$$

the class of $h(x,x)$. This class is called the *discriminant* of (M, ι, β) and is independent of the choice of x.

Let $a \in F$. Then $a\varkappa$ is again an anti-hermitian form which satisfies (2.4.4). We can replace $\varkappa$ by $a\varkappa$ in the definition of (2.4.6) without changing the discriminant. We denote by

$$\operatorname{inv}(M, \iota, \beta) \in \{\pm 1\} \tag{2.4.7}$$

the image of $\mathfrak{d}_{K/F}(M, \iota, \beta)$ by the canonical isomorphism $F^\times / \operatorname{Nm}_{K/F} K^\times \simeq \{\pm 1\}$.

Let r be a local CM-type of rank 2. Let E be the reflex field. Let $O_E \longrightarrow k$ an algebra structure of the algebraically closed field k. Let (X, ι, λ) be a local triple of CM-type r over k. Let (M, ι, β) be the associated Dieudonné module with its polarization β. Then we set

$$\operatorname{inv}(X, \iota, \lambda) := \operatorname{inv}(M, \iota, \beta).$$

For CM-triples of CM-type r, we use also the adjusted invariant $\operatorname{inv}^r(X, \iota, \lambda) = \operatorname{inv}^r(M, \iota, \beta)$, cf. section 8.2. In the case at hand we have

$$\operatorname{inv}^r(M, \iota, \beta) = \begin{cases} (-1)^{d-1} \operatorname{inv}(M, \iota, \beta), & \text{for } r \text{ special,} \\ (-1)^{d} \operatorname{inv}(M, \iota, \beta), & \text{for } r \text{ banal.} \end{cases} \tag{2.4.8}$$

2.5. **Uniqueness of framing objects.** In this subsection, we discuss the existence and uniqueness of *framing objects* that are used in the formulation of the formal moduli problems. The proofs of these statements are given later in the paper.

Let r be a generalized local CM-type of rank 2 for K/F. Let $\bar{k}$ be an algebraic closure of the residue field κ_E of O_E. Consider CM-triples (X, ι, λ) over $\bar{k}$ which satisfy (KC_r) and (EC_r).

(i) *Assume that r is special. If K/F is ramified, then a local CM-triple of type r over $\bar{k}$ as above such that the polarization is principal and with r-adjusted invariant -1 is isoclinic. When K/F is unramified, then a local CM-triple of type r over $\bar{k}$ as above such that the polarization is almost principal has r-adjusted invariant -1 and is isoclinic. In either case, any two such CM-triples are isogenous by a O_K-linear quasi-isogeny of height zero that preserves the polarizations.*

Furthermore, the group of O_K-linear self-isogenies of such a local CM-triple, preserving the polarization, can be identified with the unitary group of the split K/F-hermitian space C of dimension 2.

The assertions concerning slopes follow from Corollary 4.3.3. The uniqueness assertion is in the ramified case the content of Proposition 5.2.12, and in the unramified case of Proposition 5.3.6. The last part of the assertion follows from the fact that the contraction functor is an equivalence of categories.

(ii), a) *Let r be banal non-split. Any local CM-triple of type r over $\bar{k}$ as above is isoclinic. The group of O_K-linear self-isogenies of such a local CM-triple, preserving the polarization, can be identified with the unitary group of a K/F-hermitian space C of dimension* 2. *When K/F is unramified and the polarization is principal, then the hermitian space C is split and the r-adjusted invariant is* 1; *when K/F is ramified and the polarization is principal, then the hermitian space C is non-split and the r-adjusted invariant is -1; when K/F is unramified and the polarization is almost principal, then the hermitian space C is non-split and the r-adjusted invariant is -1. The case K/F ramified and almost principal polarization does not occur. Any two CM-triples with the same r-adjusted invariant are isogenous by a O_K-linear quasi-isogeny of height zero that preserves the polarizations.*

The assertions concerning slopes follow from Corollary 4.3.3. The uniqueness assertion is the content of Proposition 4.5.14. Similar arguments apply to the banal split case.

(ii), b) *Let r be banal split. Then the p-divisible group underlying a local CM-triple of type r is the direct product of two isoclinic p-divisible groups of slope λ, resp. $1-\lambda$, where λ depends only on r. Any two local CM-triples of type r over $\bar{k}$ are isogenous by a O_K-linear quasi-isogeny of height zero that preserves the polarizations. The group of O_K-linear self-isogenies of such a local CM-triple, preserving the polarization, can be identified with* $\mathrm{Res}_{F/\mathbb{Q}_p}(\mathrm{GL}_2)$.

Here is a chart for the various possibilities.

type r	K/F	inv^r	polarization	type C
special	ramified	-1	principal	split
special	unramified	-1	almost principal	split
banal	ramified	-1	principal	non-split
banal	unramified	1	principal	split
banal	unramified	-1	almost principal	non-split
banal	split	1	principal	GL_2/F

TABLE 1. Framing objects

Remark 2.5.1. The statement (i) above generalizes [18, Prop. 5.4]. However, the proof of the uniqueness assertion given there is incomplete. Note that for a local CM-type of the first kind in the sense of loc. cit. we have imposed $F = \mathbb{Q}_p$; therefore the condition in loc. cit. that $\varepsilon = \mathrm{inv}(X, \iota, \lambda) = -1$ implies that the associated hermitian space (C, h) is split (in this case the r-adjusted invariant coincides with the invariant).

Remark 2.5.2. The statement (i) is closely related to the fact that $B(G, \{\mu\})$ has only one element, cf. [16], §6. Here $G = \mathrm{Res}_{F/\mathbb{Q}_p}(\mathrm{GU})$ is the linear algebraic group over $\mathbb{Q}_p$ associated to the group of unitary similitudes of the non-split hermitian space of dimension 2 over K, and $\{\mu\}$ is the conjugacy class of cocharacters with component $(1,0)$ for φ_0 and central component for $\varphi \neq \varphi_0$. In fact, it seems that the essential contents of the calculations in section 8.3 is to show that the Frobenius element of a local CM-triple of type r with r-adjusted invariant -1 over $\bar{k}$ defines an element in $B(G, \mu)$.

2.6. **Formal moduli spaces.** In this subsection we are going to define RZ-spaces of formal local CM-triples, and formulate our main results about them. We fix K/F as before.

First let r be special, so that K/F is a field extension. We fix a local CM-triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of type r over $\bar{\kappa}_E$ as in (i) of subsection 2.5 (a *framing object*). We assume that if K/F is ramified, then $\lambda_{\mathbb{X}}$ is principal and that, if K/F is unramified, then $\lambda_{\mathbb{X}}$ is almost principal. Then, in either case, the r-adjusted invariant equals -1. We identify $\bar{\kappa}_E$ with the residue class field of $O_{\breve{E}}$, the ring of integers in the completion of the maximal unramified extension of E. Let $(\mathrm{Sch}/O_{\breve{E}})$ be the category of $O_{\breve{E}}$-schemes S such that the ideal sheaf $\pi\mathcal{O}_S$ is locally nilpotent.

Definition 2.6.1. We set $\mathtt{h} = 0$ if K/F is ramified, and $\mathtt{h} = 1$ if K/F is unramified.

We define a functor $\mathcal{M}_{K/F,r}$ on $(\mathrm{Sch}/O_{\breve{E}})$. A point of $\mathcal{M}_{K/F,r}(S)$ consists of an isomorphism class of the following data:

(1) Two local CM-pairs (X_0, ι_0), (X_1, ι_1) of CM-type r over S which satisfy the Eisenstein conditions (EC_r) relative to a fixed uniformizer π of F and the Kottwitz condition (KC_r).

(2) Two isogenies of p-divisible O_K-modules

$$X_0 \longrightarrow X_1 \longrightarrow X_0,$$

which have both height $2f\mathtt{h}$ and such that the composite is $\iota_0(\pi)^{\mathtt{h}}\,\mathrm{id}_{X_0}$.

(3) An isomorphism of p-divisible O_K-modules

$$\varkappa : X_1 \xrightarrow{\sim} X_0^{\wedge}.$$

We require that the composite $\lambda\colon X_0 \longrightarrow X_1 \xrightarrow{\sim} X_0^\wedge$ is a polarization of X_0, i.e., this map is anti-symmetric, and that this polarization is principal when K/F is ramified, and almost principal when K/F is unramified.[3]

(4) A quasi-isogeny of height zero of p-divisible O_K-modules

$$\rho_X : X_0 \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \bar{S},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $\lambda_{|X_0 \times_S \overline{S}}$ by a scalar in $F^\times$, locally on $\overline{S}$. Here $\overline{S} = S \otimes_{O_{\breve{E}}} \bar{\kappa}_E$. We call ρ_X a *framing.*

We denote the data above simply by $(X, \iota, \lambda, \rho)$. Another datum (X', λ', ρ') defines the same point of $\mathcal{M}_{K/F,r}(S)$ iff there are O_K-isomorphisms $X_0 \xrightarrow{\sim} X_0'$ and $X_1 \xrightarrow{\sim} X_1'$ which commute with the data (2) and (4) above. This implies that the isomorphism $X_0 \xrightarrow{\sim} X_0'$ respects the polarizations up to a factor in $O_F^\times$.

To ease the notation we write $\mathcal{M}_r = \mathcal{M}_{K/F,r}$. If R is a p-adic $O_{\breve{E}}$-algebra we set $\mathcal{M}_r(R) = \varprojlim_n \mathcal{M}_r(\operatorname{Spec} R/p^n R)$.

It follows by the methods of [27] that $\mathcal{M}_r$ is representable by a formal scheme which is locally formally of finite type over $\operatorname{Spf} O_{\breve{E}}$. Let J be the algebraic group over $\mathbb{Q}_p$ of unitary K-linear quasi-automorphisms of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ which preserve the polarization up to a scalar in $\mathbb{Q}_p^\times$. Let J^1 denote the derived group of J. Then $J^1(\mathbb{Q}_p)$ acts on the functor $\mathcal{M}_r$ by changing the framing. It follows from (i) in subsection 2.5 that J^1 can be identified with $\operatorname{Res}_{F/\mathbb{Q}_p}(\mathrm{SU})$, where SU denotes the quasi-split special unitary group in two variables over F. Note that SU is isomorphic to SL_2/F.

The first main result in the local case can now be stated as follows.

Theorem 2.6.2. *Let r be special. Then the functor $\mathcal{M}_{K/F,r}$ is represented by $\widehat{\Omega}_F \hat{\otimes}_{O_F,\varphi_0} O_{\breve{E}}$. More precisely, there exists a unique isomorphism of formal schemes*

$$\mathcal{M}_{K/F,r} \simeq \widehat{\Omega}_F \hat{\otimes}_{O_F,\varphi_0} O_{\breve{E}},$$

which is equivariant with respect to a fixed identification $J^1(\mathbb{Q}_p) \simeq \mathrm{SL}_2(F)$. In particular, $\mathcal{M}_{K/F,r}$ is flat over $\operatorname{Spf} O_{\breve{E}}$ with semi-stable reduction.

Now let r be banal. Fix a local CM-triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\overline{k}$ as in (ii) a) or (ii) b) in subsection 2.5. We write the height of $\lambda_{\mathbb{X}}$ as $2f\mathtt{h}$, where $\mathtt{h} \in \{0, 1\}$. We assume that $\mathtt{h} = 0$ when r is banal split, or when r is banal and K/F is a ramified field extension. Recall from from (ii) a) in subsection 2.5 that, when r is non-split, there is a hermitian space $C = C(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ attached to $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. By Proposition 4.5.14, the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is uniquely defined up to isogeny by the r-adjusted invariant $\operatorname{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) = \operatorname{inv}(C) \in \{\pm 1\}$ (see Proposition 8.3.6 for this last identity). To make our statements uniform, we set $\operatorname{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) = 1$ in the banal split case.

We may now define a variant for banal r of the functor $\mathcal{M}_{K/F,r}$ of Definition 2.6.1. Since the functor depends not only on K/F but also on $\operatorname{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$, we denote this functor by $\mathcal{M}_{K/F,r,\varepsilon}$, where $\operatorname{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) = \varepsilon$. When r is banal split, we have $\varepsilon = 1$; when r is banal non-split and K/F is unramified, then $\varepsilon = (-1)^{\mathtt{h}}$, cf. Proposition 4.5.14.

Let $S \in (\mathrm{Sch}/O_{\breve{E}})$. A point of $\mathcal{M}_{K/F,r,\varepsilon}(S)$ consists of an isomorphism class of exactly the same data as in Definition 2.6.1.

Theorem 2.6.3. *Let r be banal, and let $\varepsilon \in \{\pm 1\}$. The formal scheme $\mathcal{M}_{K/F,r,\varepsilon}$ is isomorphic to $(\operatorname{Spf} O_{\breve{E}}) \times (J(\mathbb{Q}_p)^o/C_{\overline{M}})$, where $J(\mathbb{Q}_p)^o$ denotes the subgroup of elements of $J(\mathbb{Q}_p)$ which preserve the polarization up to a scalar in $\mathbb{Z}_p^\times$. More precisely, there exists a unique isomorphism of formal schemes*

$$\mathcal{M}_{K/F,r,\varepsilon} \simeq (\operatorname{Spf} O_{\breve{E}}) \times (J(\mathbb{Q}_p)^o/C_{\overline{M}}),$$

which is equivariant for the action of $J(\mathbb{Q}_p)^o$. In particular, $\mathcal{M}_{K/F,r,\varepsilon}$ is formally étale over $\operatorname{Spf} O_{\breve{E}}$.

[3]It can be proved that, when K/F is unramified, the fact that $\operatorname{Ker} \lambda \subset X_0[\pi]$ follows automatically from the assumption that $\deg \lambda = p^{2f}$, cf. Proposition 5.3.7. We impose the condition $\operatorname{Ker} \lambda \subset X_0[\pi]$ in order to make transparent that the moduli problem $\mathcal{M}_{K/F,r}$ is of the kind considered in [27].

Here, when r is non-split, $J(\mathbb{Q}_p)^o$ can be identified with the group of K-linear automorphisms of $C = C(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ preserving the hermitian form up to a factor in $O_F^\times$, and $C_{\overline{M}}$ is the stabilizer in $J(\mathbb{Q}_p)^o$ of a lattice $\overline{M}$ in C which is self-dual when $\mathtt{h} = 0$ and almost self-dual when $\mathtt{h} = 1$. When r is split, then $J(\mathbb{Q}_p)^o$ can be identified with the group of automorphisms of the two-dimensional standard F-vector space of dimension 2 with determinant in $O_F^\times$, and $C_{\overline{M}}$ is the stabilizer in $J(\mathbb{Q}_p)^o$ of the standard lattice $\overline{M}$.

In the later part of the paper, we write simply $J(\mathbb{Q}_p)^o/C_{\overline{M}}$ for the formal scheme $(\mathrm{Spf}\ O_{\breve{E}}) \times (J(\mathbb{Q}_p)^o/C_{\overline{M}})$ over $\mathrm{Spf}\, O_{\breve{E}}$.

3. Background on Display Theory

In this section, $K/\mathbb{Q}_p$ is an arbitrary finite field extension with ring of integers $O = O_K$, and Nilp_O will denote the category of O-algebras R such that p is nilpotent in R. We recall the classification of strict formal p-divisible O-modules over $R \in \mathrm{Nilp}_O$ proved in [1]. A main ingredient is the Ahsendorf functor, which we present in a new form which is better suited for our applications.

3.1. **Displays.** We fix a prime element $\pi \in O$. We denote by q the number of elements in the residue class field κ of O_K.

Definition 3.1.1. ([1, Def. 3.1], [20], [34]) Let R be an O-algebra. A frame $\mathcal{F}$ for R consists of the following data:

(1) An O-algebra S and a surjective O-algebra homomorphism $S \to R$. We denote the kernel by I.
(2) An O-algebra endomorphism $\sigma : S \to S$.
(3) A σ-linear map of S-modules $\dot\sigma : I \to S$.

The following conditions are required.

(i) $I + pS$ is contained in the radical of S.
(ii) $\sigma(s) \equiv s^q \mod \pi S$ for all $s \in S$.
(iii) $\dot\sigma(I)$ generates S as an S-module.

We will denote a frame by $\mathcal{F} = (S, I, R, \sigma, \dot\sigma)$ and we will sometimes make the identification $S/I = R$.

A *morphism of O-frames* $\alpha\colon \mathcal{F} = (S, I, R, \sigma, \dot\sigma) \longrightarrow \mathcal{F}' = (S', I', R', \sigma', \dot\sigma')$ is an O-algebra homomorphism $\alpha\colon S \longrightarrow S'$ such that $\alpha(I) \subset I'$ and such that

$$\dot\sigma'(\alpha(a)) = \alpha(\dot\sigma(a)), \quad a \in I.$$

The last equation implies that

$$\sigma'(\alpha(s)) = \alpha(\sigma(s)), \quad s \in S.$$

Let $\mathcal{F} = (S, I, R, \sigma, \dot\sigma)$ be an O-frame. Then there exists a unique element $\theta \in S$ in the radical of S such that

$$\sigma(a) = \theta\dot\sigma(a), \quad \text{for all } a \in I, \tag{3.1.1}$$

cf. [1, Lem. 3.2]. In the frames below we have $\theta = \pi$.

Example 3.1.2. Let R be a p-adic O-algebra. Then the *relative Witt ring* $W_O(R)$ with respect to the chosen uniformizer $\pi \in O$ is a p-adic O-algebra. The *relative Witt polynomials*

$$\mathbf{w}_{O,n} = X_0^{q^n} + \pi X_1^{q^{n-1}} + \pi^2 X_2^{q^{n-2}} + \ldots + \pi^{n-1} X_{n-1}^q + \pi^n X_n,$$

define O-algebra homomorphisms $\mathbf{w}_{O,n} : W_O(R) \longrightarrow R$. We denote by F and V the Frobenius and the Verschiebung acting on $W_O(R)$, cf. [9]. In the case where $k = R$ is a perfect field, the ring $W_O(k)$ is the complete discrete valuation with residue class field k which is unramified over O.

The *relative Witt frame* for R is the O-frame defined as

$$\mathcal{W}_O(R) = (W_O(R), VW_O(R), R, \sigma, \dot\sigma). \tag{3.1.2}$$

Here $\sigma = F : W_O(R) \longrightarrow W_O(R)$ is the Frobenius endomorphism written as $\sigma(\xi) = {}^F\xi$, and $\dot{\sigma}({}^V\xi) = \xi$, for $\xi \in W_O(R)$. We use also the notation $\dot{F} := \dot{\sigma}$ and $I_O(R) = VW_O(R)$. If $K = \mathbb{Q}_p$ and $\pi = p$, we obtain the classical ring of Witt vectors $W(R) = W_{\mathbb{Z}_p}(R)$. We write $\mathcal{W}(R)$ for the $\mathbb{Z}_p$-frame $\mathcal{W}_{\mathbb{Z}_p}(R)$.

Example 3.1.3. Let $S \longrightarrow R$ be a surjective homomorphism of p-adic O-algebras. We assume that the kernel $\mathfrak{a}$ is endowed with divided powers relative to O ([1], 1.2.2). They make sense out of the expression "a^q/π". We also call this an *O-pd*-thickening. We denote by $\mathfrak{a}_{[F^n]}$ the ideal $\mathfrak{a}$ considered as an $W_O(S)$-module via restriction of scalars relative to $\mathbf{w}_{O,n} : W_O(S) \longrightarrow S$. The divided powers give rise to *divided Witt polynomials* $\dot{\mathbf{w}}_{O,n}$. They are homomorphisms of $W_O(S)$-modules $\dot{\mathbf{w}}_{O,n} : W_O(\mathfrak{a}) \longrightarrow \mathfrak{a}_{[F^n]}$ such that $\pi^n\dot{\mathbf{w}}_{O,n} = \mathbf{w}_{O,n}$. They give rise to an isomorphism of $W_O(S)$-modules

$$\prod_{n\geq 0} \dot{\mathbf{w}}_{O,n} : W_O(\mathfrak{a}) \xrightarrow{\sim} \prod_{n\geq 0} \mathfrak{a}_{[F^n]},$$

cf. [1], 1.2.2. The inverse image in $W_O(\mathfrak{a})$ of an element $[a, 0, 0, \ldots]$ from the right hand side is called the *logarithmic Teichmüller representative* of $a \in \mathfrak{a}$. The logarithmic Teichmüller representatives of elements of $\mathfrak{a}$ form an ideal $\tilde{\mathfrak{a}} \subset W_O(S)$. The ideal $\mathcal{J} = \tilde{\mathfrak{a}} \oplus I_O(S)$ is the kernel of the composition

$$W_O(S) \xrightarrow{\mathbf{w}_{O,0}} S \longrightarrow R.$$

Then $\dot{F} : I_O(S) \longrightarrow W_O(S)$ extends uniquely to a F-linear homomorphism $\dot{F} : \mathcal{J} \longrightarrow W_O(S)$ such that $\dot{F}(\tilde{\mathfrak{a}}) = 0$. We define the relative *Witt frame* for $S \longrightarrow R$ as

$$\mathcal{W}_O(S/R) = (W_O(S), \mathcal{J}, R, F, \dot{F}). \tag{3.1.3}$$

This is an O-frame. Later we use the more precise notation

$$I_O(S/R) = \mathcal{J} = W_O(\mathfrak{a}) + I_O(S).$$

Definition 3.1.4 ([1], Def. 3.3)**.** Let $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ be an O-frame. An *$\mathcal{F}$-display* $\mathcal{P} = (P, Q, F, \dot{F})$ consists of the following data: a finitely generated projective S-module P, a submodule $Q \subset P$, and two σ-linear maps

$$F \colon P \longrightarrow P, \quad \dot{F} \colon Q \longrightarrow P.$$

The following conditions are required.

(i) $IP \subset Q$.

(ii) The factor module P/Q is a finitely generated projective R-module.

(iii) The following relation holds for $a \in I$ and $x \in P$,

$$\dot{F}(ax) = \dot{\sigma}(a)F(x).$$

(iv) $\dot{F}(Q)$ generates P as an S-module.

(v) The projective R-module $\operatorname{Lie}\mathcal{P} = P/Q$ lifts to a finitely generated projective S-module. It is called the *Lie algebra* of $\mathcal{P}$.

If the rank of $\operatorname{Lie}\mathcal{P}$ is constant, we call it the *dimension of* $\mathcal{P}$. If the S-module P is of constant rank, we call it the *height of* $\mathcal{P}$. If we want to be precise, we say $\mathcal{F}$-height.

$\mathcal{F}$-displays form a category in the obvious way. In the case $O = \mathbb{Z}_p$ and $\mathcal{F} = \mathcal{W}(R)$ for a p-adic ring R, we speak simply of a *display* over R. Displays for general frames $\mathcal{F}$ were originally called $\mathcal{F}$-windows, cf. [1, Def. 3.3]. We note that for the O-frames $\mathcal{W}_O(R)$, the condition (v) of Definition 3.1.4 is automatically satisfied, cf. [33, Lem. 2].

Example 3.1.5. For each O-frame $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ we have the *multiplicative $\mathcal{F}$-display*

$$\mathcal{P}_m = \mathcal{P}_{m,\mathcal{F}} = (S, I, \sigma, \dot{\sigma}).$$

Example 3.1.6. Let $\mathcal{P}$ be an $\mathcal{F}$-display. Let $\varepsilon \in S$ be a unit. The display

$$\mathcal{P}(\varepsilon) = (P, Q, \varepsilon F, \varepsilon\dot{F}).$$

is called the *twist* of $\mathcal{P}$ by ε.

Recall the element θ from (3.1.1). The conditions in Definition 3.1.4 imply that

$$F(y) = \theta \dot{F}(y), \quad y \in Q. \tag{3.1.4}$$

We can always find a direct sum decomposition $P = T \oplus L$ such that $Q = IT \oplus L$. Such a decomposition is called a *normal decomposition* of P. The σ-linear homomorphism

$$\Phi := F_{|T} \oplus \dot{F}_{|L} \colon T \oplus L \longrightarrow P \tag{3.1.5}$$

is a σ-linear isomorphism, i.e., (3.1.5) corresponds to the *linearization isomorphism*,

$$F^\sharp \oplus \dot{F}^\sharp \colon (S \otimes_{\sigma,S} T) \oplus (S \otimes_{\sigma,S} L) \xrightarrow{\sim} P. \tag{3.1.6}$$

Conversely, an arbitrary σ-linear isomorphism (3.1.5) defines an $\mathcal{F}$-display in the obvious way.

For each display $\mathcal{P}$ there is a homomorphism of S-modules ([1, Def. 3.3])

$$V^\sharp : P \longrightarrow S \otimes_{\sigma,S} P \tag{3.1.7}$$

which is uniquely determined by

$$V^\sharp(s\dot{F}y) = s \otimes y, \quad V^\sharp(Fx) = \theta \otimes x, \quad x \in P, \ y \in Q, \ s \in S.$$

We have

$$V^\sharp \circ F^\sharp = \theta \, \mathrm{id}_{S \otimes_{\sigma,S} P}, \quad F^\sharp \circ V^\sharp = \theta \, \mathrm{id}_P.$$

Any morphism of O-frames $\alpha \colon \mathcal{F} \longrightarrow \mathcal{F}'$ defines a base change functor α_* from the category of $\mathcal{F}$-displays to the category of $\mathcal{F}'$-displays as follows, cf. [1, Def. 3.8]. Let $\mathcal{P}$ be an $\mathcal{F}$-display. Then we define $\alpha_*(\mathcal{P}) = \mathcal{P}' = (P', Q', F', \dot{F}')$ as follows:

$$P' = S' \otimes_S P, \quad Q' = \mathrm{Ker}\big(S' \otimes_S P \longrightarrow R' \otimes_R (P/Q)\big), \quad F' = \sigma' \otimes F \colon P' \longrightarrow P', \tag{3.1.8}$$

and $\dot{F}' = \sigma' \otimes \dot{F} \colon Q' \longrightarrow P'$.

If $\mathcal{P}$ is given in terms of a normal decomposition (3.1.5), we obtain $\mathcal{P}'$ from the σ'-linear extension of Φ,

$$\Phi' \colon (S' \otimes_S T) \oplus (S' \otimes_S L) \longrightarrow P'.$$

Example 3.1.7. The base change of the multiplicative display for the frame $\mathcal{F}$ under $\alpha \colon \mathcal{F} \longrightarrow \mathcal{F}'$ is the multiplicative display for $\mathcal{F}'$.

If R is a perfect ring of characteristic p with an O-algebra structure, the category of $\mathcal{W}_O(R)$-displays is equivalent with the more classical category of Dieudonné modules. We describe this equivalence in its natural generality.

Definition 3.1.8. (a) A *perfect O-frame* is an O-frame $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ such that $\sigma : S \longrightarrow S$ is bijective.

If $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ is a perfect O-frame, then there is an element $u \in I$, such that $\dot{\sigma}(u) = 1$. Then $\sigma(u) = \theta$ is the element from (3.1.1). Furthermore, $\dot{\sigma} : I \longrightarrow S$ is bijective, $I = Su$ and the elements u and θ are non zero divisors.

(b) A *Dieudonné module* (M, F, V) for the perfect O-frame $\mathcal{F}$ consists of a finitely generated projective S-module M and two additive maps $F : M \longrightarrow M$, $V : M \longrightarrow M$ such that the following conditions are satisfied.

(i) $F(sx) = \sigma(s)F(x), \quad V(sx) = \sigma^{-1}(s)V(x), \quad x \in P, \ s \in S.$

(ii) $F \circ V = \theta \, \mathrm{id}_M, \quad V \circ F = u \, \mathrm{id}_M.$

(iii) The R-module M/VM is projective and lifts to a finitely generated projective S-module.

If R is a perfect O-algebra, then $\mathcal{F} = \mathcal{W}_O(R)$ is a perfect O-frame and we have $u = {}^V1 = \pi = \theta$.

Proposition 3.1.9. *Let $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ be a perfect O-frame. Let $u, \theta \in S$ as defined above. Then the category of Dieudonné modules for $\mathcal{F}$ is equivalent to the category of $\mathcal{F}$-displays.*

Proof. Let (M, F, V) be a Dieudonné module. Since u and θ are not zero divisors, the maps $F : M \longrightarrow M$ and $V : M \longrightarrow M$ are injective. Therefore we can define a display $(P, Q, F, \dot{F})$ by setting

$$P = M, \ Q = VM, \ F = F, \ \dot{F} = V^{-1}.$$

Conversely, let $(P, Q, F, \dot{F})$ be a display. We set $(M, F) := (P, F)$. We have the bijective map

$$\nu : S \otimes_{\sigma,S} P \longrightarrow P, \quad \nu(s \otimes x) := \sigma^{-1}(s)x.$$

Then we define $V = \nu \circ V^\sharp$. More explicitly, we have

$$V(s\dot{F}(y)) = \sigma^{-1}(s)y, \quad y \in Q, \ s \in S.$$

This implies that $V(P) = Q$. Moreover, we obtain

$$FV(s\dot{F}y) = F(\sigma^{-1}(s)y) = sF(y) = \theta s\dot{F}y$$
$$VF(x) = V(\dot{F}(ux)) = ux.$$

Therefore (M, F, V) is a Dieudonné module. □

In our basic example $\mathcal{F} = \mathcal{W}_O(R)$ for a perfect O-algebra R, we can replace the condition (iii) above by the weaker condition that M/VM is a projective R-module. We note that for this frame $F \circ V = \pi \, \mathrm{id}_M, V \circ F = \pi \, \mathrm{id}_M$.

We refer to [1, Def. 3.3] or [35] for the definition of a *nilpotent* $\mathcal{F}$-display. If R is a perfect O-algebra, a $\mathcal{W}_O(R)$-display is nilpotent iff for the corresponding Dieudonné module (M, F, V) the endomorphism V of $M/\pi M$ is nilpotent. For an arbitrary O-algebra R such that π is nilpotent in R, a $\mathcal{W}_O(R)$-display $\mathcal{P}$ is nilpotent iff for any homomorphism of O-algebras to a perfect field $R \longrightarrow k$, the base change of $\mathcal{P}$ by the morphism of frames $\mathcal{W}_O(R) \longrightarrow \mathcal{W}_O(k)$ is nilpotent.

Definition 3.1.10. Let R be an O-algebra. Let X be a p-divisible group over R endowed with a $\mathbb{Z}_p$-algebra homomorphism $\iota : O \longrightarrow \operatorname{End} X$. We call the action ι *strict* if the induced action on $\operatorname{Lie} X$ coincides with the O-action on this R-module given by restriction of scalars $O \longrightarrow R$. We say that (X, ι) is a strict p-divisible O-module.

The following main result of [1] was known before for $O = \mathbb{Z}_p$ [33], [21].

Theorem 3.1.11 ([1], Thm. 1.1). *Let $R \in \mathrm{Nilp}_O$. There is an equivalence of categories*

$$\big(\textit{nilpotent } \mathcal{W}_O(R)\textit{-displays}\big) \longrightarrow \big(\textit{strict formal } p\textit{-divisible } O\textit{-modules over } R\big)$$

which is functorial in R.

The theorem extends to p-adic R if we require the properties "nilpotent" and "formal" only after base change to R/pR.

A nilpotent $\mathcal{W}_O(R)$-display gives rise to a crystal, as follows. Let $S \longrightarrow R$ be a *O-pd*-thickening, cf. Example 3.1.3. We assume that p is nilpotent in S. The ring homomorphism $W_O(S) \longrightarrow W_O(R)$ defines a morphism of O-frames

$$\mathcal{W}_O(S/R) \longrightarrow \mathcal{W}_O(R).$$

Theorem 3.1.12 ([20], [33]). *Let $S \longrightarrow R$ be an O-pd-thickening such that p is nilpotent in S. The base change functor*

$$\big(\textit{nilpotent } \mathcal{W}_O(S/R)\textit{-displays}\big) \longrightarrow \big(\textit{nilpotent } \mathcal{W}_O(R)\textit{-displays}\big)$$

is an equivalence of categories. □

Remark 3.1.13. In the case $O = \mathbb{Z}_p$, Lau [19] has defined a functor

$$\big(p\textit{-divisible groups over } R\big) \to \big(\mathcal{W}(S/R)\textit{-displays}\big) \tag{3.1.9}$$

which gives a quasi-inverse of the functor in Theorem 3.1.12, when restricted to formal p-divisible groups. In particular this functor associates to an arbitrary p-divisible group over R a display.

Let $\mathcal{P}$ be a nilpotent $\mathcal{W}_O(R)$-display. Let $\tilde{\mathcal{P}}$ be the unique $\mathcal{W}_O(S/R)$-display associated to $\mathcal{P}$ by Theorem 3.1.12. Then we set

$$\mathbb{D}_{\mathcal{P}}(S) = \tilde{P}/I_O(S)\tilde{P}. \tag{3.1.10}$$

This is a finitely generated projective S-module. It is a crystal in the following sense. If $S' \longrightarrow R$ is a another O-*pd*-thickening such that p in nilpotent in S' and $S' \longrightarrow S$ is a morphism of O-*pd*-thickenings, then there is a canonical isomorphism

$$S \otimes_{S'} \mathbb{D}_{\mathcal{P}}(S') \cong \mathbb{D}_{\mathcal{P}}(S).$$

This crystal corresponds to the Grothendieck-Messing crystal of a p-divisible group via Theorem 3.1.11. From Theorem 3.1.12 one obtains the Grothendieck-Messing criterion for displays in the following formulation.

Corollary 3.1.14. *Let $\mathcal{P}$ be a nilpotent $\mathcal{W}_O(R)$-display. Let $S \longrightarrow R$ be an O-pd-thickening. Each $\mathcal{W}_O(S)$-display $\tilde{\mathcal{P}}$ which lifts $\mathcal{P}$ defines a lifting $\widetilde{\mathrm{Fil}} := \tilde{Q}/I_O(S)\tilde{P} \subset \mathbb{D}_{\mathcal{P}}(S)$ of the Hodge filtration $\mathrm{Fil} := Q/I_O(R)P \subset \mathbb{D}_{\mathcal{P}}(R)$.*

For a fixed O-pd-thickening $S \longrightarrow R$, consider the category of pairs $(\mathcal{P}, \widetilde{\mathrm{Fil}})$, where $\mathcal{P}$ is a nilpotent display and $\widetilde{\mathrm{Fil}} \subset \mathbb{D}_{\mathcal{P}}(S)$ is a lifting of the Hodge filtration associated to $\mathcal{P}$. The functor which maps a pair $(\mathcal{P}, \tilde{\mathcal{P}})$ to the pair $(\mathcal{P}, \widetilde{\mathrm{Fil}})$ is an equivalence of categories. □

The following fact is well-known, but we give a proof.

Lemma 3.1.15. *Let R be a p-adic O-algebra. Let $\mathcal{P}$ be a $\mathcal{W}_O(R)$-display. Let $\tilde{O}$ be a discrete valuation ring which is a finite extension of O. Let*

$$\tilde{O} \longrightarrow \mathrm{End}\,\mathcal{P},$$

be an O-algebra homomorphism. Then P is a locally on $\mathrm{Spec}\,R$ a free $\tilde{O} \otimes_O W_O(R)$-module.

Let $S \longrightarrow R$ be an O-pd-thickening such that p is nilpotent in S. We assume that $\mathcal{P}$ is nilpotent. Then $\mathbb{D}_{\mathcal{P}}(S)$ is locally on $\mathrm{Spec}\,S$ a free $\tilde{O} \otimes_{\mathbb{Z}_p} S$-module.

Proof. We start with the case where $S = R = k$ is a perfect field which contains the residue class field of $\tilde{O}$. Let $\tilde{O}^t$ be the maximal unramified extension of O contained in $\tilde{O}$. Let σ be the Frobenius automorphism of $\tilde{O}^t$ relative to O. To each O-algebra homomorphism $\psi : \tilde{O}^t \longrightarrow k$ there is a unique Frobenius equivariant O-algebra homomorphism

$$\tilde{\psi} : \tilde{O}^t \longrightarrow W_O(k)$$

which induces ψ when composed with $\mathbf{w}_{O,0} : W_O(k) \longrightarrow k$. This follows from the remark after the definition of $W_O(R)$, cf. Example 3.1.2. The decomposition

$$\tilde{O} \otimes_O W_O(k) = \prod_{\psi} \tilde{O} \otimes_{\tilde{O}^t, \tilde{\psi}} W_O(k)$$

induces a decomposition

$$P = \oplus_{\psi} P_{\psi}.$$

Each P_{ψ} is a free module over the discrete valuation ring $\tilde{O} \otimes_{\tilde{O}^t, \tilde{\psi}} W_O(k)$. The Frobenius $F \colon P \longrightarrow P$ induces maps $P_{\psi} \longrightarrow P_{\psi\sigma}$. This shows that all P_{ψ} have the same rank as $W_O(k)$-modules. This proves the case where $R = k$ is a perfect field containing the residue class field of $\tilde{O}$.

Now let k be an arbitrary field of characteristic p. It suffices to show that $P \otimes_{W_O(k)} k$ is a free $\tilde{O} \otimes_O k$-module. By base change $\alpha \colon k \longrightarrow \bar{k}$ this follows from the previous case because $P \otimes_{W_O(k)} \bar{k} = \alpha_*(P) \otimes \bar{k}$ is a free $\tilde{O} \otimes_O \bar{k}$-module.

If R is a local ring with residue class field of characteristic p we conclude by Nakayama's lemma that $P \otimes_{W_O(R)} R$ is a free $\tilde{O} \otimes_O R$-module. The generalization to arbitrary R is immediate. This proves the first assertion of the Lemma.

If $\mathcal{P}$ is nilpotent, the crystal $\mathbb{D}_{\mathcal{P}}$ is defined. The case $\mathbb{D}_{\mathcal{P}}(R) = P/I_O(R)P$ was proved above. For arbitrary $S \longrightarrow R$ we can apply again Nakayama's lemma. □

Remark 3.1.16. Let R be a ring such that p is nilpotent in R. Let X be a p-divisible group over R with a ring homomorphism

$$O \longrightarrow \operatorname{End} X.$$

Let $S \longrightarrow R$ be a nilpotent pd-thickening. Then the value of the Grothendieck-Messing crystal $\mathbb{D}_X(S)$ is locally on $\operatorname{Spec} S$ a free $O \otimes_{\mathbb{Z}_p} S$-module. This can be shown by the same arguments as above.

Finally we discuss isogenies of $\mathcal{W}_O(R)$-displays, where R is an O-algebra such that p is nilpotent in R. We assume moreover that $\operatorname{Spec} R$ is connected. Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a morphism of displays of the same height and dimension, cf. the remark after Definition 3.1.4. Locally on $\operatorname{Spec} R$ the $W_O(R)$-modules P_1 and P_2 are free of the same rank. We may choose a basis in each of these modules and write $\det \alpha \in W_O(R)$. This is locally defined up to a unit in $W_O(R)$. More invariantly one can write exterior powers.

Definition 3.1.17. A morphism of $\mathcal{W}_O(R)$-displays of the same height and dimension $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is called an isogeny if $\det \alpha \neq 0$.

Proposition 3.1.18 ([35], Prop. 17.6.2.)**.** *Let R be an O-algebra such that p is nilpotent in R and such that* $\operatorname{Spec} R$ *is connected. Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an isogeny of $\mathcal{W}_O(R)$-displays. Then there exists a natural number* $\mathtt{h} \in \mathbb{Z}_{\geq 0}$ *such that locally on* $\operatorname{Spec} R$

$$\det \alpha = \pi^{\mathtt{h}} \epsilon, \quad \epsilon \in W_O(R)^\times.$$

□

We call $\mathtt{h}$ the *O-height* of α, and write $\mathtt{h} = \operatorname{height}_O \alpha$. If $O = \mathbb{Z}_p$, we write simply $\operatorname{height} \alpha$. An abbreviation for the Proposition is:

$$\operatorname{height}_O \alpha = \operatorname{ord}_\pi \det \alpha.$$

Proposition 3.1.19 ([35], Prop. 17.6.4.)**.** *Assume that the ideal of nilpotent elements in R is nilpotent and that* $\operatorname{Spec} R$ *is connected. Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an isogeny of O-height* $\mathtt{h}$*. Then there exists locally on* $\operatorname{Spec} R$ *a morphism of $\mathcal{W}_O(R)$-displays $\beta : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$ such that*

$$\beta \circ \alpha = \pi^{\mathtt{h}} \operatorname{id}_{\mathcal{P}_1}, \quad \alpha \circ \beta = \pi^{\mathtt{h}} \operatorname{id}_{\mathcal{P}_2}.$$

□

Proposition 3.1.20. *With the assumptions of Proposition 3.1.18, let $a : X_1 \longrightarrow X_2$ be a morphism of strict formal p-divisible O-modules over R. Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be the induced morphism of the associated $\mathcal{W}_O(R)$-displays, cf. Theorem 3.1.11. The morphism a is an isogeny of height* $\mathtt{h}$ *if and only if α is an isogeny of height* $\mathtt{h}$*.*

Proof. This can be reduced to the case of a perfect field $R = k$ where it is well-known by Dieudonné theory. □

Let R be an O-algebra and let $a\colon X_1 \longrightarrow X_2$ be a morphism of strict formal p-divisible O-modules. By Theorem 3.1.11, there is an associated morphism $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ of $\mathcal{W}_O(R)$-displays. We set

$$\operatorname{height}_O a = \operatorname{height}_O \alpha, \quad \operatorname{height}_O X_1 = \operatorname{height}_{\mathcal{W}_O(R)} \mathcal{P}_1. \tag{3.1.11}$$

The last height was defined after Definition 3.1.4. It is equal to the O-height of the endomorphism of $\mathcal{P}_1$ given by multiplication by π. We also write

$$\operatorname{height}_O \mathcal{P}_1 = \operatorname{height}_O(\pi|\mathcal{P}_1) = \operatorname{height}_{\mathcal{W}_O(R)} \mathcal{P}_1.$$

3.2. **Bilinear forms of displays.** Let $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ be an O-frame and let $\theta \in S$ be the element from (3.1.1).

Definition 3.2.1. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}$ be $\mathcal{F}$-displays. A *bilinear form of $\mathcal{F}$-displays*

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}$$

is a bilinear form of S-modules

$$\beta : P_1 \times P_2 \longrightarrow P \tag{3.2.1}$$

with the following properties:

(i) The restriction of β to $Q_1 \times Q_2$ takes values in Q.
(ii) For $y_1 \in Q_1$ and $y_2 \in Q_2$,

$$\dot{F}\beta(y_1, y_2) = \beta(\dot{F}_1 y_1, \dot{F}_2 y_2).$$

We will denote the O-module of all bilinear forms by

$$\mathrm{Bil}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}).$$

Lemma 3.2.2. *The following equations hold*

$$\begin{aligned} F\beta(x_1, y_2) &= \beta(F_1 x_1, \dot{F}_2 y_2), \qquad x_1 \in P_1,\ y_2 \in Q_2,\\ F\beta(y_1, x_2) &= \beta(\dot{F}_1 y_1, F_2 x_2), \qquad y_1 \in Q_1,\ x_2 \in P_2,\\ \theta F\beta(x_1, x_2) &= \beta(F_1 x_1, F_2 x_2), \qquad x_1 \in P_1,\ x_2 \in P_2. \end{aligned}$$

Proof. We omit the verification which is, for classical displays, contained in [33]. □

Let R be a perfect O-algebra and let $\mathcal{F} = \mathcal{W}_O(R)$. Then we may equivalently consider Dieudonné modules (P, F, V) and (P_i, F_i, V_i) for $i = 1, 2$, cf. Proposition 3.1.9. We can reformulate the Definition 3.2.1 as follows: A bilinear form of Dieudonné modules is a bilinear form of $W_O(R)$-modules $\beta : P_1 \times P_2 \longrightarrow P$ such that

$$\beta(V_1 x_1, V_2 x_2) = V\beta(x_1, x_2). \tag{3.2.2}$$

Proposition 3.2.3. *Let $\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}$ be a bilinear form of $\mathcal{F}$-displays. Let $\alpha : \mathcal{F} \longrightarrow \mathcal{F}'$ be a morphism of frames. Denote by $\mathcal{P}'_1$, $\mathcal{P}'_2$, and $\mathcal{P}'$ the displays obtained by base change with respect to α. Because $P'_i = S' \otimes_S P_i$, and $P' = S' \otimes_S P$, there is an induced S'-bilinear form $\beta' : P'_1 \times P'_2 \longrightarrow P'$. This is a bilinear form of $\mathcal{F}'$-displays*

$$\mathcal{P}'_1 \times \mathcal{P}'_2 \longrightarrow \mathcal{P}'.$$

Proof. We omit the straightforward verification. □

Let $\mathcal{P} = (P, Q, F, \dot{F})$ be an $\mathcal{F}$-display. We are going to define the *dual $\mathcal{F}$-display* $\mathcal{P}^\vee = (P^\vee, Q^\vee, F^\vee, \dot{F}^\vee)$. For an S-module M, we define $M^* = \mathrm{Hom}_S(M, S)$. We set $P^\vee := P^*$, and

$$Q^\vee = \{\psi \in P^\vee \mid \psi(Q) \subset I\}.$$

We note that we have a natural perfect pairing

$$P/IP \times P^\vee/IP^\vee \longrightarrow R.$$

We deduce that $Q^\vee/IP^\vee$ is the orthogonal complement of Q/IP and is therefore a direct summand of P/IP. We claim that there are σ-linear maps

$$F^\vee : P^\vee \longrightarrow P^\vee, \quad \dot{F}^\vee : Q^\vee \longrightarrow P^\vee$$

which are uniquely determined by the following conditions. We denote by $< \, , \, > : P \times P^\vee \longrightarrow S$ the natural perfect pairing. Then we require for $x \in P,\ y \in Q, \phi \in P^\vee,\ \psi \in Q^\vee$:

$$\begin{aligned} &< \dot{F}(y), F^\vee(\phi) > = \sigma(< y, \phi >), \quad < F(x), F^\vee(\phi) > = \theta\sigma(< x, \phi >),\\ &< \dot{F}(y), \dot{F}^\vee(\psi) > = \dot{\sigma}(< y, \psi >), \quad < F(x), \dot{F}^\vee(\psi) > = \sigma(< x, \psi >). \end{aligned} \tag{3.2.3}$$

Since $\dot{F}$ is a σ-linear surjection, the maps $F^\vee$ and $\dot{F}^\vee$ are uniquely determined by these identities. To verify the existence of these maps, we consider a normal decomposition,

$$P = T \oplus L.$$

Let $L^\vee \subset P^\vee$ be the orthogonal complement of L and $T^\vee \subset P^\vee$ the orthogonal complement of T. Hence there are canonical isomorphisms

$$L^\vee \cong T^*, \quad T^\vee \cong L^*.$$

We obtain the normal decomposition

$$P^\vee = T^\vee \oplus L^\vee, \quad Q^\vee = IT^\vee \oplus L^\vee.$$

For $\psi \in L^\vee = T^*$, we set

$$\dot{F}^\vee(\psi)(\dot{F}(\ell)) = 0, \quad \dot{F}^\vee(\psi)(\dot{F}(t)) = \sigma(\psi(t)), \quad \ell \in L,\ t \in T.$$

This definition makes sense because of the linearization isomorphism (3.1.6). Finally, we define $F^\vee(\phi)$ for $\phi \in T^\vee = L^*$ by the equations

$$F^\vee(\phi)(\dot{F}(\ell)) = \sigma(\phi(\ell)), \quad F^\vee(\phi)(F(t)) = 0.$$

One verifies that, with these definitions, the identities (3.2.3) are satisfied. It follows from the symmetry of the equations (3.2.3) that we have a natural isomorphism

$$\mathcal{P} \cong (\mathcal{P}^\vee)^\vee.$$

By the equations (3.2.3) we have a natural bilinear form of displays

$$\mathcal{P} \times \mathcal{P}^\vee \longrightarrow \mathcal{P}_m \tag{3.2.4}$$

with values in the multiplicative display $\mathcal{P}_m = \mathcal{P}_{m,\mathcal{F}}$. If $\mathcal{P}'$ is another $\mathcal{F}$-display, the bilinear form (3.2.4) induces an isomorphism

$$\mathrm{Hom}_{\mathcal{F}\text{-displays}}(\mathcal{P}', \mathcal{P}^\vee) \xrightarrow{\sim} \mathrm{Bil}(\mathcal{P}' \times \mathcal{P}, \mathcal{P}_m). \tag{3.2.5}$$

We deduce a variant of the Grothendieck-Messing criterion. Let $\mathcal{P}$ and $\mathcal{P}'$ be $\mathcal{W}_O(R)$-displays such that $\mathcal{P}^\vee$ and $\mathcal{P}'$ are nilpotent. Let $S \longrightarrow R$ be a O-*pd*-thickening in Nilp_O, cf. Example 3.1.3. We denote by $\mathcal{P}^\vee_{\mathrm{rel}}$ and $\mathcal{P}'_{\mathrm{rel}}$ the associated $\mathcal{W}_O(S/R)$-displays, which exist by Theorem 3.1.12. We define $\mathcal{P}_{\mathrm{rel}} = (\mathcal{P}^\vee_{\mathrm{rel}})^\vee$, where the last $^\vee$ denotes the dual in the category of $\mathcal{W}_O(S/R)$-displays. We set $\mathbb{D}_\mathcal{P}(S) = P_{\mathrm{rel}}/I(S)P_{\mathrm{rel}}$. Then we obtain a crystal which is dual to the crystal $\mathbb{D}_{\mathcal{P}^\vee}(S)$, cf. (3.1.10). This crystal agrees with $\mathbb{D}_\mathcal{P}(S)$ defined earlier, if $\mathcal{P}$ is nilpotent. It follows form (3.2.5) that each bilinear from

$$\beta : \mathcal{P}' \times \mathcal{P} \longrightarrow \mathcal{P}_{m,\mathcal{W}_O(R)} \tag{3.2.6}$$

induces a bilinear form

$$\beta_{\mathrm{rel}} : \mathcal{P}'_{\mathrm{rel}} \times \mathcal{P}_{\mathrm{rel}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_O(S/R)}$$

and, in particular, a S-bilinear form

$$\beta_{\mathrm{crys}} : \mathbb{D}_{\mathcal{P}'}(S) \times \mathbb{D}_\mathcal{P}(S) \longrightarrow S. \tag{3.2.7}$$

Proposition 3.2.4. *Let $R \in \mathrm{Nilp}_O$ and let $S \longrightarrow R$ be an O-pd-thickening in Nilp_O. Let $\mathcal{P}$ and $\mathcal{P}'$ be $\mathcal{W}_O(R)$-displays and assume that $\mathcal{P}^\vee$ and $\mathcal{P}'$ are nilpotent. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ be liftings which correspond to liftings of the Hodge filtrations $\widetilde{\mathrm{Fil}} \subset \mathbb{D}_\mathcal{P}(S)$ and $\widetilde{\mathrm{Fil}}' \subset \mathbb{D}_{\mathcal{P}'}(S)$, cf. Corollary 3.1.14. Then a bilinear form $\beta : \mathcal{P}' \times \mathcal{P} \longrightarrow \mathcal{P}_{m,\mathcal{W}_O(R)}$ lifts to a bilinear form $\tilde{\beta} : \tilde{\mathcal{P}}' \times \tilde{\mathcal{P}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_O(S)}$ iff*

$$\beta_{\mathrm{crys}}(\widetilde{\mathrm{Fil}}', \widetilde{\mathrm{Fil}}) = 0.$$

□

Proof. This is a consequence of Corollary 3.1.14 and (3.2.5). □

We go back to an arbitrary O-frame $\mathcal{F}$ and add a remark on the map $V^\sharp$, cf. (3.1.7). If P is an S-module we set $P^{(\sigma)} = S \otimes_{\sigma,S} P$. If P is projective and finitely generated, the perfect pairing $<\, ,\, >$ induces a perfect pairing

$$\begin{aligned} <\, ,\, >_{(\sigma)} : P^{(\sigma)} \times (P^*)^{(\sigma)} &\longrightarrow S \\ (s_1 \otimes x, s_2 \otimes \phi) &\longmapsto s_1 s_2 \sigma(\phi(x)). \end{aligned}$$

Let $\mathcal{P}$ be an $\mathcal{F}$-display and let $\mathcal{P}^\vee$ be the dual display. The maps $(F^\vee)^\sharp$ and $V^\sharp$ are dual in the following sense

$$< V^\sharp x, s \otimes \phi >_{(\sigma)} = < x, (F^\vee)^\sharp (s \otimes \phi) > .$$

Definition 3.2.5. A *polarization* of an $\mathcal{F}$-display $\mathcal{P}$ is a bilinear form

$$\beta : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}_{m,\mathcal{F}}$$

such that the underlying bilinear form $P \times P \longrightarrow S$ is alternating and its determinant is non-zero.

If $\mathcal{F} = \mathcal{W}_O(R)$, the *height* of β is the height of the associated isogeny $\mathcal{P} \longrightarrow \mathcal{P}^\vee$ (cf. Proposition 3.1.18). We write $\mathrm{height}_O\, \beta$ for the height of β. Then we have $\mathrm{height}_O\, \beta = \mathrm{ord}_\pi \det \beta$ in the notation of Proposition 3.1.18. The polarization is called *principal* if $\mathrm{height}_O(\beta) = 0$.

Remark 3.2.6. Let X be a strict formal p-divisible O-module over $R \in \mathrm{Nilp}_O$. Let $\mathcal{P}$ be the $\mathcal{W}_O(R)$-display of X in the sense of Theorem 3.1.11. If the dual display $\mathcal{P}^\vee$ is nilpotent, it corresponds to a strict formal p-divisible O-module $X^\vee$, called the *O-dual* of X. In this case, a polarization is given by an anti-symmetric O-module homomorphism $X \longrightarrow X^\vee$.

3.3. **The Ahsendorf functor.** We will give here an alternative definition of the Ahsendorf functor of [1] which is better suited to our purposes. One step of this definition is contained in the Appendix of [22]. We use a Lubin-Tate frame introduced by Mihatsch in loc. cit., but for us it will be important to make a specific choice, cf. Definition 3.3.8.

Let $\mathbb{Q}_p \subset \mathfrak{k} \subset K$ be a subfield. We denote by $\mathfrak{o}$ the ring of integers in $\mathfrak{k}$. We fix a prime element $\varpi \in \mathfrak{o}$. If R is an $\mathfrak{o}$-algebra we denote by $W_\mathfrak{o}(R)$ the Witt vectors relative to $\mathfrak{o}$ and ϖ. The Frobenius and the Verschiebung will be denoted by $\mathfrak{f}$ and $\mathfrak{v}$. We set $[K : \mathfrak{k}] = ef$ where e is the ramification index and f is the inertia index. Beginning with section 4 we will only consider the case where $\mathfrak{k} = \mathbb{Q}_p$.

For an O-algebra R we have the Drinfeld homomorphism

$$\mu : W_\mathfrak{o}(R) \longrightarrow W_O(R), \tag{3.3.1}$$

cf. [9, Prop. 1.2]. It is functorial in R and satisfies $\mathbf{w}_{O,n}(\mu(\xi)) = \mathbf{w}_{\mathfrak{o},fn}(\xi)$, for $\xi \in W_\mathfrak{o}(R)$. This implies the following properties:

$$\mu({}^{\mathfrak{f}^f}\xi) = {}^F\mu(\xi), \quad \mu({}^{\mathfrak{v}}\xi) = \frac{\varpi}{\pi}\,{}^V(\mu({}^{\mathfrak{f}^{f-1}}\xi)), \quad \mu([u]) = [u], \tag{3.3.2}$$

for $\xi \in W_\mathfrak{o}(R)$, $u \in R$. The last equation says that the Teichmüller representative $[u] \in W_\mathfrak{o}(R)$, is mapped by μ to the Teichmüller representative $[u] \in W_O(R)$.

We have $\mu(I_\mathfrak{o}(R)) \subset I_O(R)$. Therefore we may rewrite the second equation of (3.3.2) as

$${}^{\dot{F}}(\mu(\eta)) = \frac{\varpi}{\pi}\mu({}^{\mathfrak{f}^{f-1}\dot{\mathfrak{f}}}\eta), \quad \eta \in I_\mathfrak{o}(R). \tag{3.3.3}$$

The following definition extends Definition 3.1.10 to the relative case.

Definition 3.3.1. Let $\mathcal{F} = (S, I, R, \sigma, \dot{\sigma})$ be an $\mathfrak{o}$-frame, where R is a p-adic O-algebra. Let $\mathcal{P} = (P, Q, F, \dot{F})$ be an $\mathcal{F}$-display. *A strict O-action on $\mathcal{P}$* is a homomorphism of $\mathfrak{o}$-algebras $O \longrightarrow \mathrm{End}\,\mathcal{P}$ such that the induced action on the R-module P/Q coincides with the O-module structure on P/Q obtained by restriction of scalars $O \longrightarrow R$.

For a p-adic O-algebra R we will define a functor

$$\mathfrak{A}_{O/\mathfrak{o},R} : \left(\begin{array}{c}\mathcal{W}_\mathfrak{o}(R)\text{-displays} \\ \text{with strict } O\text{-action}\end{array}\right) \longrightarrow \Big(\mathcal{W}_O(R)\text{-displays}\Big). \tag{3.3.4}$$

We call this functor *the Ahsendorf functor*. The image of a $\mathcal{W}_\mathfrak{o}(R)$-display $\mathcal{P}$ as in Definition 3.3.1 will be denoted by $\mathcal{P}_{\mathrm{a}} = \mathfrak{A}_{O/\mathfrak{o},R}(\mathcal{P})$. The main theorem on the Ahsendorf functor is:

Theorem 3.3.2. *Let R be an O-algebra such that p is nilpotent in R. The Ahsendorf functor induces an equivalence of categories*

$$\mathfrak{A}_{O/\mathfrak{o},R} : \left(\begin{array}{c}\textit{nilpotent } \mathcal{W}_\mathfrak{o}(R)\textit{-displays} \\ \textit{with strict } O\textit{-action}\end{array}\right) \longrightarrow \Big(\textit{nilpotent } \mathcal{W}_O(R)\textit{-displays}\Big).$$

Furthermore, the Ahsendorf functor canonically associates to a bilinear form

$$\beta : \mathcal{P}' \times \mathcal{P}'' \longrightarrow \mathcal{P} \tag{3.3.5}$$

of $\mathcal{W}_\mathfrak{o}(R)$-displays with a strict O-actions such that β is also O-bilinear, a bilinear form of $\mathcal{W}_O(R)$-displays

$$\mathcal{P}'_{\mathrm{a}} \times \mathcal{P}''_{\mathrm{a}} \longrightarrow \mathcal{P}_{\mathrm{a}}.$$

Proof. The first statement is the main result of [1]. The second statement is shown in Proposition 3.3.15. ∎

Remark 3.3.3. By Theorem 3.3.2 we obtain a functor

$$\Big(\text{strict formal } p\text{-divisible } O\text{-modules over } R\Big) \longrightarrow \Big(\mathcal{W}_O(R)\text{-displays}\Big), \tag{3.3.6}$$

which is defined as follows. By [19], [35] there is a functor from the first category to the category of $\mathcal{W}(R)$-displays with a strict O-action. Composing this with $\mathfrak{A}_{O/\mathbb{Z}_p,R}$ we obtain (3.3.6). In particular this gives a quasi-inverse functor to the functor of Theorem 3.1.11.

We will now define the Ahsendorf functor. We denote by $K^t \subset K$ the maximal subextension which is unramified over $\mathfrak{k}$. Let O^t be the ring of integers of K^t. We consider the Witt vectors $W_{O^t}(R)$ with respect to the prime element $\varpi \in O^t$. The Frobenius resp. the Verschiebung acting on $W_{O^t}(R)$ will be denoted by F' and V'. We will define $\mathfrak{A}_{O/\mathfrak{o},R}$ as the composite of two functors

$$\begin{aligned} \mathfrak{A}_{O^t/\mathfrak{o},R} : &\begin{pmatrix} \mathcal{W}_{\mathfrak{o}}(R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{W}_{O^t}(R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix}, \\ \mathfrak{A}_{O/O^t,R} : &\begin{pmatrix} \mathcal{W}_{O^t}(R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix} \longrightarrow \Big(\mathcal{W}_O(R) - \text{displays}\Big). \end{aligned} \tag{3.3.7}$$

We begin with the definition of $\mathfrak{A}_{O^t/\mathfrak{o},R}$.

Lemma 3.3.4. *Let S be an O-algebra which has no π-torsion. Let $\tau : S \longrightarrow S$ be a O-algebra homomorphism such that*

$$\tau(s) \equiv s^q \pmod{\pi}.$$

Let $u_0, u_1, \ldots, u_n, \ldots \in S$. Then there exists $\xi \in W_O(S)$ such that $\mathbf{w}_{O,n}(\xi) = u_n$ for all n iff

$$\tau(u_{n-1}) \equiv u_n \mod \pi^n S, \quad \text{for} \quad n \geq 1.$$

The element ξ is uniquely determined.

Proof. The proof is up to obvious changes identical with the proof for the classical case $O = \mathbb{Z}_p$, cf. [2, IX, §1, 2, Lemme 2]. □

We denote by $\sigma \in \mathrm{Gal}(K^t/\mathfrak{k})$ the Frobenius automorphism. By Lemma 3.3.4, there is a homomorphism $\lambda : O^t \longrightarrow W_{\mathfrak{o}}(O^t)$, defined by $\mathbf{w}_{\mathfrak{o},n}(\lambda(a)) = \sigma^n(a)$ for $a \in O^t$ and all n. We obtain a ring homomorphism

$$\varkappa : O^t \xrightarrow{\lambda} W_{\mathfrak{o}}(O^t) \longrightarrow W_{\mathfrak{o}}(R). \tag{3.3.8}$$

We introduce the *Ahsendorf frame* with respect to the unramified extension $O^t/\mathfrak{o}$ for a p-adic O^t-algebra R,

$$\mathcal{A}_{\mathfrak{o}}(R) = (W_{\mathfrak{o}}(R), I_{\mathfrak{o}}(R), R, \mathfrak{f}^f, \mathfrak{f}^{f-1}\dot{\mathfrak{f}}). \tag{3.3.9}$$

This is an O^t-frame via $\varkappa$.

Let $\mathcal{P} = (P, Q, F, \dot{F})$ be a $\mathcal{W}_{\mathfrak{o}}(R)$-display with a strict O-action. We set

$$P_m = \{x \in P \mid \iota(a)x = \varkappa(\sigma^m(a))x, \text{ for } a \in O^t\}, \quad m \in \mathbb{Z}/f\mathbb{Z}.$$

The $W_{\mathfrak{o}}(R)$-module P decomposes as

$$P = \oplus_{m\in\mathbb{Z}/f\mathbb{Z}} P_m. \tag{3.3.10}$$

There is a similiar decomposition for Q. The maps F and $\dot{F}$ of $\mathcal{P}$ are graded of degree one,

$$F : P_m \longrightarrow P_{m+1}, \quad \dot{F} : Q_m \longrightarrow P_{m+1}.$$

If the action ι is strict, we have $Q_m = P_m$ for $m \neq 0$. Then we define the $\mathcal{A}_{\mathfrak{o}}(R)$-display $\mathcal{P}_{\mathrm{ua}}$:

$$P_{\mathrm{ua}} = P_0, \quad Q_{\mathrm{ua}} = Q_0, \quad F_{\mathrm{ua}} = \dot{F}^{f-1}F, \quad \dot{F}_{\mathrm{ua}} = \dot{F}^f. \tag{3.3.11}$$

It is clear that O acts strictly on $\mathcal{P}_{\mathrm{ua}}$.

It follows from (3.3.2) that $\mu : W_{\mathfrak{o}}(R) \longrightarrow W_{O^t}(R)$ induces a morphism of O^t-frames

$$\mu : \mathcal{A}_{\mathfrak{o}}(R) \longrightarrow \mathcal{W}_{O^t}(R). \tag{3.3.12}$$

By base change we obtain from $\mathcal{P}_{\mathrm{ua}}$ a $\mathcal{W}_{O^t}(R)$-display $\mathcal{P}_{\mathrm{t}} = \mu_*(\mathcal{P}_{\mathrm{ua}})$. The strict action of O on $\mathcal{P}_{\mathrm{ua}}$ induces a strict action of O on $\mathcal{P}_{\mathrm{t}}$ because the tangent space remains unchanged by this base change.

Definition 3.3.5. The Ahsendorf functor $\mathfrak{A}_{O^t/\mathfrak{o},R}$ is the functor which associates to a $\mathcal{W}_{\mathfrak{o}}(R)$-display $\mathcal{P}$ with a strict O-action the $\mathcal{W}_O(R)$-display $\mathcal{P}_{\mathrm{t}}$ defined above.

The Ahsendorf functor is compatible with bilinear forms as follows. Let $\beta : \mathcal{P}' \times \mathcal{P}'' \longrightarrow \mathcal{P}$ as in (3.3.5). Because β is O^t-bilinear, β induces for each $m \in \mathbb{Z}/f\mathbb{Z}$ a pairing

$$\beta : P'_m \times P''_m \longrightarrow P_m,$$

and P'_i and P''_j are orthogonal for $i \neq j$. For $y' \in Q'_0$ and $y'' \in Q''_0$ we find

$$\beta((\dot{F}')^f y', (\dot{F}'')^f y'') = \dot{F}^f \beta(y', y'').$$

Therefore the restriction of β,

$$\beta_{\mathrm{ua}} : P'_0 \times P''_0 \longrightarrow P_0$$

induces a bilinear form of $\mathcal{A}_{\mathfrak{o}}(R)$-displays

$$\mathcal{P}'_{\mathrm{ua}} \times \mathcal{P}''_{\mathrm{ua}} \longrightarrow \mathcal{P}_{\mathrm{ua}}.$$

Applying Proposition 3.2.3, we obtain a bilinear form in the category of $\mathcal{W}_{O^t}(R)$-displays,

$$\beta_{\mathrm{t}} : \mathcal{P}'_{\mathrm{t}} \times \mathcal{P}''_{\mathrm{t}} \longrightarrow \mathcal{P}_{\mathrm{t}}. \tag{3.3.13}$$

Now we define $\mathfrak{A}_{O/O^t,R}$. First we introduce the Lubin-Tate frame. We choose a finite normal extension L of K^t which contains K. We set $\Phi = \mathrm{Hom}_{K^t\text{-Alg}}(K, L)$. Let $\varphi_0 : K \longrightarrow L$ be the identical embedding.

Let $\mathbf{E}_K \in O^t[T]$ be the Eisenstein polynomial of $\pi \in O$ over O^t. In $O_L[T]$ it decomposes as

$$\mathbf{E}_K(T) = \prod_{\varphi \in \Phi} (T - \varphi(\pi)).$$

We set

$$\mathbf{E}_{K,0}(T) = \prod_{\varphi \in \Phi, \varphi \neq \varphi_0} (T - \varphi(\pi)) \in O_L[T]$$

One sees easily that $\mathbf{E}_{K,0} \in O[T]$. We lift these polynomials via $\mathbf{w}_0$ to the ring of Witt vectors,

$$\begin{aligned} \tilde{\mathbf{E}}_K(T) &= \textstyle\prod_{\varphi \in \Phi} (T - [\varphi(\pi)]) \in W_{O^t}(O^t)[T], \\ \tilde{\mathbf{E}}_{K,0}(T) &= \textstyle\prod_{\varphi \in \Phi, \varphi \neq \varphi_0} (T - [\varphi(\pi)]) \in W_{O^t}(O)[T]. \end{aligned} \tag{3.3.14}$$

The Frobenius F' and the Verschiebung V' act via the second factor on $O \otimes_{O^t} W_{O^t}(R)$. We set

$$\dot{F}' = (V')^{-1} : O \otimes_{O^t} I_{O^t}(R) \longrightarrow O \otimes_{O^t} W_{O^t}(R).$$

Proposition 3.3.6. *The element*

$${}^{\dot{F}'}(\tilde{\mathbf{E}}_K(\pi \otimes 1)) \in O \otimes_{O^t} W_{O^t}(O^t)$$

is a unit of the form

$$(\frac{\pi^e}{\varpi} \otimes 1)\delta, \quad \delta \in O \otimes_{O^t} W_{O^t}(O^t) \tag{3.3.15}$$

such that $\delta - (1 \otimes 1)$ lies in the kernel of

$$O \otimes_{O^t} W_{O^t}(O^t) \longrightarrow O \otimes_{O^t} W_{O^t}(\kappa),$$

and hence in the radical of $O \otimes_{O^t} W_{O^t}(O^t)$.

Proof. The element is defined because $\mathrm{id} \otimes \mathbf{w}_{O^t,0} : O \otimes_{O^t} W_{O^t}(O^t) \longrightarrow O$ maps $\tilde{\mathbf{E}}_K(\pi \otimes 1)$ to 0. In the following computation we pass to $O \otimes_{O^t} W_{O^t}(O_L)$. We find from the definitions:

$$\begin{aligned} {}^{\dot{F}'}(\tilde{\mathbf{E}}_K(\pi \otimes 1)) &= {}^{\dot{F}'}(\prod_{\varphi \in \Phi} (\pi \otimes 1 - 1 \otimes [\varphi(\pi)]) \\ &= \frac{1}{\varpi} \prod_{\varphi \in \Phi} (\pi \otimes 1 - 1 \otimes [\varphi(\pi)^q]) = \frac{1}{\varpi} \sum_{i=0}^{e} \pi^{e-i} \otimes {}^{F'} s_i. \end{aligned} \tag{3.3.16}$$

Here we denote by s_i the elementary symmetric polynomial of degree i evaluated at the e arguments $[\varphi(\pi)]$. By definition $s_0 = 1$. We claim that for $i > 0$ the elements ${}^{F'} s_i \in W_{O^t}(O^t)$ are divisible by ϖ. Clearly $\mathbf{w}_{O^t,0}(s_i)$ is divisible by π. On the other hand, $\mathbf{w}_{O^t,0}(s_i) \in O^t$ and therefore is divisible by ϖ. We find expressions in $W_{O^t}(O^t)$,

$$s_i = [\varpi c_i] + {}^{V'}\xi_i, \quad c_i \in O^t, \ \xi_i \in W_{O^t}(O^t).$$

Therefore ${}^{F'}s_i = [\varpi^q][c_i^q] + \varpi\xi^i$ is divisible by ϖ. Indeed, using Lemma 3.3.4, one shows as in the proof of [33, Lem. 28] that ϖ divides $[\varpi^q]$.

Now we may write the last term of (3.3.16) as

$$\frac{\pi^e}{\varpi} \otimes 1 + \sum_{i=1}^{e} \pi^{e-i} \otimes \frac{{}^{F'}s_i}{\varpi}.$$

Finally, $\frac{{}^{F'}s_i}{\varpi}$ lies for $i > 0$ in the kernel of $W_{O^t}(O^t) \longrightarrow W_{O^t}(\kappa)$. Indeed, the elements $[\varphi(\pi)] \in W_{O^t}(O_L)$ are mapped to zero in $W_{O^t}(\kappa_L)$ and therefore a fortiori the symmetric functions s_i. We conclude that s_i and then ${}^{F'}s_i$ become zero in $W_{O^t}(\kappa)$ for $i > 0$. Because ϖ is not a zero divisor in $W_{O^t}(\kappa)$ the elements $\frac{{}^{F'}s_i}{\varpi}$ are then also in the kernel. □

We write in the ring $W_O(O)$,

$$\pi - [\pi] = {}^{V}\varepsilon. \tag{3.3.17}$$

One checks that $\varepsilon \in W_O(O)$ is a unit. If we apply F to the last equation, we obtain

$$\pi - [\pi^q] = \pi\varepsilon. \tag{3.3.18}$$

In particular $[\pi^q]$ is divisible by π.

Lemma 3.3.7. *The image of the element* $\left({}^{\dot{F}'}\tilde{\mathbf{E}}_K(\pi \otimes 1)\right)^{-1} \cdot {}^{F'}\tilde{\mathbf{E}}_{K,0}(\pi \otimes 1)$ *under the Drinfeld homomorphism*

$$\mu : O \otimes_{O^t} W_{O^t}(O) \longrightarrow W_O(O)$$

equals $\varepsilon^{-1}(\varpi/\pi)$.

Proof. It is enough to show the same assertion for $O \otimes_{O^t} W_{O^t}(O_L) \longrightarrow W_O(O_L)$. The image of ${}^{\dot{F}'}\tilde{\mathbf{E}}_K(\pi \otimes 1)$ by the last map is $\varpi^{-1}\prod_{\varphi}(\pi - [\varphi(\pi)^q])$. Here we used that ϖ is not a zero divisor in the participating rings. Our assertion is equivalent with the equation

$$\varpi^{-1}\prod_{\varphi}(\pi - [\varphi(\pi)^q])\varepsilon^{-1}\frac{\varpi}{\pi} = \prod_{\varphi \neq \varphi_0}(\pi - [\varphi(\pi)^q]).$$

But this is a consequence of (3.3.18). □

The free $W_{O^t}(R)$-module $O \otimes_{O^t} W_{O^t}(R)$ has the basis

$$1 \otimes 1,\ \pi \otimes 1 - 1 \otimes [\pi], \ldots, \pi^m \otimes 1 - 1 \otimes [\pi]^m, \ldots, \pi^{e-1} \otimes 1 - 1 \otimes [\pi]^{e-1}. \tag{3.3.19}$$

To ease the notation, here $[\pi]$ denotes the Teichmüller representative of the image of π by the morphism $O \longrightarrow R$. Let

$$\mathcal{J} = \operatorname{Ker}\left(O \otimes_{O^t} W_{O^t}(R) \longrightarrow R\right),$$

where the map is induced by $\mathbf{w}_0 : W_{O^t}(R) \longrightarrow R$. The ideal $\mathcal{J}$ is contained in the radical of $O \otimes_{O^t} W_{O^t}(R)$. As a $W_{O^t}(R)$-module, $\mathcal{J}$ is the direct sum of $O \otimes_O I_O(R)$ and the direct summand generated by the last $e-1$ elements of (3.3.19). In particular we obtain

$$\mathcal{J} = O \otimes_{O^t} I_{O^t}(R) + (\pi \otimes 1 - 1 \otimes [\pi])(O \otimes_{O^t} W_{O^t}(R)). \tag{3.3.20}$$

We define maps $\sigma_{\mathrm{lt}} : O \otimes_{O^t} W_{O^t}(R) \longrightarrow O \otimes_{O^t} W_{O^t}(R)$, $\dot{\sigma}_{\mathrm{lt}} : \mathcal{J} \longrightarrow O \otimes_{O^t} W_{O^t}(R)$ by

$${}^{\sigma_{\mathrm{lt}}}\xi = {}^{F'}\xi, \quad {}^{\dot{\sigma}_{\mathrm{lt}}}\eta = ({}^{\dot{F}'}\tilde{\mathbf{E}}_K)^{-1}\,{}^{\dot{F}'}(\tilde{\mathbf{E}}_{K,0}\eta), \quad \xi \in O \otimes_{O^t} W_{O^t}(R),\ \eta \in \mathcal{J}.$$

The map $\dot{\sigma}_{\mathrm{lt}} : \mathcal{J} \longrightarrow O \otimes_{O^t} W_{O^t}(R)$ is σ_{lt}-linear. Then we obtain

$${}^{\dot{\sigma}_{\mathrm{lt}}}(\pi \otimes 1 - 1 \otimes [\pi]) = ({}^{\dot{F}'}\tilde{\mathbf{E}}_K)^{-1}\,{}^{\dot{F}'}(\tilde{\mathbf{E}}_K) = 1. \tag{3.3.21}$$

Definition 3.3.8. (comp. [22, Def. 2.7]) The *Lubin-Tate frame* for O is the O-frame

$$\mathcal{F}_{\mathrm{lt}}(R) := (O \otimes_{O^t} W_{O^t}(R), \mathcal{J}, R, \sigma_{\mathrm{lt}}, \dot{\sigma}_{\mathrm{lt}}).$$

This is indeed an O-frame: the only thing we need to check is

$$^{\sigma_{\mathrm{lt}}}\xi \equiv \xi^q \ (\bmod\ (\pi\otimes 1)O\otimes_{O^t} W_{O^t}(R)),$$

and this follows because $a \equiv a^q \bmod \pi$ for $a \in O$ and $^{F'}\eta \equiv \eta^q \bmod \varpi$ for $\eta \in W_{O^t}(R)$.

We remark that by (3.3.21)

$$(\pi\otimes 1 - 1\otimes[\pi^q])\ ^{\dot\sigma_{\mathrm{lt}}}\eta = \ ^{\sigma_{\mathrm{lt}}}\eta, \quad \eta\in\mathcal{J}.$$

Now we start with a $\mathcal{W}_{O^t}(R)$-display $\mathcal{P}$ with a strict O-action. The last condition can be reformulated as

$$\mathcal{J}P \subset Q. \tag{3.3.22}$$

We refer to [1, Prop. 2.26] for the proof of the following lemma.

Lemma 3.3.9. *Let P be a $W_{O^t}(R)$-module with an action of O, i.e., a homomorphism of O-algebras*

$$O \longrightarrow \mathrm{End}_{W_{O^t}(R)} P.$$

Assume that locally on $\operatorname{Spec} R$ *the $W_{O^t}(R)$-module P is free. Then P is locally on* $\operatorname{Spec} R$ *a finitely generated free $O\otimes_{O^t} W_{O^t}(R)$-module.* □

Lemma 3.3.10. *Let $\mathcal{P}$ be a $\mathcal{W}_{O^t}(R)$-display with a strict O-action. Let $x\in P$. By (3.3.22) $(\pi\otimes 1 - 1\otimes[\pi])x \in Q$. The following equation holds,*

$$Fx = \big(\ ^{\dot F'}\tilde{\mathbf{E}}_K(\pi\otimes 1)\big)^{-1}\cdot\ ^{F'}\tilde{\mathbf{E}}_{K,0}(\pi\otimes 1)\cdot \dot F\big((\pi\otimes 1 - 1\otimes[\pi])x\big).$$

Proof. From the definition of the polynomials $\tilde{\mathbf{E}}_K$ and $\tilde{\mathbf{E}}_{K,0}$, we find since $\varphi_0(\pi)=\pi$,

$$\tilde{\mathbf{E}}_K(\pi\otimes 1) = \tilde{\mathbf{E}}_{K,0}(\pi\otimes 1)\cdot(\pi\otimes 1 - 1\otimes[\pi]).$$

Therefore

$$\begin{aligned}\dot F(\tilde{\mathbf{E}}_K(\pi\otimes 1)x) &= \dot F(\tilde{\mathbf{E}}_{K,0}(\pi\otimes 1))\cdot(\pi\otimes 1 - 1\otimes[\pi])x)\\ &= \ ^{F'}(\tilde{\mathbf{E}}_{K,0}(\pi\otimes 1))\dot F((\pi\otimes 1 - 1\otimes[\pi])x).\end{aligned}$$

Because $\tilde{\mathbf{E}}_K(\pi\otimes 1)\in O\otimes_{O^t} I_{O^t}(R)$, we obtain

$$\dot F(\tilde{\mathbf{E}}_K(\pi\otimes 1)x) = \ ^{\dot F'}(\tilde{\mathbf{E}}_K(\pi\otimes 1))Fx.$$

We conclude by Lemma 3.3.6. □

We now associate to the $\mathcal{W}_{O^t}(R)$-display $\mathcal{P}=(P,Q,F,\dot F)$ with a strict O-action a $\mathcal{F}_{\mathrm{lt}}$-display $\mathcal{P}_{\mathrm{lt}}=(P_{\mathrm{lt}},Q_{\mathrm{lt}},F_{\mathrm{lt}},\dot F_{\mathrm{lt}})$. We set $P_{\mathrm{lt}}=P$, $Q_{\mathrm{lt}}=Q$, $\dot F_{\mathrm{lt}}=\dot F$, and

$$F_{\mathrm{lt}}(x) = \dot F((\pi\otimes 1 - 1\otimes[\pi])x), \quad x\in P. \tag{3.3.23}$$

Proposition 3.3.11. *$\mathcal{P}_{\mathrm{lt}}$ is an $\mathcal{F}_{\mathrm{lt}}(R)$-display.*

Proof. The only thing we have not checked is the equation

$$\dot F_{\mathrm{lt}}(\eta x) = \ ^{\dot\sigma_{\mathrm{lt}}}\eta F_{\mathrm{lt}}x, \quad \eta\in\mathcal{J}. \tag{3.3.24}$$

We begin with the case $\eta = \ ^{V'}\xi$. We apply Lemma 3.3.10

$$\dot F_{\mathrm{lt}}(\eta x) = \dot F(\ ^{V'}\xi x) = \xi F(x) = \xi(\ ^{\dot F}\tilde{\mathbf{E}}_K)^{-1}\cdot\ ^{F}\tilde{\mathbf{E}}_{K,0}\cdot\dot F((\pi\otimes 1 - 1\otimes[\pi])x). \tag{3.3.25}$$

By definition

$$^{\dot\sigma_{\mathrm{lt}}}\eta = (\ ^{\dot F'}\tilde{\mathbf{E}}_K)^{-1}\ ^{\dot F'}(\tilde{\mathbf{E}}_{K,0}\ ^{V}\xi) = (\ ^{\dot F'}\tilde{\mathbf{E}}_K)^{-1}\ ^{F'}(\tilde{\mathbf{E}}_{K,0})\xi.$$

Using the definition (3.3.23), we can write the right hand side of (3.3.25) as $^{\dot\sigma_{\mathrm{lt}}}\eta F_{\mathrm{lt}}(x)$, hence we are done in this case.

Next we consider the case where $\eta = (\pi\otimes 1 - 1\otimes[\pi])\xi$. Then we find

$$\dot F_{\mathrm{lt}}(\eta x) = \dot F((\pi\otimes 1 - 1\otimes[\pi])\xi x) = \ ^{F}\xi F_{\mathrm{lt}}(x).$$

But by (3.3.21) we have $^{\dot\sigma_{\mathrm{lt}}}((\pi\otimes 1 - 1\otimes[\pi])\xi) = \ ^{F'}\xi$. Therefore in this case (3.3.24) is true as well. □

We use the same symbol $\varepsilon \in W_O(R)^\times$ for the image of the element $\varepsilon \in W_O(O)^\times$ defined by (3.3.17). We define the frame

$$\mathcal{W}_O^\varepsilon(R) = (W_O(R), I_O(R), F, \varepsilon^{-1}\dot{F}). \tag{3.3.26}$$

We note that the categories of displays over $\mathcal{W}_O(R)$ and $\mathcal{W}_O^\varepsilon(R)$ are canonically isomorphic. Indeed, if $\mathcal{P} = (P, Q, F, \dot{F})$ is a $\mathcal{W}_O(R)$-display, then $\mathcal{P} = (P, Q, \varepsilon F, \dot{F})$ is a $\mathcal{W}_O^\varepsilon(R)$-display.

Recall that we denote the Frobenius and the Verschiebung acting on $W_{O^t}(R)$ by F' and V'. We consider the Drinfeld homomorphism $\mu : W_{O^t}(R) \longrightarrow W_O(R)$, cf. (3.3.1). This is a functorial ring homomorphism such that $\mathbf{w}'_n(\mu(\xi)) = \mathbf{w}_n(\xi)$ which has the following properties

$$\mu({}^{F'}\xi) = {}^F\mu(\xi), \quad \mu({}^{V'}\xi) = \frac{\varpi}{\pi}\, {}^V\mu(\xi), \quad \mu([a]) = [a], \text{ for } a \in R. \tag{3.3.27}$$

The Drinfeld homomorphism extends to a ring homomorphism

$$\mu : O \otimes_{O^t} W_{O^t}(R) \longrightarrow W_O(R) \tag{3.3.28}$$

which we denote by the same letter.

Proposition 3.3.12. *The Drinfeld homomorphism induces a morphism of O-frames*

$$\mu : \mathcal{F}_{\mathrm{lt}}(R) \longrightarrow \mathcal{W}_O^\varepsilon(R).$$

Proof. We have to check that the image of $\mathcal{J} \subset O \otimes_{O^t} W_{O^t}(R)$ by μ is contained in $I_O(R)$. This is immediate because $\mu(\pi \otimes 1 - 1 \otimes [\pi]) = \pi - [\pi] = {}^V\varepsilon$. It remains to prove the equations for $\xi \in O \otimes_{O^t} W_{O^t}(R)$ and $\eta \in \mathcal{J}$,

$$\mu({}^{\sigma_{\mathrm{lt}}}\xi) = {}^F\mu(\xi), \quad \mu({}^{\dot{\sigma}_{\mathrm{lt}}}\eta) = \varepsilon^{-1}\, {}^{\dot{F}}\mu(\eta). \tag{3.3.29}$$

The first equation follows from (3.3.27). To prove the second equation, it is enough to consider the following two cases separately: $\eta = {}^{V'}\xi$ and $\eta = (\pi \otimes 1 - 1 \otimes [\pi])\xi$. In the first case we have

$${}^{\dot{\sigma}_{\mathrm{lt}}}\eta = ({}^{\dot{F}'}\tilde{\mathbf{E}}_K)^{-1}\, {}^{\dot{F}'}(\tilde{\mathbf{E}}_{K,0}\, {}^{V'}\xi) = ({}^{\dot{F}'}\tilde{\mathbf{E}}_K)^{-1}({}^{F'}\tilde{\mathbf{E}}_{K,0})\xi.$$

Applying Lemma 3.3.7, we obtain

$$\mu({}^{\dot{\sigma}_{\mathrm{lt}}}\eta) = \varepsilon^{-1}(\varpi/\pi)\mu(\xi).$$

On the other hand, we have by (3.3.27)

$$\varepsilon^{-1}\, {}^{\dot{F}}\mu({}^{V'}\xi) = \varepsilon^{-1}\, {}^{\dot{F}}((\varpi/\pi)\, {}^V\mu(\xi)) = \varepsilon^{-1}((\varpi/\pi)\mu(\xi),$$

as desired.

Now we consider the case $\eta = (\pi \otimes 1 - 1 \otimes [\pi])\xi$. We have

$$\mu((\pi \otimes 1 - 1 \otimes [\pi])\xi) = (\pi - [\pi])\mu(\xi) = {}^V\varepsilon\mu(\xi) = {}^V(\varepsilon\, {}^F\mu(\xi)).$$

We obtain

$$\varepsilon^{-1}\, {}^{\dot{F}}\mu(\eta) = {}^F\mu(\xi).$$

On the other hand, we find by (3.3.21) and (3.3.27)

$$\mu({}^{\dot{\sigma}_{\mathrm{lt}}}\eta) = \mu({}^{F'}\xi) = {}^F\mu(\xi),$$

as desired. □

Starting now with a $\mathcal{W}_{O^t}(R)$-display $\mathcal{P} = (P, Q, F, \dot{F})$ with a strict O-action, we have the associated $\mathcal{F}_{\mathrm{lt}}(R)$-display $\mathcal{P}_{\mathrm{lt}} = (P_{\mathrm{lt}}, Q_{\mathrm{lt}}, F_{\mathrm{lt}}, \dot{F}_{\mathrm{lt}})$, cf. Proposition 3.3.11. After taking the base change by the morphism of frames of Proposition 3.3.12, we obtain a $\mathcal{W}_O^\varepsilon(R)$-display $\mathcal{P}_{\mathrm{a}}^\varepsilon = (P_{\mathrm{a}}^\varepsilon, Q_{\mathrm{a}}^\varepsilon, F_{\mathrm{a}}^\varepsilon, \dot{F}_{\mathrm{a}}^\varepsilon)$. Then $\mathcal{P}_{\mathrm{a}} = (P_{\mathrm{a}}^\varepsilon, Q_{\mathrm{a}}^\varepsilon, \varepsilon^{-1}F_{\mathrm{a}}^\varepsilon, \dot{F}_{\mathrm{a}}^\varepsilon)$ is an $\mathcal{W}_O(R)$-display.

Definition 3.3.13. The Ahsendorf functor $\mathfrak{A}_{O/O^t,R}$ is the functor which associates to the $\mathcal{W}_{O^t}(R)$-display $\mathcal{P}$ with a strict O-action the $\mathcal{W}_O(R)$-display $\mathcal{P}_{\mathrm{a}}$ defined above.

From the construction we obtain that

$$P_{\mathrm{a}} = W_O(R) \otimes_{O\otimes_{O^t} W_{O^t}(R)} P, \tag{3.3.30}$$

and that Q_{a} is the kernel of the natural map

$$W_O(R) \otimes_{O\otimes_{O^t} W_{O^t}(R)} P \longrightarrow P/Q.$$

We note that the canonical map $P \longrightarrow P_{\mathrm{a}}$ induces a map $Q \longrightarrow Q_{\mathrm{a}}$.

Proposition 3.3.14. *Let $\mathcal{P}$ be a $\mathcal{W}_{O^t}(R)$-display with a strict action of O. Let $\mathcal{P}_{\mathrm{a}} = \mathfrak{A}_{O/O^t,R}(\mathcal{P})$ be its image by the Ahsendorf functor. The following diagram is commutative*

$$\begin{array}{ccc} Q & \xrightarrow{\dot{F}} & P \\ \downarrow & & \downarrow \\ Q_{\mathrm{a}} & \xrightarrow[\dot{F}_{\mathrm{a}}]{} & P_{\mathrm{a}} \end{array}$$

Proof. This follows from the definition of $\mathcal{P}_{\mathrm{lt}}$ before Proposition 3.3.11 and the definition of base change (via the morphism of Proposition 3.3.12). □

We note that this diagram determines the map $\dot{F}_{\mathrm{a}}$ uniquely. Indeed, consider the following equation in P_{a} under the identification (3.3.30),

$${}^V\varepsilon \otimes x = 1 \otimes (\pi \otimes 1 - 1 \otimes [\pi])x.$$

Applying $\dot{F}_{\mathrm{a}}$, we obtain from the diagram that

$$\varepsilon F_{\mathrm{a}}(1 \otimes x) = 1 \otimes \dot{F}((\pi \otimes 1 - 1 \otimes [\pi])x).$$

This shows that F_{a} is uniquely determined. Because the image of Q and $I_O(R)P_{\mathrm{a}}$ generate Q_{a} as a $W_O(R)$-module, the map $\dot{F}_{\mathrm{a}}$ is then also uniquely determined.

We return to the notation that $\mathcal{P}$ is an $\mathfrak{o}$-display with a strict O-action. Applying the functors $\mathfrak{A}_{O^t/\mathfrak{o},R}$ and $\mathfrak{A}_{O/O^t,R}$, we obtain first $\mathcal{P}_{\mathrm{t}}$ and then $\mathcal{P}_{\mathrm{a}}$. We find by our definitions that, with the notation of (3.3.10),

$$P_{\mathrm{a}} = W_O(R) \otimes_{\left(O\otimes_{O^t,\varkappa} W_{\mathfrak{o}}(R)\right)} P_0 = W_O(R) \otimes_{\left(O\otimes_{\mathfrak{o}} W_{\mathfrak{o}}(R)\right)} P. \tag{3.3.31}$$

We note that $P_0 = (O \otimes_{O^t,\varkappa} W_{\mathfrak{o}}(R)) \otimes_{O\otimes_{\mathfrak{o}} W_{\mathfrak{o}}(R)} P$.

We already noted that the Ahsendorf functor $\mathfrak{A}_{O^t/\mathfrak{o},R}$ is compatible with bilinear forms. Similar remarks are also valid for the Ahsendorf functor $\mathfrak{A}_{O/O^t,R}$: first one checks that the functor $\mathcal{P} \mapsto \mathcal{P}_{\mathrm{lt}}$ is compatible with bilinear forms of displays, and then applies Proposition 3.2.3 for the compatibility of base change with bilinear forms. Taken together with (3.3.13), we obtain the following property of the Ahsendorf functor $\mathfrak{A}_{O/\mathfrak{o},R}$.

Proposition 3.3.15. *Consider a bilinear form of $\mathcal{W}_{\mathfrak{o}}(R)$-displays,*

$$\beta : \mathcal{P}' \times \mathcal{P}'' \longrightarrow \mathcal{P},$$

which is also O-bilinear. Then the bilinear form $\beta : P' \times P'' \longrightarrow P$ induces by (3.3.31) a $W_O(R)$-bilinear form $\beta_{\mathrm{a}} : P'_{\mathrm{a}} \times P''_{\mathrm{a}} \longrightarrow P_{\mathrm{a}}$. The bilinear form β_{a} is a bilinear form of $\mathcal{W}_O(R)$-displays,

$$\beta_{\mathrm{a}} : \mathcal{P}'_{\mathrm{a}} \times \mathcal{P}''_{\mathrm{a}} \longrightarrow \mathcal{P}_{\mathrm{a}}.$$

□

Remark 3.3.16. In the case where $R = k$ is a perfect field, the description of the Ahsendorf functor is very simple. We consider the functor $\mathfrak{A}_{O/\mathbb{Z}_p,k}$ which is relevant for us. As a prime element of $\mathbb{Z}_p$ we choose p. The element $\varepsilon \in W_O(k)$ is 1. As above, we denote by $\mathfrak{f}$, resp. $\mathfrak{v}$, the Frobenius, resp. the Verschiebung, of the ring of Witt vectors $W(k)$. In this case the morphism (3.3.12) of frames $\mu : \mathfrak{A}_{O/\mathbb{Z}_p,k} \longrightarrow \mathcal{W}_{O^t}(k)$, and the morphism of frames $\mu : \mathcal{F}_{\mathrm{lt}}(k) \longrightarrow \mathcal{W}_O(k)$ of Proposition 3.3.12 are isomorphisms. Therefore we identify $\mathcal{W}_O(k)$ with the frame

$$(O \otimes_{O^t} W(k), \pi O \otimes_{O^t} W(k), k, \mathfrak{f}^f, \mathfrak{f}^f \pi^{-1}). \tag{3.3.32}$$

This is a perfect frame with $u = \theta = \pi$, cf. Definition 3.1.8.

Let (P, F, V) be a $\mathcal{W}(k)$-Dieudonné module with a strict O-action. We have the decomposition $P = \oplus_m P_m$ cf. (3.3.10). The summand P_0 is an $O \otimes_{O^t} W(k)$-module. Since the action of O is strict, we find

$$\pi P_0 \subset Q_0 = V^f P_0.$$

Therefore we can define

$$V_{\mathrm{a}} = V^f, \ F_{\mathrm{a}} = V^{-f}\pi : P_0 \longrightarrow P_0. \tag{3.3.33}$$

Then $(P_0, F_{\mathrm{a}}, V_{\mathrm{a}})$ is a Dieudonné module for the frame (3.3.32). It is the image of (P, F, V) by the Ahsendorf functor $\mathfrak{A}_{O/\mathbb{Z}_p,k}$.

Proposition 3.3.17. *Let* $R \in \mathrm{Nilp}_O$. *We assume that* $\operatorname{Spec} R$ *is connected. Let* $\mathcal{P}$ *be a* $\mathcal{W}(R)$-*display with a strict* O-*action, and let* $\mathcal{P}_{\mathrm{a}}$ *be the image by the Ahsendorf functor* $\mathfrak{A}_{O/\mathbb{Z}_p,k}$. *Then*

$$\operatorname{height} \mathcal{P} = [O : \mathbb{Z}_p] \operatorname{height}_O \mathcal{P}_{\mathrm{a}}.$$

The right hand side denotes the height of the $\mathcal{W}_O(R)$-*display* $\mathcal{P}_{\mathrm{a}}$ *in the sense of Definition 3.1.4.*

Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *be an isogeny of* $\mathcal{W}(R)$-*displays with strict* O-*action, and let* $\alpha_{\mathrm{a}} : \mathcal{P}_{1,a} \longrightarrow \mathcal{P}_{2,a}$ *be the image by the Ahsendorf functor. Then*

$$\operatorname{height} \alpha = [O^t : \mathbb{Z}_p] \operatorname{height}_O \alpha_{\mathrm{a}}.$$

Let $R = k$ *be a perfect field. Let* $\lambda_1 < \ldots < \lambda_m$ *be the slopes of* $\mathcal{P}$. *Then the slopes of* $\mathcal{P}_{\mathrm{a}}$ *are* $[O : \mathbb{Z}_p]\lambda_1, \ldots, [O : \mathbb{Z}_p]\lambda_m$. *The display* $\mathcal{P}$ *with its strict* O-*action is isogenous to a direct sum of displays with a strict* O-*action* $\oplus_{i=1}^m \mathcal{P}(\lambda_i)$ *such that* $\mathcal{P}(\lambda_i)$ *is isoclinic of slope* λ_i.

Proof. It suffices to consider the case where $R = k$ is a perfect field. Then it is a consequence of the description of the Ahsendorf functor given above, cf. (3.3.33). □

In the end of this subsection, we relate explicitly the deformation theory of a display with a strict O-action and its image by the Ahsendorf functor. Let $S \longrightarrow R$ be an epimorphism of O-algebras which are p-adic. We assume that the kernel $\mathfrak{a}$ of this epimorphism is endowed with divided powers γ relative to $\mathfrak{o}$. Then γ induces also divided powers γ_{t} on $\mathfrak{a}$ relative to O^t. Indeed, let $q_{\mathfrak{o}}$ be the number of elements in the residue class field of $\mathfrak{o}$. Then we set

$$\gamma_{\mathrm{t}}(a) = \gamma(a) a^{q - q_{\mathfrak{o}}} = ''a^q/\varpi'', \quad a \in \mathfrak{a} \tag{3.3.34}$$

By setting $\gamma_{\mathrm{a}}(a) = \gamma_{\mathrm{t}}(a)(\varpi/\pi)$, we obtain divided powers γ_{a} relative to O on $\mathfrak{a}$.

Let $\mathcal{P} = (P, Q, F, \dot{F})$ be a $\mathcal{W}_{\mathfrak{o}}(S/R)$-display with a strict action

$$\iota : O \longrightarrow \operatorname{End} \mathcal{P}.$$

The definition of strictness is literally the same as Definition 3.3.1. Since $(S \longrightarrow R, \gamma_{\mathrm{a}})$ is an O-*pd*-thickening, the O-frame $\mathcal{W}_O(S/R)$ is defined, cf. Example 3.1.3. The Ahsendorf functor generalizes to a *relative Ahsendorf functor*

$$\mathfrak{A}_{O/\mathfrak{o},S/R} : \begin{pmatrix} \mathcal{W}_{\mathfrak{o}}(S/R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix} \longrightarrow \Big(\mathcal{W}_O(S/R) - \text{displays}\Big). \tag{3.3.35}$$

The construction is the same but uses some additional arguments, which we will indicate now.

We define the *relative Ahsendorf frame* for $S \longrightarrow R$,

$$\mathcal{A}_{\mathfrak{o}}(S/R) = (W_{\mathfrak{o}}(S), I_{\mathfrak{o}}(S/R), R, \mathfrak{f}^f, \mathfrak{f}^{f-1}\dot{\mathfrak{f}}), \tag{3.3.36}$$

where $\dot{\mathfrak{f}} : I_{\mathfrak{o}}(S/R) \longrightarrow W_{\mathfrak{o}}(S)$ is defined as in Example 3.1.3. This is an O^t-frame by the homomorphism $\varkappa : O^t \longrightarrow W_{\mathfrak{o}}(S)$.

From the O^t-action on the $W_{\mathfrak{o}}(S)$-module P we obtain a decomposition, comp (3.3.10),

$$P = \oplus_{m \in \mathbb{Z}/f\mathbb{Z}} P_m, \quad Q = \oplus_{m \in \mathbb{Z}/f\mathbb{Z}} Q_m. \tag{3.3.37}$$

We obtain an $\mathcal{A}_{\mathfrak{o}}(S/R)$-display $\mathcal{P}_{\mathrm{ua}} = (P_{\mathrm{ua}}, Q_{\mathrm{ua}}, F_{\mathrm{ua}}, \dot{F}_{\mathrm{ua}})$ by the formulas (3.3.11).

Lemma 3.3.18. *The Drinfeld homomorphism* $\mu : W_{\mathfrak{o}}(S) \longrightarrow W_{O^t}(S)$ *induces a morphism of frames*

$$\mathcal{A}_{\mathfrak{o}}(S/R) \longrightarrow \mathcal{W}_{O^t}(S/R). \tag{3.3.38}$$

Proof. We have to prove the formula

$$^{\dot{F}'}(\mu(\eta)) = \mu(\ ^{\mathfrak{f}^{f-1}\dot{\mathfrak{f}}}\eta), \quad \eta \in I_{\mathfrak{o}}(S/R).$$

For $\eta \in I_{\mathfrak{o}}(S)$, this is (3.3.3). Therefore the formula follows if we show that $\mu : W_{\mathfrak{o}}(S) \longrightarrow W_{O^t}(S)$ maps logarithmic Teichmüller representatives of elements in $\mathfrak{a}$ to logarithmic Teichmüller representatives. Let $\dot{\mathbf{w}}_{\mathfrak{o},n}$ be the divided Witt polynomials defined by γ and let $\dot{\mathbf{w}}_{O^t,n}$ be the divided Witt polynomials defined by γ_{t}. It follows from the definition of the Drinfeld homomorphism (3.3.1) that

$$\dot{\mathbf{w}}_{O^t,n}(\mu(\xi)) = \varpi^{(f-1)n}\dot{\mathbf{w}}_{\mathfrak{o},fn}(\xi), \quad \xi \in W_{\mathfrak{o}}(\mathfrak{a}). \tag{3.3.39}$$

This is verified by reducing to a universal case where $\mathfrak{a}$ is without p-torsion. If now $\xi = \tilde{a} \in W_{\mathfrak{o}}(\mathfrak{a})$ is a logarithmic Teichmüller representative, the right hand side of (3.3.39) is 0 for $n \neq 0$, and is a for $n = 0$. This shows that $\mu(\mathfrak{a})$ is the logarithmic Teichmüller representative of a in $W_{O^t}(\mathfrak{a})$. □

Applying now base change to $\mathcal{P}_{\mathrm{ua}}$ relative to (3.3.38), we obtain a $\mathcal{W}_{O^t}(S/R)$-display $\mathcal{P}_{\mathrm{t}}$ with a strict O-action. The assignment $\mathcal{P} \mapsto \mathcal{P}_{\mathrm{t}}$ defines the functor

$$\mathfrak{A}_{O^t/\mathfrak{o},S/R} : \begin{pmatrix} \mathcal{W}_{\mathfrak{o}}(S/R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{W}_{O^t}(S/R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix}.$$

Next we define the functor

$$\mathfrak{A}_{O/O^t,S/R} : \begin{pmatrix} \mathcal{W}_{O^t}(S/R) - \text{displays} \\ \text{with strict } O\text{-action} \end{pmatrix} \longrightarrow \Big(\mathcal{W}_{O}(S/R) - \text{displays}\Big). \tag{3.3.40}$$

We begin with the definition of the relative Lubin-Tate frame $\mathcal{F}_{\mathrm{lt}}(S/R)$. We start with the frame $\mathcal{W}_{O^t}(S/R) = (W_{O^t}(S), I_{O^t}(S/R), F', \dot{F}')$. Recall that $I_{O^t}(S/R) = \tilde{\mathfrak{a}} \oplus I_{O^t}(S)$, where the ideal $\tilde{\mathfrak{a}}$ consists of the logarithmic Teichmüller representatives $\tilde{a}$ of elements $a \in \mathfrak{a}$, with respect to the divided powers γ_t. We have by definition $\dot{F}'(\tilde{a}) = 0$. Tensoring with $O \otimes_{O^t}$, we obtain

$$\begin{aligned} F' &: O \otimes_{O^t} W_{O^t}(S) &&\longrightarrow O \otimes_{O^t} W_{O^t}(S), \\ \dot{F}' &: O \otimes_{O^t} I_{O^t}(S/R) &&\longrightarrow O \otimes_{O^t} W_{O^t}(S). \end{aligned}$$

We define an ideal in $O \otimes_{O^t} W_{O^t}(S)$,

$$\mathcal{J}(S/R) = O \otimes_{O^t} I_{O^t}(S/R) + (\pi \otimes 1 - 1 \otimes [\pi])(O \otimes_{O^t} W_{O^t}(S)). \tag{3.3.41}$$

For an element $\eta \in \mathcal{J}(S/R)$ we find

$$\tilde{\mathbf{E}}_{K,0}(\pi \otimes 1)\eta \in O \otimes_{O^t} I_{O^t}(S/R).$$

Indeed, the factor ring $O \otimes_{O^t} W_{O^t}(S)/O \otimes_{O^t} I_{O^t}(S/R) = O \otimes_{O^t} R$ is annihilated by $\mathbf{E}_K(\pi \otimes 1)$. As before in the definition of the Lubin-Tate frame, we define maps $\sigma_{\mathrm{lt}} : O \otimes_{O^t} W_{O^t}(S) \longrightarrow O \otimes_{O^t} W_{O^t}(S)$ and $\dot{\sigma}_{\mathrm{lt}} : \mathcal{J}(S/R) \longrightarrow O \otimes_{O^t} W_{O^t}(S)$ by

$$^{\sigma_{\mathrm{lt}}}\xi = \ ^{F'}\xi, \quad ^{\dot{\sigma}_{\mathrm{lt}}}\eta = \ ^{\dot{F}'}\tilde{\mathbf{E}}_K(\pi \otimes 1)^{-1}\ ^{\dot{F}'}(\tilde{\mathbf{E}}_{K,0}(\pi \otimes 1)\eta),$$

with $\xi \in O \otimes_{O^t} W_{O^t}(S)$, $\eta \in \mathcal{J}(S/R)$. The justification of the following definition is analogous to the justification of Definition 3.3.8 of the Lubin-Tate frame.

Definition 3.3.19. The O-frame

$$\mathcal{F}_{\mathrm{lt}}(S/R) := (O \otimes_{O^t} W_{O^t}(S), \mathcal{J}(S/R), \sigma_{\mathrm{lt}}, \dot{\sigma}_{\mathrm{lt}})$$

is called the *relative Lubin-Tate frame* corresponding to the epimorphism of O-algebras $S \longrightarrow R$ and the divided powers γ_t relative to O^t on the kernel $\mathfrak{a}$.

Lemma 3.3.20. *The Drinfeld homomorphism*

$$\mu : O \otimes_{O^t} W_{O^t}(S) \longrightarrow W_O(S)$$

defines a morphism of O-frames

$$\mathcal{F}_{\mathrm{lt}}(S/R) \longrightarrow \mathcal{W}_O^{\varepsilon}(S/R). \tag{3.3.42}$$

The last frame is defined by (3.3.26).

Proof. One can argue exactly as in the proof of Proposition 3.3.12, but we need that

$$\mu : W_{O^t}(\mathfrak{a}) \longrightarrow W_O(\mathfrak{a})$$

maps logarithmic Teichmüller representatives $\tilde{a} \in W_{O^t}(\mathfrak{a})$ with respect to the O^t-divided powers γ_t to logarithmic Teichmüller representatives $\tilde{a} \in W_O(\mathfrak{a})$ with respect to the O-divided powers γ_{a}. This is a consequence of the following relation of divided Witt polynomials,

$$\dot{\mathbf{w}}_{O,n}(\mu(\alpha)) = \left(\frac{\varpi}{\pi}\right)^n \dot{\mathbf{w}}_{O^t,n}(\alpha), \quad \alpha \in W_{O^t}(\mathfrak{a}).$$

Again we may restrict to the p-torsionfree case, where this formula follows immediately from the definition of μ, cf. (3.3.1). □

Let $\mathcal{P}_{\mathrm{t}}$ be a $\mathcal{W}_{O^t}(S/R)$-display with a strict O-action. Then the R-module $P_{\mathrm{t}}/Q_{\mathrm{t}}$ is annihilated by $\pi \otimes 1 - 1 \otimes [\pi]$. As in Proposition 3.3.11, we define $F_{\mathrm{lt}} : P \longrightarrow P$ by

$$F_{\mathrm{lt}}(x) = \dot{F}_{\mathrm{t}}((\pi \otimes 1 - 1 \otimes [\pi])x), \quad x \in P.$$

We set $P_{\mathrm{lt}} = P_{\mathrm{t}}$, $Q_{\mathrm{lt}} = Q_{\mathrm{t}}$, $\dot{F}_{\mathrm{lt}} = \dot{F}_{\mathrm{t}}$. Then we obtain a $\mathcal{F}_{\mathrm{lt}}(S/R)$-display $\mathcal{P}_{\mathrm{lt}} = (P_{\mathrm{lt}}, Q_{\mathrm{lt}}, F_{\mathrm{lt}}, \dot{F}_{\mathrm{lt}})$. If we apply the base change by (3.3.42), we obtain a $\mathcal{W}_O^{\varepsilon}(S/R)$-display $\mathcal{P}_{\mathrm{a}}^{\varepsilon} = (P_{\mathrm{a}}^{\varepsilon}, Q_{\mathrm{a}}^{\varepsilon}, F_{\mathrm{a}}^{\varepsilon}, \dot{F}_{\mathrm{a}}^{\varepsilon})$. The assignement $\mathcal{P}_{\mathrm{t}} \mapsto \mathcal{P}_{\mathrm{a}} = (P_{\mathrm{a}}^{\varepsilon}, Q_{\mathrm{a}}^{\varepsilon}, \varepsilon^{-1}F_{\mathrm{a}}^{\varepsilon}, \dot{F}_{\mathrm{a}}^{\varepsilon})$ is the desired relative Ahsendorf functor $\mathfrak{A}_{O/O^t,S/R}$.

Proposition 3.3.21. *Let $\mathcal{P}$ be a $\mathcal{W}_{\mathfrak{o}}(S/R)$-display with a strict O-action. Let $\mathcal{P}_{\mathrm{a}}$ be the $\mathcal{W}_O(S/R)$-display associated to it by the relative Ahsendorf functor $\mathfrak{A}_{O/\mathfrak{o},S/R}$. Then there is a canonical isomorphism*

$$P_{\mathrm{a}}/I_O(S)P_{\mathrm{a}} \cong S \otimes_{O\otimes_{\mathfrak{o}}S} (P/I_{\mathfrak{o}}(S)P). \tag{3.3.43}$$

Proof. This is an immediate consequence of (3.3.31) □

With the notation $P = P_{\mathrm{ua}}$ of (3.3.37), we may write

$$P_{\mathrm{a}}/I_O(S)P_{\mathrm{a}} = P_{\mathrm{ua}}/I_{\mathfrak{o}}(S)P_{\mathrm{ua}} + (\pi \otimes 1 - 1 \otimes [\pi])P_{\mathrm{ua}}. \tag{3.3.44}$$

To see this, one uses that $\pi \otimes 1 - 1 \otimes \pi$ generates the kernel of the canonical map $O \otimes_{O^t} O \longrightarrow O$ as an ideal. We see that $P_{\mathrm{a}}/I_O(S)P_{\mathrm{a}}$ is the biggest quotient of $P_{\mathrm{ua}}/I_{\mathfrak{o}}(S)P_{\mathrm{ua}}$ such that the action via ι and via the structure homomorphism $O \longrightarrow S$ agree.

Let R be an O-algebra R such that π is nilpotent in R. Let $\mathcal{P}$ be a $\mathcal{W}_{\mathfrak{o}}(R)$-display with strict O-action. We assume that $\mathcal{P}$ is nilpotent. Then $\mathcal{P}_{\mathrm{a}}$ is also nilpotent. Then there is a crystal $\mathbb{D}_{\mathcal{P}}$ on the category of $\mathfrak{o}$-*pd*-thickenings and a crystal $\mathbb{D}_{\mathcal{P}_{\mathrm{a}}}$ on the category of O-*pd*-thickenings associated to these displays.

Corollary 3.3.22. *Let $\mathcal{P}$ be a nilpotent $\mathcal{W}_{\mathfrak{o}}(R)$-display with a strict O-action. Then the image $\mathcal{P}_{\mathrm{a}}$ by the Ahsendorf functor is a nilpotent $\mathcal{W}_O(R)$-display. Let $S \longrightarrow R$ be a surjective map of O-algebras which are p-adic. Assume that the kernel $\mathfrak{a}$ of this epimorphism is endowed with divided powers γ relative to $\mathfrak{o}$. Let γ_{a} be the corresponding O-divided powers on $\mathfrak{a}$. There is a canonical isomorphism*

$$\mathbb{D}_{\mathcal{P}_{\mathrm{a}}}(S, \gamma_{\mathrm{a}}) = S \otimes_{(O\otimes_{\mathfrak{o}}S)} \mathbb{D}_{\mathcal{P}}(S, \gamma).$$

Proof. Indeed, $\mathbb{D}_{\mathcal{P}}(S)$ is computed from a $\mathcal{W}_{\mathfrak{o}}(S/R)$-display $\tilde{\mathcal{P}}$ which lifts $\mathcal{P}$ and which is unique up to isomorphism. But then the relative Ahsendorf functor applied to $\tilde{\mathcal{P}}$ gives a $\mathcal{W}_O(S/R)$-display $\tilde{\mathcal{P}}_{\mathrm{a}}$ which lifts $\mathcal{P}_{\mathrm{a}}$. We conclude by Proposition 3.3.21. □

Corollary 3.3.23. *With the notation of Corollary 3.3.22, the Ahsendorf functor $\mathfrak{A}_{O/\mathfrak{o},S}$ defines a bijection between the liftings of $\mathcal{P}$ to a $\mathcal{W}_{\mathfrak{o}}(S)$-display with a strict O-action and the liftings of $\mathcal{P}_{\mathrm{a}}$ to a $\mathcal{W}_O(S)$-display.*

Proof. We show that each lifting of $\mathcal{P}_{\mathrm{a}}$ is in the essential image of $\mathfrak{A}_{O/\mathfrak{o},S}$. A lifting $\tilde{\mathcal{P}}_{\mathrm{a}}$ of $\mathcal{P}_{\mathrm{a}}$ corresponds, by Grothendieck-Messing for nilpotent displays, to a direct summand $U_{\mathrm{a}} \subset \mathbb{D}_{\mathcal{P}_{\mathrm{a}}}(S)$. Let $U \subset \mathbb{D}_{\mathcal{P}}$ be the preimage of U_{a} by the natural epimorphism $\mathbb{D}_{\mathcal{P}} \longrightarrow \mathbb{D}_{\mathcal{P}_{\mathrm{a}}}$. Then U defines a lifting of $\mathcal{P}$ which is mapped by the Ahsendorf functor to $\tilde{\mathcal{P}}_{\mathrm{a}}$. □

It is straightforward to deduce from the last Corollary Ahsendorf's Theorem 3.3.2 for an artinian local ring with perfect residue class field, i.e., we reproved a special case of [1].

3.4. **The Lubin-Tate display.** Let K be a finite extension of $\mathbb{Q}_p$. Let $K^t \subset K$ be the maximal subextension which is unramified over $\mathbb{Q}_p$. We denote by $O^t \subset O$ the rings of integers and by κ the common residue class field. We fix a prime element $\pi \in O$. Let L be a normal extension of $\mathbb{Q}_p$ which contains K. We set $L^t = K^t$. Let $\Phi = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K, L)$ and $\Psi = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K^t, L)$. We denote by $\varphi_0 \in \Phi$ and $\psi_0 \in \Psi$ the identical embeddings. We denote by Φ_ψ the preimage of ψ by the restriction map $\Phi \longrightarrow \Psi$. We define

$$\mathbf{E}_\psi(T) = \prod_{\varphi \in \Phi_\psi} (T - \varphi(\pi)) \in O_L[T].$$

Clearly this polynomial has coefficients in $O_{L^t} \subset O_L$. Let $\mathbf{E} \in O^t[T]$ be the Eisenstein polynomial of π in the extension K/K^t. Then $\mathbf{E}_\psi$ is the image of $\mathbf{E}$ by ψ in $O_{L^t}[T]$. We consider the surjective O_L-algebra homomorphismus

$$O_L[T] \longrightarrow O \otimes_{O^t,\psi} O_L,$$

which maps T to $\pi \otimes 1$. Then $\mathbf{E}_\psi(\pi \otimes 1) = 0$.

We lift the polynomials $\mathbf{E}_\psi$ to the Witt ring

$$\tilde{\mathbf{E}}_\psi(T) = \prod_{\varphi \in \Phi_\psi} (T - [\varphi(\pi)]) \in W(O_{L^t})[T].$$

We consider the decomposition

$$O \otimes_{\mathbb{Z}_p} O_{L^t} = \prod_{\psi \in \Psi} O \otimes_{O^t,\psi} O_{L^t}.$$

Let $\sigma \in \mathrm{Gal}(K^t/\mathbb{Q}_p)$ be the Frobenius automorphism. We have the morphism $\lambda : O^t \longrightarrow W(O^t)$ from (3.3.8). We define $\tilde{\psi}$ as the composite

$$\tilde{\psi} : O^t \xrightarrow{\lambda} W(O^t) \xrightarrow{W(\psi)} W(O_{L^t}).$$

Then we obtain the decomposition

$$O \otimes_{\mathbb{Z}_p} W(O_{L^t}) = \prod_{\psi \in \Psi} O \otimes_{O^t,\tilde{\psi}} W(O_{L^t}). \tag{3.4.1}$$

Let $\tilde{\mathbf{E}}_\psi(\pi \otimes 1)$ be the image of $\tilde{\mathbf{E}}_\psi$ by the homomorphism $W(O_{L^t})[T] \longrightarrow O \otimes_{O^t,\tilde{\psi}} W(O_{L^t})$ which maps T to $\pi \otimes 1$. Since $\mathbf{E}_\psi(\pi \otimes 1) = 0$, we conclude that

$$\tilde{\mathbf{E}}_\psi(\pi \otimes 1) \in O \otimes_{O^t,\tilde{\psi}} I(O_{L^t}). \tag{3.4.2}$$

For an arbitrary O_{L^t}-algebra R, the decomposition (3.4.1) induces

$$O \otimes_{\mathbb{Z}_p} W(R) = \prod_{\psi \in \Psi} O \otimes_{O^t,\tilde{\psi}} W(R). \tag{3.4.3}$$

The Frobenius and the Verschiebung act on the left hand side via the second factor, and this induces on the right hand side the maps

$$\begin{array}{rcl} F\colon O \otimes_{O^t,\tilde{\psi}} W(R) & \longrightarrow & O \otimes_{O^t,\widetilde{\psi\sigma}} W(R) \\ a \otimes \xi & \longmapsto & a \otimes {}^F\xi \\ V\colon O \otimes_{O^t,\widetilde{\psi\sigma}} W(R) & \longrightarrow & O \otimes_{O^t,\tilde{\psi}} W(R) \\ a \otimes \xi & \longmapsto & a \otimes {}^V\xi \end{array} \tag{3.4.4}$$

We note that $\widetilde{\psi\sigma} = \tilde{\psi} \circ \sigma$. We will write $\dot{F} = V^{-1} : O \otimes_{O^t,\tilde{\psi}} I(R) \longrightarrow O \otimes_{O^t,\widetilde{\psi\sigma}} W(R)$.

Proposition 3.4.1. *The element* ${}^{\dot{F}}\tilde{\mathbf{E}}_\psi(\pi \otimes 1) \in O \otimes_{O^t,\widetilde{\psi\sigma}} W(O_{L^t})$ *is a unit of the form*

$${}^{\dot{F}}\tilde{\mathbf{E}}_\psi(\pi \otimes 1) = (\frac{\pi^e}{p} \otimes 1)\delta, \quad \delta \in O \otimes_{O^t,\widetilde{\psi\sigma}} W(O_{L^t}), \tag{3.4.5}$$

where $\delta - 1 \otimes 1$ *is in the kernel of* $O \otimes_{O^t,\widetilde{\psi\sigma}} W(O_{L^t}) \longrightarrow O \otimes_{O^t,\widetilde{\psi\sigma}} W(\kappa_{L^t})$. *In particular,* $\delta - 1 \otimes 1$ *lies in the radical of* $O \otimes_{O^t,\widetilde{\psi\sigma}} W(O_{L^t})$.

Proof. The proof is the same as that of Proposition 3.3.6. □

The polynomial $\mathbf{E}_{\psi_0}$ has the decomposition

$$\mathbf{E}_{\psi_0}(T) = (T-\pi)\cdot \mathbf{E}_{\psi_0,0}(T).$$

These polynomials have coefficients in $K \subset L$. Recall that $\varphi_0(\pi) = \pi$. We set

$$\tilde{\mathbf{E}}_{\psi_0,0}(T) = \prod_{\varphi\in\Phi_{\psi_0},\varphi\neq\varphi_0} (T - [\varphi(\pi)]) \in W(O_L)[T].$$

This polynomial lies in $W(O)[T] \subset W(O_L)[T]$. We set

$$P_{\mathcal{L}} = O \otimes_{\mathbb{Z}_p} W(O) = \oplus_{\psi\in\Psi} O \otimes_{O^t,\tilde{\psi}} W(O). \tag{3.4.6}$$

We denote by $Q_{\psi_0} \subset O \otimes_{O^t,\tilde{\psi_0}} W(O)$ the kernel of the map

$$O \otimes_{O^t,\tilde{\psi_0}} W(O) \xrightarrow{\mathrm{id}\otimes\mathbf{w}_0} O \otimes_{O^t,\psi_0} O \xrightarrow{\text{mult.}} O. \tag{3.4.7}$$

Lemma 3.4.2.

$$Q_{\psi_0} = \{x \in O \otimes_{O^t,\tilde{\psi_0}} W(O) \mid \tilde{\mathbf{E}}_{\psi_0,0}(\pi\otimes 1)x \in O \otimes_{O^t,\tilde{\psi_0}} I(O)\}$$

Proof. Let $\bar{Q}_{\psi_0}$ be the kernel of the second map of (3.4.7). Then we can reformulate our assertion as

$$\bar{Q}_{\psi_0} = \{x \in O \otimes_{O^t,\psi_0} O \mid \mathbf{E}_{\psi_0,0}(\pi\otimes 1)x = 0\}.$$

We write

$$O \otimes_{O^t,\psi_0} O \simeq \big(O^t[T]/\mathbf{E}_{\psi_0}(T)O^t[T]\big) \otimes_{O^t,\psi_0} O \simeq O[T]/\mathbf{E}_{\psi_0}(T)O[T].$$

We see that $\bar{Q}_{\psi_0} = (T-\pi)O[T]/\mathbf{E}_{\psi_0}(T)O[T]$ which coincides with $\mathbf{E}_{\psi_0,0}(T)O[T]/\mathbf{E}_{\psi_0}(T)O[T]$. □

We let F and V act via the second factor on $O \otimes_{\mathbb{Z}_p} W(O)$ and therefore on the right hand side of (3.4.6) by the formulas (3.4.4).

Definition 3.4.3. The *Lubin-Tate display* is the $\mathcal{W}(O)$-display $\mathcal{L} = (P_{\mathcal{L}}, Q_{\mathcal{L}}, F_{\mathcal{L}}, \dot{F}_{\mathcal{L}})$, defined as follows. Let $P_{\mathcal{L}} = O \otimes_{\mathbb{Z}_p} W(O)$. Then $P_{\mathcal{L}} = \oplus_\psi P_{\psi,\mathcal{L}}$ with $P_{\psi,\mathcal{L}} = O \otimes_{O^t,\tilde{\psi}} W(O)$. Set $Q_\psi = P_{\psi,\mathcal{L}}$ for $\psi \neq \psi_0$ and $Q_{\psi_0} \subset P_{\psi_0,\mathcal{L}}$ as above, and define

$$Q_{\mathcal{L}} = \oplus_{\psi\in\Psi} Q_\psi \subset P_{\mathcal{L}}.$$

The maps $F_{\mathcal{L}}$ and $\dot{F}_{\mathcal{L}}$ are defined as the direct sum of the following maps for all ψ. For $\psi \neq \psi_0$ we define

$$\begin{array}{llll} F_{\mathcal{L}}: & O \otimes_{O^t,\tilde{\psi}} W(O) & \longrightarrow & O \otimes_{O^t,\widetilde{\psi\sigma}} W(O), \\ & x & \longmapsto & {}^F(\tilde{\mathbf{E}}_\psi(\pi\otimes 1)x), \end{array} \quad \begin{array}{llll} \dot{F}_{\mathcal{L}}: & O \otimes_{O^t,\tilde{\psi}} W(O) & \longrightarrow & O \otimes_{O^t,\widetilde{\psi\sigma}} W(O). \\ & y & \longmapsto & {}^{\dot{F}}(\tilde{\mathbf{E}}_\psi(\pi\otimes 1)y) \end{array}$$

For ψ_0 we define

$$\begin{array}{llll} F_{\mathcal{L}}: & O \otimes_{O^t,\tilde{\psi_0}} W(O) & \longrightarrow & O \otimes_{O^t,\widetilde{\psi_0\sigma}} W(O), \\ & x & \longmapsto & {}^F(\tilde{\mathbf{E}}_0(\pi\otimes 1)x), \end{array} \quad \begin{array}{llll} \dot{F}_{\mathcal{L}}: & Q_{\psi_0} & \longrightarrow & O \otimes_{O^t,\widetilde{\psi_0\sigma}} W(O). \\ & y & \longmapsto & {}^{\dot{F}}(\tilde{\mathbf{E}}_0(\pi\otimes 1)y) \end{array}$$

The action of O by multiplication via the first factor on $O \otimes_{\mathbb{Z}_p} W(O)$ defines a strict O-action on $\mathcal{L}$. If R is a p-adic O-algebra we denote by $\mathcal{L}_R$ the base change of $\mathcal{L}$ via the morphism of frames $\mathcal{W}(O) \longrightarrow \mathcal{W}(R)$.

The tuple $(P_{\mathcal{L}}, Q_{\mathcal{L}}, F_{\mathcal{L}}, \dot{F}_{\mathcal{L}})$ is indeed a $\mathcal{W}(O)$-display. The only non-trivial point is that $\dot{F}_{\mathcal{L}}$ is surjective. But this follows from Proposition 3.4.1.

We will now apply the Ahsendorf functor $\mathfrak{A}_{O/\mathbb{Z}_p,O}$ to the Lubin-Tate display $\mathcal{L}$. We use the previous section in the case $\mathfrak{o} = \mathbb{Z}_p$ and $\varpi = p$. We set $\psi_i = \psi_0\sigma^i : O^t \longrightarrow O$, where ψ_0 is the identity. We have the decomposition

$$P_{\mathcal{L}} = \oplus_{i=0}^{f-1} P_{\mathcal{L},\psi_i}, \quad P_{\mathcal{L},\psi_i} = O \otimes_{O^t,\tilde{\psi_i}} W(O).$$

We denote the Frobenius on $W(O)$ by F. In the last section we have associated to $\mathcal{L}$ an $\mathcal{A}(O)$-display $\mathcal{P}_{\mathrm{ua}}$ for the O^t-frame

$$\mathcal{A}_{\mathbb{Z}_p}(O) = \mathcal{A}(O) = (W(O), I(O), F^f, F^{f-1}\dot{F}).$$

By definition $P_{\mathrm{ua}} = P_{\mathcal{L},\psi_0}$, $Q_{\mathrm{ua}} = Q_{\mathcal{L},\psi_0}$, and $\dot{F}_{\mathrm{ua}} = \dot{F}^f_{\mathcal{L}}$. More explicitely, for $y \in Q_{\mathrm{ua}}$,

$$\dot{F}_{\mathrm{ua}}(y) = \ {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_{f-1}}(\pi \otimes 1)\ {}^{F\dot{F}}\tilde{\mathbf{E}}_{\psi_{f-2}}(\pi \otimes 1) \cdot \ldots \cdot\ {}^{F^{f-2}\dot{F}}\tilde{\mathbf{E}}_{\psi_1}(\pi \otimes 1)\ {}^{F^{f-1}\dot{F}}(\tilde{\mathbf{E}}_0(\pi \otimes 1)y).$$

We set

$$\mathbf{n} = \ {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_{f-1}}(\pi \otimes 1)\ {}^{F\dot{F}}\tilde{\mathbf{E}}_{\psi_{f-2}}(\pi \otimes 1) \cdot \ldots \cdot\ {}^{F^{f-2}\dot{F}}\tilde{\mathbf{E}}_{\psi_1}(\pi \otimes 1)\ {}^{F^{f-1}\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1).$$

Then we may write

$$\dot{F}_{\mathrm{ua}}(y) = \mathbf{n}\big(\ {}^{F^{f-1}\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\big)^{-1}\ {}^{F^{f-1}\dot{F}}(\tilde{\mathbf{E}}_0(\pi \otimes 1)y). \tag{3.4.8}$$

To $\mathcal{P}_{\mathrm{ua}}$ we apply base change with respect to the Drinfeld morphism $\mu : \mathcal{A}(O) \longrightarrow \mathcal{W}_{O^t}(O)$. We obtain the $\mathcal{W}_{O^t}(O)$-display $\mathcal{P}_{\mathrm{t}}$, where

$$P_{\mathrm{t}} = O \otimes_{O^t} W_{O^t}(O)$$

and where Q_{t} is the kernel of the homomorphism $O \otimes_{O^t} W_{O^t}(O) \longrightarrow O$ induced by $\mathbf{w}_{O^t,0}$, cf. (3.4.7). The polynomials $\tilde{\mathbf{E}}_{\psi_0}$ and $\tilde{\mathbf{E}}_{\psi_0,0}$ are mapped by μ to the polynomials $\tilde{\mathbf{E}}_K$ and $\tilde{\mathbf{E}}_{K,0}$ of (3.3.14). We denote by $\mathbf{n}_{\mathrm{lt}}$ the image of $\mathbf{n}$ by μ. Therefore we obtain

$$\dot{F}_{\mathrm{t}}(y) = \mathbf{n}_{\mathrm{lt}}\big(\ {}^{\dot{F}'}\tilde{\mathbf{E}}_K(\pi \otimes 1)\big)^{-1}\ {}^{\dot{F}'}(\tilde{\mathbf{E}}_{K,0}(\pi \otimes 1)y), \quad y \in Q_{\mathrm{t}}.$$

By Proposition 3.3.11, we associate to $\mathcal{P}_{\mathrm{t}}$ a $\mathcal{F}_{\mathrm{lt}}(O)$-display $\mathcal{P}_{\mathrm{lt}}$. In terms of the Lubin-Tate frame (comp. Definition 3.3.8), we may rewrite the last equation as

$$\dot{F}_{\mathrm{lt}}(y) = \mathbf{n}_{\mathrm{lt}}\dot{\sigma_{\mathrm{lt}}}(y), \quad y \in Q_{\mathrm{t}} = Q_{\mathrm{lt}} = \mathcal{J}.$$

Proposition 3.4.4. *The Ahsendorf functor $\mathfrak{A}_{O/\mathbb{Z}_p,O}$ maps the Lubin-Tate display $\mathcal{L}$ to a $\mathcal{W}_O(O)$-display which is canonically isomorphic to $\mathcal{P}_{m,\mathcal{W}_O(O)}(\pi^{ef}/p^f)$, i.e., the twist of the multiplicative display by $\pi^{ef}/p^f \in O \subset W_O(O)$, cf. Example 3.1.6.*

Proof. Let $\mathcal{P}_{m,\mathfrak{lt}}$ be the multiplicative $\mathcal{F}_{\mathrm{lt}}(O)$-display. The above identities show that the display $\mathcal{L}_{\mathfrak{t}}$ is equal to the $\mathcal{F}_{\mathrm{lt}}(O)$-display $\mathcal{P}_{m,\mathfrak{lt}}(\mathbf{n}_{\mathrm{lt}})$. Applying to this display the base change by the Drinfeld morphism of frames, cf. Proposition 3.3.12, we obtain essentially (i.e., neglecting ε) the image of $\mathcal{L}$ by the Ahsendorf functor. By Proposition 3.4.1 the image of the element $\mathbf{n}$ by the map $O \otimes_{O^t,\tilde{\psi}_0} W(O) \longrightarrow O \otimes_{O^t,\psi_0} W(\kappa) = O$ is π^{ef}/p^f. This implies that the image of $\mathbf{n}_{\mathrm{lt}} \in O \otimes_{O^t} W_{O^t}(O)$ in $O \otimes_{O^t} W_{O^t}(\kappa)$ is π^{ef}/p^f. By the following Lemma there is a uniquely determined unit $\xi \in O \otimes_{O^t} W_{O^t}(O)$ such that $\xi\mathbf{n}_{\mathrm{lt}} = (\pi^{ef}/p^f \otimes 1)\ {}^{F^f}\xi$. This gives a canonical isomorphism $\mathcal{P}_{m,\mathfrak{lt}}(\mathbf{n}_{\mathrm{lt}}) \xrightarrow{\sim} \mathcal{P}_{m,\mathfrak{lt}}(\pi^{ef}/p^f \otimes 1)$. The base change with respect to a morphism of frames maps the multiplicative display to the multiplicative display, cf. Example 3.1.7. Therefore the last display is mapped by the base change of Proposition 3.3.12 to the $\mathcal{W}_O^{\varepsilon}(O)$-display

$$\big(W_O(O), I_O(O), F, \varepsilon^{-1}\dot{F}\big)(\pi^{ef}/p^f \otimes 1\big).$$

Here we denote by F the Frobenius acting on $W_O(O)$ and by $\dot{F}$ the inverse of the Verschiebung. The element ε is defined by (3.3.17).

Therefore the definition of the Ahsendorf functor gives $\mathcal{P}_{m,\mathcal{W}_O(O)}(\varepsilon^{-1}(\pi^{ef}/p^f \otimes 1))$ as the image of $\mathcal{L}$. The image of ε by the homomorphism $W_O(O) \longrightarrow W_O(\kappa)$ is 1. A variant of the next Lemma shows that there is a unique $\xi \in W_O(O)$ such that ${}^{F}\xi\xi = \varepsilon$. This shows that the last display is canonically isomorphic to $\mathcal{P}_{m,\mathcal{W}_O(O)}(\pi^{ef}/p^f \otimes 1)$. □

Lemma 3.4.5. *Let $\alpha \in O \otimes_{O^t} W_{O^t}(O)$ be a unit whose image in $O \otimes_{O^t} W_{O^t}(\kappa)$ is 1. Then there exists a unique unit $\xi \in O \otimes_{O^t} W_{O^t}(O)$ whose image in $O \otimes_{O^t} W_{O^t}(\kappa)$ is 1 and such that*

$${}^{F^f}\xi \cdot \xi^{-1} = \alpha.$$

Proof. One proves this by induction on n for $O \otimes_{O^t} W_{O^t}(O/\pi^n O)$. Alternatively one can use Grothendieck-Messing for frames due to Lau and show that the multiplicative display of $\mathcal{F}_{\mathrm{lt}}(\kappa)$ has no nontrivial deformation with respect to $\mathcal{F}_{\mathrm{lt}}(O) \longrightarrow \mathcal{F}_{\mathrm{lt}}(\kappa)$. □

Remark 3.4.6. Let k a perfect field which is an extension of κ. We regard it as an O-algebra via the residue class map $O \longrightarrow \kappa$. Then we can describe the Dieudonné module $(P_{\mathcal{L}_k}, F_{\mathcal{L}_k}, V_{\mathcal{L}_k})$

of $\mathcal{L}_k$ as follows. Let $\tilde{\psi}_0 : O^t = W(\kappa) \longrightarrow W(k)$ be the canonical map. The set Ψ consists of the maps $\tilde{\psi}_0\sigma^i$ where σ is the Frobenius of the extension $O^t/\mathbb{Z}_p$. We have

$$P_{\mathcal{L}_k} = O \otimes_{\mathbb{Z}_p} W(k) = \prod_{\tilde{\psi}\in\Psi} O \otimes_{O^t,\tilde{\psi}} W(k).$$

The Frobenius and the Verschiebung

$$O \otimes_{O^t,\tilde{\psi}} W(k) \underset{V_{\mathcal{L}_k}}{\overset{F_{\mathcal{L}_k}}{\rightleftarrows}} O \otimes_{O^t,\tilde{\psi}\sigma} W(k) \tag{3.4.9}$$

are defined as follows:

$$\begin{aligned} F_{\mathcal{L}_k}(x_\psi) &= \pi^e\ {}^F x_\psi, & V_{\mathcal{L}_k}(x_{\psi\sigma}) &= (p/\pi^e)\ {}^{F^{-1}} x_{\psi\sigma}, \qquad \text{for } \psi \neq \psi_0,\\ F_{\mathcal{L}_k}(x_{\psi_0}) &= \pi^{e-1}\ {}^F x_{\psi_0}, & V_{\mathcal{L}_k}(x_{\psi_0\sigma}) &= (p/\pi^{e-1})\ {}^{F^{-1}} x_{\psi_0\sigma}. \end{aligned}$$

The upper left index F denotes the action of the Frobenius via the second factor on $O \otimes_{\mathbb{Z}_p} W(k)$. This description follows easily because $\tilde{\mathbf{E}}_\psi(T) = T^e$, and therefore $\tilde{\mathbf{E}}_\psi(\pi \otimes 1) = \pi^e \otimes 1 \in O \otimes_{O^t,\tilde{\psi}} W(k)$ and $\tilde{\mathbf{E}}_{\psi_0,0}(\pi \otimes 1) = \pi^{e-1} \otimes 1$. Moreover, we have $Q_{\psi_0} = \pi O \otimes_{O^t,\tilde{\psi}} W(k)$.

We have identified the frame $\mathcal{W}_O(k)$ with

$$\big(O \otimes_{O^t,\tilde{\psi}_0} W(k), \pi O \otimes_{O^t,\tilde{\psi}_0} W(k), k, F^f, F^f\pi^{-1}\big).$$

By Remark 3.3.16, the Ahsendorf functor associates to the Dieudonné module $(P_{\mathcal{L}_k}, F_{\mathcal{L}_k}, V_{\mathcal{L}_k})$ of $\mathcal{L}_k$ the $\mathcal{W}_O(k)$-Dieudonné module

$$\big(O \otimes_{O^t,\tilde{\psi}_0} W(k), \frac{\pi^{ef}}{p^f} F^f, \frac{p^f}{\pi^{ef-1}} F^{-f}\big). \tag{3.4.10}$$

This is equal to the Dieudonné module of the twisted multiplicative display $\mathcal{P}_{m,\mathcal{W}_O(k)}(\pi^{ef}/p^f)$, in agreement with Proposition 3.4.4.

In the end of this subsection we discuss the Faltings dual of a display $\mathcal{P} = (P, Q, F, \dot{F})$ with a strict O-action over an O-algebra R. We begin with a recipe how to construct such $\mathcal{P}$. We consider the decomposition induced by (3.4.3),

$$P = \oplus_\psi P_\psi, \quad Q = Q_{\psi_0} \oplus (\oplus_{\psi\neq\psi_0} P_\psi).$$

Let

$$J_{\psi_0} = \operatorname{Ker}\big(O \otimes_{O^t,\tilde{\psi}_0} W(R) \longrightarrow R\big), \tag{3.4.11}$$

where the map is induced by the structure homomorphism $O \longrightarrow R$ and the homomorphism $\mathbf{w}_0$. Then

$$J_{\psi_0} = O \otimes_{O^t,\tilde{\psi}_0} I(R) + (\pi \otimes 1 - 1 \otimes [\pi])(O \otimes_{O^t,\tilde{\psi}_0} W(R)),$$

cf. (3.3.20). To find a normal decomposition $P = T \oplus L$, we start with

$$P_{\psi_0} = T_{\psi_0} \oplus L_{\psi_0}, \quad Q_{\psi_0} = J_{\psi_0} T_{\psi_0} \oplus L_{\psi_0}.$$

We define $\phi_{T_{\psi_0}} : T_{\psi_0} \longrightarrow P_{\psi_0\sigma}$,

$$\phi_{T_{\psi_0}}(t) = \dot{F}\big((\pi \otimes 1 - 1 \otimes [\pi])t\big).$$

Then we set $T = T_{\psi_0} \subset P$ and $L = L_{\psi_0} \oplus (\oplus_{\psi\neq\psi_0} P_\psi)$. Let $\phi_L : L \longrightarrow P$ be the restriction of $\dot{F}$ to L and let $\phi_T = \phi_{T_{\psi_0}}$. The restriction to L_{ψ_0} is denoted by $\phi_{L_{\psi_0}}$.

Lemma 3.4.7. *The map*

$$\phi_T \oplus \phi_L : T \oplus L \longrightarrow P$$

is an F-linear $O \otimes_{\mathbb{Z}_p} W(R)$-module isomorphism, where the Frobenius F acts on $O \otimes_{\mathbb{Z}_p} W(R)$ via the second factor.

Proof. For $\psi \neq \psi_0$ the map $\dot{F}_\psi : P_\psi \longrightarrow P_{\psi\sigma}$ is an F-linear isomorphism. Therefore it suffices to show that the map

$$\phi_T \oplus \phi_{L_{\psi_0}} : T_{\psi_0} \oplus L_{\psi_0} \longrightarrow P_{\psi_0\sigma} \tag{3.4.12}$$

is an F-linear isomorphism or, equivalently, an F-linear epimorphism. Since $P_{\psi_0\sigma}$ is generated by $\dot{F}(Q_{\psi_0})$ it suffices to show that the following elements are in the image of the linearization of (3.4.12):

$$\dot{F}(\ell),\ \dot{F}((\pi \otimes 1 - 1 \otimes [\pi])t),\ \dot{F}(\,^V\eta t), \quad \ell \in L_{\psi_0}, t \in T, \eta \in O \otimes_{\mathbb{Z}_p} W(R).$$

For the first two elements this is obvious. For the last element this follows from the formula

$$Fx = \big(\,^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\big)^{-1} \cdot\ ^F\tilde{\mathbf{E}}_0(\pi \otimes 1) \cdot \dot{F}\big((\pi \otimes 1 - 1 \otimes [\pi])x\big), \quad x \in P_{\psi_0}, \tag{3.4.13}$$

which is proved in the same way as Lemma 3.3.10. □

We omit the proof of the following proposition.

Proposition 3.4.8. *Let R be a p-adic O-algebra. Let T_{ψ_0}, L_{ψ_0} be $O \otimes_{O^t, \tilde{\psi}_0} W(R)$-modules and let P_ψ for $\psi \neq \psi_0$ be $O \otimes_{O^t, \tilde{\psi}} W(R)$-modules which are free locally on* Spec R*. Set $P_{\psi_0} = T_{\psi_0} \oplus L_{\psi_0}$. Assume given F-linear isomorphisms*

$$\phi_{T_{\psi_0}} \oplus \phi_{L_{\psi_0}} : T_{\psi_0} \oplus L_{\psi_0} \longrightarrow P_{\psi_0\sigma}, \quad \phi_\psi : P_\psi \longrightarrow P_{\psi\sigma}, \text{ for } \psi \neq \psi_0,$$

cf. (3.4.4) for the meaning of F-linear.

Then there exists a unique display $\mathcal{P} = (P, Q, F, \dot{F})$ with a strict action of O over R such that $P = \oplus_\psi P_\psi$ and $Q = J_{\psi_0} T_{\psi_0} \oplus (\oplus_{\psi \neq \psi_0} P_\psi)$ with $T = T_{\psi_0}$ and $L = L_{\psi_0} \oplus (\oplus_{\psi \neq \psi_0} P_\psi)$ and such that $\phi_T = \phi_{T_{\psi_0}} : T \longrightarrow P$ and $\phi_L = \phi_{\psi_0} \oplus (\oplus_{\psi \neq \psi_0} \phi_\psi) : L \longrightarrow P$ are given by the display structure of $\mathcal{P}$ as in Lemma 3.4.7. □

Let $\mathcal{P}$ be a display with a strict action by O over R, as above. Then we set

$$P^\nabla = \mathrm{Hom}_{O \otimes_{\mathbb{Z}_p} W(R)}(P, O \otimes_{\mathbb{Z}_p} W(R)) = \oplus_\psi P^\nabla_\psi, \quad \text{where}$$
$$P^\nabla_\psi = \mathrm{Hom}_{O \otimes_{O^t, \tilde{\psi}} W(R)}(P_\psi, O \otimes_{O^t, \tilde{\psi}} W(R)).$$

We define

$$Q^\nabla_{\psi_0} = \{\hat{x} \in P^\nabla_{\psi_0} \mid \hat{x}(Q_{\psi_0}) \subset J_{\psi_0}\} \subset P^\nabla_{\psi_0}, \quad Q^\nabla = Q^\nabla_{\psi_0} \oplus (\oplus_{\psi \neq \psi_0} P^\nabla_\psi).$$

Let

$$\langle\ ,\ \rangle_{\mathrm{can}} : P \times P^\nabla \longrightarrow O \otimes_{\mathbb{Z}_p} W(R) \tag{3.4.14}$$

be the canonical perfect $O \otimes_{\mathbb{Z}_p} W(R)$-bilinear form. It induces pairings $P_\psi \times P^\nabla_\psi \longrightarrow O \otimes_{O^t, \tilde{\psi}} W(R)$. Under the perfect R-bilinear form

$$P_{\psi_0}/J_{\psi_0} P_{\psi_0} \times P^\nabla_{\psi_0}/J_{\psi_0} P^\nabla_{\psi_0} \longrightarrow (O \otimes_{O^t, \tilde{\psi}} W(R))/J_{\psi_0} \simeq R,$$

the R-submodules $Q_{\psi_0}/J_{\psi_0} P_{\psi_0}$ and $Q^\nabla_{\psi_0}/J_{\psi_0} P^\nabla_{\psi_0}$ are orthogonal complements.

Proposition 3.4.9. *Let $\mathcal{P}$ be a display with a strict O-action over R. Let P^∇ and Q^∇ as above. Then there are unique F-linear homomorphisms of $O \otimes_{\mathbb{Z}_p} W(R)$-modules*

$$F^\nabla : P^\nabla \longrightarrow P^\nabla, \quad \dot{F}^\nabla : Q^\nabla \longrightarrow P^\nabla$$

such that $\mathcal{P}^\nabla = (P^\nabla, Q^\nabla, F^\nabla, \dot{F}^\nabla)$ becomes a display and such that the bilinear form (3.4.14) defines a bilinear form of displays with a strict O-action,

$$\langle\ ,\ \rangle_{\mathrm{can}} : \mathcal{P} \times \mathcal{P}^\nabla \longrightarrow \mathcal{L}. \tag{3.4.15}$$

We call $\mathcal{P}^\nabla$ the Faltings dual *of $\mathcal{P}$.*

Proof. It follows from the definition that

$$\langle Q, Q^\nabla \rangle_{\mathrm{can}} \subset Q_{\mathcal{L}} \subset O \otimes_{\mathbb{Z}_p} W(R).$$

We will define a display $\mathcal{P}^\nabla = (P^\nabla, Q^\nabla, F^\nabla, \dot{F}^\nabla)$. Then we will show that

$$\langle \dot{F}y, \dot{F}^\nabla \hat{y} \rangle_{\mathrm{can}} = \dot{F}_{\mathcal{L}} \langle y, \hat{y} \rangle_{\mathrm{can}} \quad y \in Q,\ \hat{y} \in Q^\nabla. \tag{3.4.16}$$

This will show that (3.4.15) is a bilinear form of displays with an O-action. Since the pairing (3.4.14) is perfect, the map $\dot{F}^\nabla$ is uniquely determined by (3.4.16).

We begin by defining $\mathcal{P}^{\nabla}$. We chose a normal decomposition

$$P_{\psi_0} = T_{\psi_0} \oplus L_{\psi_0}, \quad Q_{\psi_0} = J_{\psi_0} T_{\psi_0} \oplus L_{\psi_0} \tag{3.4.17}$$

Let $T_{\psi_0}^{\nabla}$ be the orthogonal complement of T_{ψ_0} and let $L_{\psi_0}^{\nabla}$ be the orthogonal complement of L_{ψ_0} with respect to the perfect bilinear form

$$P_{\psi_0} \times P_{\psi_0}^{\nabla} \longrightarrow O \otimes_{O^t \tilde{\psi}_0} W(R).$$

We consider the maps $\phi_{T_{\psi_0}}$ and $\phi_{L_{\psi_0}}$. We define maps

$$\phi_{T_{\psi_0}^{\nabla}} : T_{\psi_0}^{\nabla} \longrightarrow P_{\psi_0 \sigma}^{\nabla}, \quad \phi_{L_{\psi_0}^{\nabla}} : L_{\psi_0}^{\nabla} \longrightarrow P_{\psi_0 \sigma}^{\nabla}$$

by the equations

$$\begin{aligned} \langle \phi_{T_{\psi_0}}(t) + \phi_{L_{\psi_0}}(\ell), \phi_{T_{\psi_0}^{\nabla}}(\hat{t}) \rangle_{\mathrm{can}} &= {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\ {}^{F}\langle \ell, \hat{t} \rangle_{\mathrm{can}}, \qquad t \in T_{\psi_0}, \ell \in L_{\psi_0} \\ \langle \phi_{T_{\psi_0}}(t) + \phi_{L_{\psi_0}}(\ell), \phi_{L_{\psi_0}^{\nabla}}(\hat{\ell}) \rangle_{\mathrm{can}} &= {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\ {}^{F}\langle t, \hat{\ell} \rangle_{\mathrm{can}}, \qquad \hat{t} \in T_{\psi_0}^{\nabla},\ \hat{\ell} \in L_{\psi_0}^{\nabla}. \end{aligned} \tag{3.4.18}$$

This definition makes sense because $\phi_{T_{\psi_0}} \oplus \phi_{L_{\psi_0}} : T_{\psi_0} \oplus L_{\psi_0} \longrightarrow P_{\psi_0 \sigma}$ is an F-linear isomorphism. For $\psi \neq \psi_0$ the map $\dot{F} : P_\psi \longrightarrow P_{\psi\sigma}$ is an F-linear isomorphism. Therefore we can define $\dot{F}^{\nabla} : P_{\psi}^{\nabla} \longrightarrow P_{\psi\sigma}^{\nabla}$ by the equation

$$\langle \dot{F} z, \dot{F}^{\nabla} \hat{z} \rangle_{\mathrm{can}} = {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi}(\pi \otimes 1)\ {}^{F}\langle z, \hat{z} \rangle_{\mathrm{can}}, \quad z \in P_\psi,\ \hat{z} \in P_{\psi}^{\nabla}.$$

We now apply Proposition 3.4.8 to the modules $T_{\psi_0}^{\nabla}, L_{\psi_0}^{\nabla}$, and P_{ψ}^{∇} for $\psi \neq \psi_0$, and to the maps $\phi_{T_{\psi_0}^{\nabla}}, \phi_{L_{\psi_0}^{\nabla}}$, and $\phi_\psi = \dot{F}_{\psi}^{\nabla}$ for $\psi \neq \psi_0$. This concludes the definition of the display $\mathcal{P}^{\nabla} = (P^{\nabla}, Q^{\nabla}, F^{\nabla}, \dot{F}^{\nabla})$.

Now we verify (3.4.16). Let $\psi \neq \psi_0$. If $y \in P_\psi$ and $\hat{y} \in P_{\psi}^{\nabla}$, the right hand side is by definition

$${}^{\dot{F}}\tilde{\mathbf{E}}_{\psi}(\pi \otimes 1) \cdot {}^{F}(y, \hat{y})_{\mathrm{can}}.$$

Therefore (3.4.16) holds in this case by the definition of $\dot{F}_{\psi}^{\nabla}$. For ψ_0 we use the normal decomposition (3.4.17) and the induced normal decomposition $Q_{\psi_0}^{\nabla} = J_{\psi_0} T_{\psi_0}^{\nabla} \oplus L_{\psi_0}^{\nabla}$. Using the definition (3.4.11) of J_{ψ_0}, the identity (3.4.16) becomes for the ψ_0-components a series of equations:

(1) $\langle \dot{F}\big((\pi \otimes 1 - 1 \otimes [\pi])t\big), \dot{F}^{\nabla}\hat{\ell} \rangle_{\mathrm{can}} = \dot{F}_{\mathcal{L}} \langle (\pi \otimes 1 - 1 \otimes [\pi])t, \hat{\ell} \rangle_{\mathrm{can}}$,

(2) $\langle \dot{F}(\ {}^{V}\eta t), \dot{F}^{\nabla}\hat{\ell} \rangle_{\mathrm{can}} = \dot{F}_{\mathcal{L}} \langle\ {}^{V}\eta t, \hat{\ell} \rangle_{\mathrm{can}}, \quad \eta \in O \otimes_{O^t, \tilde{\psi}_0 \sigma} W(R)$,

(3) $\langle \dot{F}\ell, \dot{F}^{\nabla}\big((\pi \otimes 1 - 1 \otimes [\pi])\hat{t}\big) \rangle_{\mathrm{can}} = \dot{F}_{\mathcal{L}} \langle \ell, (\pi \otimes 1 - 1 \otimes [\pi])\hat{t} \rangle_{\mathrm{can}}$,

(4) $\langle \dot{F}\ell, \dot{F}^{\nabla}(\ {}^{V}\eta \hat{t}) \rangle_{\mathrm{can}} = \dot{F}_{\mathcal{L}} \langle \ell,\ {}^{V}\eta \hat{t} \rangle_{\mathrm{can}}$,

(5) $\langle \dot{F}\ell, \dot{F}^{\nabla}\hat{\ell} \rangle_{\mathrm{can}} = 0$,

(6) $\langle \dot{F}(J_{\psi_0} T_{\psi_0}), \dot{F}^{\nabla}(J_{\psi_0} T_{\psi_0}^{\nabla}) \rangle_{\mathrm{can}} = 0$.

We compute the right hand side of equation (1):

$$\mathrm{RHS}(1) = {}^{\dot{F}}\tilde{\mathbf{E}}_0(\pi \otimes 1)(\pi \otimes 1 - 1 \otimes [\pi])\ {}^{F}\langle t, \hat{\ell} \rangle_{\mathrm{can}} = {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\ {}^{F}\langle t, \hat{\ell} \rangle_{\mathrm{can}}.$$

Therefore equation (1) is exactly the second equation of (3.4.18) for $\ell = 0$. The equation (3) follows in the same way.

We prove now the equation (2). For the right hand side we find:

$$\mathrm{RHS}(2) = \eta F_{\mathcal{L}} \langle t, \hat{\ell} \rangle_{\mathrm{can}} = \eta\ {}^{F}\tilde{\mathbf{E}}_0(\pi \otimes 1)\ {}^{F}\langle t, \hat{\ell} \rangle_{\mathrm{can}}.$$

We compute the left hand side of (2) by applying (3.4.13) to $\dot{F}(\ {}^{V}\eta t) = \eta F t$:

$$\begin{aligned} \langle \dot{F}(\ {}^{V}\eta t), \dot{F}^{\nabla}\hat{\ell} \rangle_{\mathrm{can}} &= \eta\ {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)^{-1}\ {}^{F}\tilde{\mathbf{E}}_0(\pi \otimes 1) \langle \dot{F}((\pi \otimes 1 - 1 \otimes [\pi])t), \dot{F}^{\nabla}\hat{\ell} \rangle_{\mathrm{can}} \\ &= \eta\ {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)^{-1}\ {}^{F}\tilde{\mathbf{E}}_0(\pi \otimes 1)\ {}^{\dot{F}}\tilde{\mathbf{E}}_{\psi_0}(\pi \otimes 1)\ {}^{F}\langle t, \hat{\ell} \rangle_{\mathrm{can}}. \end{aligned}$$

Here the last equation follows from (1). This proves (2). In the same way we obtain (4) from equation (3). The equation (5) follows from the second equation of (3.4.18) for $t = 0$.

Finally we prove equation (6). The special case

$$\langle \dot{F}((\pi \otimes 1 - 1 \otimes [\pi])t), \dot{F}^{\nabla}((\pi \otimes 1 - 1 \otimes [\pi])\hat{t}) \rangle_{\mathrm{can}} = 0 \tag{3.4.19}$$

is exactly the first equation of (3.4.18) for $\ell = 0$. We have to show that the same holds if we replace the first argument of the bilinear form in (3.4.19) by $\dot{F}({}^V\eta t)$ or the second argument by $\dot{F}^{\nabla}({}^V\eta\hat{t})$. But this may be reduced to (3.4.19) in the same way as in the proof of equation (2). This finishes the proof of (3.4.16). □

Proposition 3.4.10. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over R with a strict O-action. Then the natural map*

$$\mathrm{Hom}_{O\text{-displays}}(\mathcal{P}_2, \mathcal{P}_1^{\nabla}) \longrightarrow \mathrm{Bil}_{O\text{-displays}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{L}_R)$$

is an isomorphism. Here we consider bilinear forms of displays with a strict O-action which are also O-bilinear.

Proof. We define the inverse map. Let β be an element from the right hand side. This is in particular a $O \otimes W(R)$-bilinear form

$$\beta : P_1 \times P_2 \longrightarrow O \otimes W(R).$$

On the other hand, we have the canonical perfect $O\otimes W(R)$-bilinear form $\langle\ ,\ \rangle_{\mathrm{can}} : P_1 \times P_1^{\nabla} \longrightarrow O\otimes W(R)$. Since this is perfect, we can define a $O\otimes W(R)$-module homomorphism $\alpha : P_2 \longrightarrow P_1^{\nabla}$ by

$$\beta(x_1, x_2) = \langle x_1, \alpha(x_2)\rangle_{\mathrm{can}}.$$

We omit the straightforward verification that α defines a morphism of displays. □

Theorem 3.4.11. *Let R be an O-algebra such that p is nilpotent in R. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over R with a strict O-action. We denote by $\mathcal{P}_{1,\mathrm{a}}$ and $\mathcal{P}_{2,\mathrm{a}}$ their images by the Ahsendorf functor $\mathfrak{A}_{O/\mathbb{Z}_p,R}$. Proposition 3.3.15 and Proposition 3.4.4 define a homomorphism*

$$\mathrm{Bil}_{O\text{-displays}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{L}_R) \longrightarrow \mathrm{Bil}_{\mathcal{W}_O(R)\text{-displays}}(\mathcal{P}_{1,\mathrm{a}} \times \mathcal{P}_{2,\mathrm{a}}, \mathcal{P}_{m,\mathcal{W}_O(R)}(\pi^{ef}/p^f)). \tag{3.4.20}$$

If the displays $\mathcal{P}_1^{\nabla}$ and $\mathcal{P}_2$ are nilpotent, the homomorphism (3.4.20) is an isomorphism. Equivalently, (3.4.20) is an isomorphism if $(\mathcal{P}_{1,\mathrm{a}})^{\vee}$ and $\mathcal{P}_{2,\mathrm{a}}$ are nilpotent $\mathcal{W}_{O_F}(R)$-displays.

Proof. We apply (3.4.20) to $\mathcal{P}_2 = \mathcal{P}_1^{\nabla}$ and the canonical bilinear form. We obtain a bilinear form

$$\mathcal{P}_{1,\mathrm{a}} \otimes (\mathcal{P}_1^{\nabla})_{\mathrm{a}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_O(O)}(\pi^{ef}/p^f),$$

which is perfect by Proposition 3.3.15. After twisting, we obtain also a perfect pairing of $\mathcal{P}_{1,\mathrm{a}}$ and $(\mathcal{P}_1^{\nabla})_{\mathrm{a}}((\pi^{ef}/p^f)^{-1})$ with values in $\mathcal{P}_{m,\mathcal{W}_O(O)}$. Therefore we have an identification with the dual display

$$(\mathcal{P}_{1,\mathrm{a}})^{\vee} \cong (\mathcal{P}_1^{\nabla})_{\mathrm{a}}\big((\pi^{ef}/p^f)^{-1}\big).$$

By Theorem 3.3.2 we have an isomorphism

$$\mathrm{Hom}_{O\text{-displays}}(\mathcal{P}_2, \mathcal{P}_1^{\nabla}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}_O(R)\text{-displays}}(\mathcal{P}_{2,\mathrm{a}}, (\mathcal{P}_1^{\nabla})_{\mathrm{a}}).$$

Here the left hand side agrees with the left hand side of (3.4.20) by Proposition 3.4.10. We have seen that the right hand side is the same as

$$\mathrm{Hom}_{\mathcal{W}_O(R)\text{-displays}}(\mathcal{P}_{2,\mathrm{a}}, \mathcal{P}_{1,\mathrm{a}}^{\wedge}(\pi^{ef}/p^f)) \cong \mathrm{Bil}_{\mathcal{W}_O(R)\text{-displays}}(\mathcal{P}_{1,\mathrm{a}} \times \mathcal{P}_{2,\mathrm{a}}, \mathcal{P}_{m,\mathcal{W}_O(R)}(\pi^{ef}/p^f)).$$

The last isomorphism follows from (3.2.5). □

Definition 3.4.12. Let $R \in \mathrm{Nilp}_O$ and let $\mathcal{P}$ be a display with a strict O-action. A *relative polarization* of $\mathcal{P}$ with respect to O is a polarization of the $\mathcal{W}_O(R)$-display $\mathcal{P}_{\mathrm{a}}$ obtained from $\mathcal{P}$ by the Ahsendorf functor, cf. Definition 3.2.5.

Let $\breve{O}$ be the completion of the maximal unramified extension of O. We consider Theorem 3.4.11 in the case of an $\breve{O}$-algebra R. We denote by $\tau \in \mathrm{Gal}(\breve{O}/O)$ the Frobenius automorphism. Since $\pi^e/p \in \breve{O}$ is a unit we find $\eta_0 \in \breve{O}^{\times}$ such that $\tau(\eta_0)\eta_0^{-1} = \pi^e/p$. By Lemma 3.3.4 there is a $\tau - F$-equivariant homomorphism $\breve{O} \longrightarrow W_O(\breve{O})$ such that the composite with $\mathbf{w}_0$ is the identity.

Let $R \in \mathrm{Nilp}_{\breve{O}}$. We denote by $\eta_{0,R}$ the image of η_0 by the homomorphism

$$\breve{O} \longrightarrow W_O(\breve{O}) \longrightarrow W_O(R).$$

Then multiplication by $\eta_{0,R}^f$ defines an isomorphism of $\mathcal{W}_O(R)$-displays,

$$\mathcal{P}_{m,\mathcal{W}_O(R)}(\pi^{ef}/p^f) \xrightarrow{\sim} \mathcal{P}_{m,\mathcal{W}_O(R)}. \tag{3.4.21}$$

Therefore we may write Theorem 3.4.11 without the twist (π^{ef}/p^f).

Corollary 3.4.13. *Let $R \in \mathrm{Nilp}_{\breve{O}}$. Fix $\eta_0 \in \breve{O}^\times$ with $\tau(\eta_0)\eta_0^{-1} = \pi^e/p$, which defines the isomorphism* (3.4.21)*. Let $\mathcal{P}$ be a display with a strict O-action over R such that $\mathcal{P}$ and $\mathcal{P}^\nabla$ are nilpotent. Then a relative polarization on $\mathcal{P}$ with respect to O is the same thing as an isogeny of displays with an O-action $\mathcal{P} \longrightarrow \mathcal{P}^\nabla$ such that the induced bilinear form*

$$\mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{L}_R$$

is alternating. □

In the situation of the corollary, $\mathcal{P}$ is the display of a formal p-divisible group X with a strict O-action and $\mathcal{P}^\nabla$ is the display of a formal p-divisible group with a strict O-action which we denote by X^∇. We call X^∇ the *Faltings dual* of X. However, we do not relate our definition to that of Faltings in [11], which operates directly in the realm of p-divisible groups. We can consider a relative polarization as an isogeny of p-divisible groups with an O-action,

$$X \longrightarrow X^\nabla. \tag{3.4.22}$$

4. The contracting functor

We return to the notation of section 2. In particular, throughout this section, K/F denotes an etale extension of degree two of a p-adic field, and r denotes a generalized CM-type of rank 2.

Let E be the reflex field of r, and let $\tilde{E} \subset \bar{\mathbb{Q}}_p$ be its normal closure. As in subsection 2.2, $E' \subset \tilde{E}$ is the composite of E and the normal closure of K^t in $\bar{\mathbb{Q}}_p$.

4.1. The aim of this section.

Definition 4.1.1. Let S be a scheme over $\mathrm{Spec}\, O_{E'}$ such that p is locally nilpotent in $\mathcal{O}_S$. We denote[4] by $\mathfrak{P}_{r,S}$ the category of local CM-pairs of type r over S which satisfy the Eisenstein conditions (EC_r). If $S = \mathrm{Spec}\, R$, we will also write $\mathfrak{P}_{r,R}$ or simply $\mathfrak{P}_R$.

The category of local CM-pairs $(\mathcal{P}, \iota)$ of type r in the sense of displays which satisfy the Eisenstein conditions will be denoted by $\mathfrak{d}\mathfrak{P}_{r,S}$, resp., $\mathfrak{d}\mathfrak{P}_{r,R}$.

We will define a functor $\mathfrak{C}'_{r,R}$ that associates to a CM-pair $(\mathcal{P}, \iota) \in \mathfrak{d}\mathfrak{P}_{r,R}$ a new display $(\mathcal{P}', \iota')$ over R with an action $\iota' : O_K \longrightarrow \mathrm{End}\,\mathcal{P}'$. When r is special relative to $\varphi_0 \colon F \longrightarrow \overline{\mathbb{Q}}_p$, cf. Definition 2.1.1, then the restriction of ι' to O_F is strict with respect to $O_F \xrightarrow{\varphi_0} O_{E'} \longrightarrow R$. If r is banal, then $\mathcal{P}'$ is étale. We will call the functor $\mathfrak{C}'_{r,R}$ the *pre-contracting functor*. Under suitable hypotheses, the pre-contracting functor will be an equivalence of categories.

We will also describe what $\mathfrak{C}'_{r,R}$ does with polarizations, and define a functor $\mathfrak{C}'^{,\mathrm{pol}}_{r,R}$ on the category $\mathfrak{P}^{\mathrm{pol}}_{r,S}$, defined as follows.

Definition 4.1.2. We denote by $\mathfrak{P}^{\mathrm{pol}}_{r,S}$ the category of polarized local CM-pairs (X, ι, λ) of type r over S such that (X, ι) satisfies the Eisenstein conditions (EC_r). If $S = \mathrm{Spec}\, R$, we also write $\mathfrak{P}^{\mathrm{pol}}_{r,R}$. We denote by $\mathfrak{d}\mathfrak{P}^{\mathrm{pol}}_{r,S}$ the corresponding category of local CM-triples in the sense of displays. Explicitly, $\mathfrak{d}\mathfrak{P}^{\mathrm{pol}}_{r,R}$ denotes the category of triples $(\mathcal{P}, \iota, \beta)$ where $(\mathcal{P}, \iota) \in \mathfrak{d}\mathfrak{P}_{r,R}$ and where $\beta \colon \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}_{m,\mathcal{W}(R)}$ is a polarization such that

$$\beta(\iota(a)x_1, x_2) = \beta(x_1, \iota(\bar{a})x_2), \quad a \in O_K,\ x_1, x_2 \in P.$$

In the sequel, we will abbreviate the last condition into saying that β is *anti-linear for the O_K-action.*

[4]The symbol $\mathfrak{P}$ is to remind us that this is a category of local CM-pairs.

In a second step, we will use the functors $\mathfrak{C}'_{r,R}$, resp. $\mathfrak{C}'^{,\mathrm{pol}}_{r,R}$, to define *contracting functors.* Here, we make a distinction between the case when r is special and the case when r is banal. In the case when r is special, the image of the contracting functor is a $\mathcal{W}_{O_F}(R)$-display $\mathcal{P}_{\mathrm{c}}$, endowed with an action ι of O_K such that $\mathcal{P}_{\mathrm{c}}$ is of height 4 and dimension 2 (and, in the polarized case, with a polarization). When r is banal, the image is a p-adic étale sheaf G in $\mathbb{Z}_p$-flat modules of rank $4d$, endowed with an action of O_K (and with a polarization form in the polarized case).

4.2. **The Kottwitz condition for CM-pairs.** The Kottwitz condition (KC_r) can be formulated in terms of polynomial functions. Let $\mathcal{L}$ be a locally free R-module equipped with an action of O_K. If S is an R-algebra, we write $\mathcal{L}_S = \mathcal{L} \otimes_R S$.

Let us assume for a moment that R is endowed with an $O_{\tilde{E}}$-algebra structure. For $\varphi \in \Phi = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K, \tilde{E})$ we consider the induced map

$$\varphi_R : O_K \xrightarrow{\varphi} O_{\tilde{E}} \longrightarrow R.$$

Definition 4.2.1. We say that $(\mathcal{L}, \iota)$ satisfies the Kottwitz condition relative to r if for each $O_{\tilde{E}}$-algebra S endowed with an O_E-algebra homomorphism $R \longrightarrow S$

$$\det{}_S(a \mid \mathcal{L}_S) = \prod_{\varphi\in\Phi} \varphi_S(a)^{r_\varphi}, \quad \text{for all } a \in O_K \otimes_{\mathbb{Z}_p} S. \tag{4.2.1}$$

Let $\mathbb{A} = \mathbb{V}(O_K)$ be the affine space over $\operatorname{Spec} \mathbb{Z}_p$. The right hand side of this equation may be considered as a polynomial function on $\mathbb{A}_{O_{\tilde{E}}}$. By base change to $\tilde{E}$, it is easily shown that this function is defined on $\mathbb{A}_{O_E}$. We note that each factor of the right hand side of (4.2.1) is a linear polynomial function such that some coefficient is a unit in $O_{\tilde{E}}$. Therefore these factors are non-zero divisors in $\Gamma(\mathbb{A}_S, \mathcal{O}_{\mathbb{A}_S})$ for each S. Here we remark that a polynomial in $S[U_1, \ldots, U_r]$ is a non-zero divisor if one of its coefficients is a non-zero divisor in S.

Because the right hand side of (4.2.1) is a polynomial function on $\mathbb{A}_{O_E}$, the condition does not depend on the $O_{\tilde{E}}$-algebra structure on S. By a theorem of Amitsur, condition (4.2.1) is equivalent to the Kottwitz condition (KC_r) of (2.2.1) (compare [28]).

For a $O_{E'}$-algebra S we have a decomposition

$$O_K \otimes_{\mathbb{Z}_p} S = \prod_{\psi\in\Psi} O_K \otimes_{O_{K^t},\psi_S} S. \tag{4.2.2}$$

Here ψ_S denotes the composite $O_{K^t} \xrightarrow{\psi} O_{E'} \longrightarrow S$. Let $\mathbf{E}_{\psi_S}$ the image of the Eisenstein polynomial $\mathbf{E} \in O_{K^t}[T]$ by the last homomorphism. We have a natural isomorphism

$$S[T]/\mathbf{E}_{\psi_S} S[T] \xrightarrow{\sim} O_K \otimes_{O_{K^t},\psi_S} S, \quad T \longmapsto \Pi \otimes 1. \tag{4.2.3}$$

Therefore we may regard an $O_K \otimes_{O_{K^t},\psi_S} S$-module M as an $S[T]$-module. If $U \in S[T]$ and $x \in M$, we write $Ux = U(\Pi \otimes 1)x$. If $U_0 \in O_{E'}[T]$, with image $U \in S[T]$, then we write simply $U_0 x = Ux$.

Returning to the R-module $\mathcal{L}$ with action by O_K, the decomposition (4.2.2) induces a decomposition

$$\mathcal{L}_S = \oplus \mathcal{L}_{S,\psi}.$$

By considering, for given ψ, an element a of (4.2.2) whose components are zero for $\psi' \neq \psi$, we obtain

$$\det{}_S(a \mid \mathcal{L}_{S,\psi}) = \prod_{\varphi|\psi} \varphi_S(a)^{r_\varphi}, \quad \text{for all } a \in O_K \otimes_{O_{K^t},\psi} S. \tag{4.2.4}$$

We call this condition ($\mathrm{KC}_{\psi,r}$). The condition (KC_r) holds iff the conditions ($\mathrm{KC}_{\psi,r}$) hold for each ψ.

We will call ψ *banal* with respect to r if $r_\varphi \in \{0, 2\}$ for each $\varphi|\psi$. We call ψ *special* with respect to r if there exists $\varphi|\psi$ such that $r_\varphi = 1$ and if for any other $\varphi'|\psi$ with $r_{\varphi'} = 1$ we have $\varphi' = \bar{\varphi}$. We note that another φ' can only exist if $\bar{\psi} = \psi$.

We also use the conditions ($\mathrm{EC}_{\psi,r}$). The meaning is clear from (2.2.16) and (2.2.18) where the two conditions ($\mathrm{EC}_{\psi_0,r}$) and ($\mathrm{EC}_{\psi,r}$) are taken together.

We consider CM-pairs (X, ι) of type r over $R \in \mathrm{Nilp}_{O_E}$, cf. section 2.3. Thus X is a p-divisible group of height $4d$ and dimension $2d$ and ι is a $\mathbb{Z}_p$-algebra homomorphism

$$\iota : O_K \longrightarrow \mathrm{End}\, X$$

such the rank condition (RC_r) is satisfied. If we speak about the Kottwitz or Eisenstein condition we refer to the induced action on $\mathrm{Lie}\, X$. We use a similar terminology when we consider CM-pairs $(\mathcal{P}, \iota)$ in the sense of displays. This means that $\mathcal{P} = (P, Q, F, \dot{F})$ is a $\mathcal{W}(R)$-display of height $4d$ and dimension $2d$ endowed with a ring homomorphism

$$\iota : O_K \longrightarrow \mathrm{End}\, \mathcal{P},$$

such that the rank condition (RC_r) is satisfied for the induced action on P/Q.

Display theory defines a functor from the category of CM-pairs (X, ι) of type r to the category of CM-pairs $(\mathcal{P}, \iota)$ of type r. We set $\mathbb{D}_{\mathcal{P}} = P/I_R P$ and $\mathcal{L}_{\mathcal{P}} = P/Q$. If $\mathcal{P}$ is the display of X, we have the identifications

$$\mathbb{D}_{\mathcal{P}} = \mathbb{D}_X(R), \quad \mathcal{L}_{\mathcal{P}} = \mathrm{Lie}\, X.$$

Here $\mathbb{D}_X(R)$ is the Grothendieck-Messing crystal evaluated at R. For an R-algebra S, we will write $\mathbb{D}_{\mathcal{P},S} := \mathbb{D}_{\mathcal{P}} \otimes_R S$ and $\mathcal{L}_{\mathcal{P},S} := \mathcal{L}_{\mathcal{P}} \otimes_R S$. If $\mathcal{P}$ is fixed, we write simply $\mathbb{D}_S$ and $\mathcal{L}_S$. If S is a $O_{E'}$-algebra, (4.2.2) gives a decomposition

$$\mathbb{D}_S = \oplus_{\psi \in \Psi} \mathbb{D}_{S,\psi}.$$

Proposition 4.2.2. *Let ψ be banal with respect to r. Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over an O_E-algebra R.*

Then the Eisenstein condition $(\mathrm{EC}_{\psi,r})$ is satisfied iff $\mathbf{E}_{A_\psi}\mathbb{D}_\psi$ is the kernel of the canonical map $\mathbb{D}_\psi \longrightarrow \mathcal{L}_{\mathcal{P},\psi}$. Moreover $(\mathrm{EC}_{\psi,r})$ implies the Kottwitz condition $(\mathrm{KC}_{\psi,r})$.

Here $\mathbf{E}_{A_\psi}$ denotes the operator $\mathbf{E}_{A_\psi}\big(\iota(\Pi)\big)$ on the module in question, for a fixed choice of Π, cf. (2.2.12).

Proof. We reduce to the case where S is an R-algebra endowed with an $O_{\tilde{E}}$-algebra structure. Then $\mathbf{E}_{A_\psi,S} \in S[T]$ is defined as the image of $\mathbf{E}_{A_\psi}$ by $O_{\tilde{E}}[T] \longrightarrow S[T]$. It acts on any $O_K \otimes_{O_{K^t},\psi}$ S-module by (4.2.3).

Via ι, we view $\mathbb{D}_S$ and $\mathcal{L}_S$ as $O_K \otimes_{\mathbb{Z}_p} S$-modules. We consider the canonical surjective map

$$\mathbb{D}_S \longrightarrow \mathcal{L}_S.$$

The decomposition (4.2.2) induces decompositions,

$$\mathbb{D}_S = \oplus_\psi \mathbb{D}_\psi, \quad \mathcal{L}_S = \oplus_\psi \mathcal{L}_\psi. \tag{4.2.5}$$

We allowed us to omit the index S on the right hand side of these equations.

The Eisenstein condition $(\mathrm{EC}_{\psi,r})$ for banal ψ says that $\mathcal{L}_\psi$ is annihilated by $\mathbf{E}_{A_\psi}$, cf. (2.2.13). Therefore it is clearly implied by the condition of the Proposition. If conversely $(\mathrm{EC}_{\psi,r})$ holds, we obtain a surjective map

$$\mathbb{D}_\psi/\mathbf{E}_{A_\psi}\mathbb{D}_\psi \longrightarrow \mathcal{L}_\psi. \tag{4.2.6}$$

By Lemma 3.1.15, $\mathbb{D}_\psi$ is locally on $\mathrm{Spec}\, S$ a free $O_K \otimes_{O_{K^t},\psi} S$-module. It has rank 2 because the height of $\mathcal{P}$ is $4d$. We may assume that

$$\mathbb{D}_\psi \cong (O_K \otimes_{O_{K^t},\psi} S)^2 = S[T]^2/\mathbf{E}_{\psi_S} S[T]^2.$$

We see that both sides of (4.2.6) are locally free S-modules of the same rank $r_\psi = \sum_{\varphi|\psi} r_\varphi$. Therefore this map is an isomorphism.

The condition $(\mathrm{KC}_{\psi,r})$ would follow from

$$\det_S(a \mid S[T]/\mathbf{E}_{A_\psi}S[T]) = \prod_{\varphi \in A_\psi} \varphi_S(a), \quad a \in S[T]. \tag{4.2.7}$$

We have

$$\mathbf{E}_{A_\psi}(T) = \prod_{\varphi \in A_\psi} (T - \varphi_S(\Pi \otimes 1)), \quad \varphi_S(a) = a(\varphi_S(\Pi \otimes 1)).$$

We obtain (4.2.7) from the following Lemma. □

Lemma 4.2.3. *Let R be a ring. Let*

$$\mathbf{E}(T) = \prod_{i=1}^{s}(T - \Pi_i), \quad \Pi_i \in R$$

be a polynomial. A polynomial $f(T) \in R[T]$ defines by multiplication an endomorphism of the free R-module $R[T]/\mathbf{E}R[T]$. Then

$$\det_R(f(T) \mid R[T]/\mathbf{E}R[T]) = \prod_{i=1}^{s} f(\Pi_i).$$

Proof. One can easily reduces the question to the case where R is a field of characteristic 0 such that $\mathbf{E}(T)$ is a product of different linear factors. For the reduction one starts with a ring homomorphism

$$\mathbb{Z}[\underline{X}, \underline{Y}] \longrightarrow R,$$

where for the first set of variables $\underline{X} = (X_1, \dots, X_s)$, X_i is mapped to Π_i and where the second set of variables $\underline{Y}$ is mapped to the coefficients of f.

If now R is a field and $\mathbf{E}$ is separable, we have a canonical isomorphism of R-modules

$$\begin{array}{rcl} R[T]/\mathbf{E}R[T] & \xrightarrow{\sim} & \oplus_{i=1}^{s} R. \\ T & \longmapsto & (\Pi_i)_i \end{array}$$

Multiplication by $f(T)$ on the left hand side acts on the right hand side on the i-th factor by multiplication by $f(\Pi_i)$. This implies the assertion. □

Corollary 4.2.4. *Let r be banal. Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over an O_E-algebra R. Then the Eisenstein condition* (EC_r) *implies the Kottwitz condition* (KC_r). □

We consider next the case where ψ is special. This means by definition that there is exactly one pair $\{\varphi, \bar{\varphi}\}$ such that $\varphi | \psi$ and $r_\varphi = r_{\bar{\varphi}} = 1$. When K/F is ramified, we have $E' = E$ and when K/F is unramified, we have $[E' : E] \leq 2$.

Proposition 4.2.5. *Let ψ be special with respect to r. Let R be a O_E-algebra such that p is nilpotent in R. Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over R which satifies the Eisenstein condition* $(\mathrm{EC}_{\psi,r})$. *Let S be a $O_{E'}$-algebra which is endowed with an O_E-algebra homomorphism $R \longrightarrow S$. Then, with the notations of (4.2.5), the canonical map*

$$\mathbb{D}_\psi / \mathbf{E}_{A_\psi} \mathbb{D}_\psi \longrightarrow \mathcal{L}_\psi / \mathbf{E}_{A_\psi} \mathcal{L}_\psi$$

is an isomorphism.

Proof. Clearly we may assume that S is a local ring with residue class field k. We postpone the verification that the assertion holds for $S = k$ (compare (4.3.17) and (4.3.20) below).

We begin with the case K/F unramified. Then $\mathrm{rank}_S \mathcal{L}_\psi = r_\psi = 2a_\psi + 1$. Let

$$f : \mathcal{L}_\psi \longrightarrow \mathcal{L}_\psi \tag{4.2.8}$$

be the S-module homomorphism given by multiplication with $\mathbf{E}_{A_\psi}$. From the case of a field, we deduce that $\dim_k \mathcal{L}/f(\mathcal{L}) \otimes_S k = 2a_\psi$. By $(\mathrm{EC}_{\psi,r})$ we have

$$\bigwedge^2 f = 0,$$

cf. (2.2.18). Therefore we can apply Lemma 4.9 of [28] with $s = 1$. This says that $\mathcal{L}_\psi / f(\mathcal{L}_\psi)$ is a free S-module of rank $2a_\psi$. Therefore the canonical map of the proposition is a surjection of free S-modules of the same rank, and hence an isomorphism.

The argument in the case K/F ramified is similiar. In this case, we have $\mathrm{rank}_S \mathcal{L}_\psi = r_\psi = 2a_\psi + 2 = 2e$. The dimension $\dim_k \mathcal{L}_\psi / f(\mathcal{L}_\psi) \otimes_S k = 2a_\psi$, as before. In this case the condition $(\mathrm{EC}_{\psi,r})$ says

$$\bigwedge^3 f = 0,$$

cf. (2.2.16). Therefore we may apply Lemma 4.9 loc.cit. with $s = 2$. We conclude as before. □

Proposition 4.2.6. *Let r be special and K/F unramified. Let R be a O_E-algebra such that p is nilpotent in R. Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over R which satifies the Eisenstein condition* (EC_r). *Then the condition* (KC_r) *is also satisfied.*

Proof. We consider an algebra S as in the last proposition. We keep the notation of (4.2.5). We need only to verify $(\mathrm{KC}_{\psi_0,r})$ since the Kottwitz condition is satisfied for ψ banal by Proposition 4.2.2. We have to verify that

$$\det_S(a \mid \mathcal{L}_{\psi_0}) = \varphi_{0,S}(a) \cdot \prod_{\varphi \in A_{\psi_0}} \varphi_S(a)^2, \quad a \in O_K \otimes_{O_{K^t},\psi_0} S. \tag{4.2.9}$$

Since $\mathbb{D}_{\psi_0}$ is locally on S a free $O_K \otimes_{O_{K^t},\psi_0} S$-module of rank 2, we obtain from the isomorphism of Proposition 4.2.5

$$\det_S(a \mid \mathcal{L}_{\psi_0}/\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) = \prod_{\varphi \in A_{\psi_0}} \varphi_S(a)^2.$$

The proposition also shows that $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ is a locally free S-module of rank 1. It follows from the Eisenstein condition (2.2.18) that this module is annihilated by $(T - \varphi_{0,S}(\Pi \otimes 1))$. Hence an element $a \in O_K \otimes_{O_{K^t},\psi_0} S = S[T]/\mathbf{E}_{\psi_0}S[T]$ acts on $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ as $\varphi_{0,S}(a)$. In particular

$$\det_S(a \mid \mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) = \varphi_{0,S}(a).$$

The formula (4.2.9) follows. □

We reformulate the Eisenstein condition in the case where K/F is unramified.

Proposition 4.2.7. *Let r be special and K/F unramified. Let $\varphi_0, \bar{\varphi}_0 \in \Phi$ be the two embeddings such that $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$, and let ψ_0, resp. $\bar{\psi}_0$, the embeddings induced by φ_0, resp. $\bar{\varphi}_0$. Let R be a $O_{E'}$-algebra, and let $(\mathcal{P}, \iota)$ be a CM-pair of type r over R. Let $\mathbb{D} = \mathbb{D}_{\mathcal{P}}(R)$. The CM-pair $(\mathcal{P}, \iota)$ satisfies the Eisenstein conditions iff the following conditions hold.*

(1) If $\psi \in \Psi$ is banal, then the canonical map

$$\mathbb{D}_\psi/\mathbf{E}_{A_\psi}\mathbb{D}_\psi \longrightarrow \operatorname{Lie}_\psi X$$

is an isomorphism.

(2) If $\psi \in \{\psi_0, \bar{\psi}_0\}$, then the canonical map

$$\mathbb{D}_\psi/\mathbf{E}_{A_\psi}\mathbb{D}_\psi \longrightarrow \operatorname{Lie}_\psi X/\mathbf{E}_{A_\psi} \operatorname{Lie}_\psi X$$

is an isomorphism.

(3) The R-modules $\mathbf{E}_{A_{\psi_0}} \operatorname{Lie}_{\psi_0}$, resp. $\mathbf{E}_{A_{\bar{\psi}_0}} \operatorname{Lie}_{\bar{\psi}_0}$, are locally free of rank 1 and O_K acts on them via

$$\varphi_{0,R} : O_K \longrightarrow O_{E'} \longrightarrow R, \quad \textit{resp.} \quad \bar{\varphi}_{0,R} : O_K \longrightarrow O_{E'} \longrightarrow R.$$

Proof. This is a consequence of Proposition 4.2.5 and the proof of Proposition 4.2.6. □

We next consider what happens to the Eisenstein conditions when passing from a display to its conjugate dual, cf. Lemma 2.3.2. We note that we already checked in loc. cit. that the condition (RC_r) is preserved. Recall that, if $(\mathcal{P}, \iota)$ is a CM-pair, the conjugate dual $(\mathcal{P}^\wedge, \iota^\wedge)$ is defined by

$$\big(\mathcal{P}^\wedge = \mathcal{P}^\vee, \quad \iota^\wedge(a) = \iota(\bar{a})^\wedge\big).$$

Corollary 4.2.8. *Let K/F be unramified and let r be arbitrary or let K/F be split. Let $(\mathcal{P}, \iota)$ be a CM-pair over an O_E-algebra R such that p is nilpotent in R. If $(\mathcal{P}, \iota)$ satisfies the Eisenstein condition* (EC_r), *then the conjugate dual $(\mathcal{P}^\wedge, \iota^\wedge)$ also satisfies* (EC_r).

Proof. We have a canonical isomorphism $\mathbb{D}^\wedge = \operatorname{Hom}_R(\mathbb{D}, R)$. The resulting perfect pairing

$$\langle\ ,\ \rangle : \mathbb{D} \times \mathbb{D}^\wedge \longrightarrow R \tag{4.2.10}$$

satisfies

$$\langle ax, \hat{x}\rangle = \langle x, \bar{a}\hat{x}\rangle, \quad a \in O_K,\ x \in \mathbb{D},\ \hat{x} \in \mathbb{D}^\wedge. \tag{4.2.11}$$

This implies that for $\psi_1 \neq \bar{\psi}_2$ the modules $\mathbb{D}_{\psi_1}$ and $\mathbb{D}^\wedge_{\psi_2}$ are orthogonal and that for any ψ the induced pairing

$$\mathbb{D}_\psi \times \mathbb{D}^\wedge_{\bar{\psi}} \longrightarrow R \tag{4.2.12}$$

is perfect. Let $\mathbb{D}^1_\psi \subset \mathbb{D}_\psi$ be the kernel of the map $\mathbb{D}_\psi \longrightarrow \mathcal{L}_\psi := \mathrm{Lie}_\psi \mathcal{P}$ and let $\mathbb{D}^{\wedge,1}_{\bar\psi} \subset \mathbb{D}^\wedge_{\bar\psi}$ be defined in the same way. By definition of the dual display, $\mathbb{D}^1_\psi$ and $\mathbb{D}^{\wedge,1}_{\bar\psi}$ are orthogonal complements of each other with respect to (4.2.12). We consider the case $\psi \in \{\psi_0, \bar\psi_0\}$. Recall that this is possible only in the non-split case. The Eisenstein condition for $\mathcal{P}$ says that we have a split filtration of direct summands of $\mathbb{D}_\psi$

$$\mathbf{S}_\psi \mathbf{E}_{A_\psi} \mathbb{D}_\psi \subset \mathbb{D}^1_\psi \subset \mathbf{E}_{A_\psi} \mathbb{D}_\psi, \tag{4.2.13}$$

such that the factor modules are locally free of rank 1. We claim that the orthogonal complement of $\mathbf{E}_{A_\psi}\mathbb{D}_\psi$ is $\mathbf{S}_{\bar\psi}\mathbf{E}_{A_{\bar\psi}}\mathbb{D}^\wedge_{\bar\psi}$. Indeed, by (4.2.11), we have

$$\langle\, \mathbf{E}_{A_\psi} x,\ \hat{x}\rangle = \langle\, x,\ \mathbf{E}_{B_{\bar\psi}} \hat{x}\rangle. \tag{4.2.14}$$

This implies that $\mathbf{E}_{A_\psi}\mathbb{D}_\psi$ and $\mathbf{S}_{\bar\psi}\mathbf{E}_{A_{\bar\psi}}\mathbb{D}^\wedge_{\bar\psi}$ are orthogonal. Because (4.2.10) is perfect we obtain a surjection of R-modules

$$\mathbb{D}_\psi / \mathbf{E}_{A_\psi}\mathbb{D}_\psi \longrightarrow \mathrm{Hom}_R(\mathbf{S}_{\bar\psi}\mathbf{E}_{A_{\bar\psi_R}}\mathbb{D}^\wedge_{\bar\psi}, R). \tag{4.2.15}$$

Recall that $\mathbb{D}_\psi$ is locally on $\mathrm{Spec}\, R$ a free $O_K \otimes_{O_{K^t},\psi} R$-module of rank 2. It follows that both sides of (4.2.15) are locally free R-modules of the same rank $2a_\psi = 2e - 2a_{\bar\psi} - 2$. Therefore this map is an isomorphism. This proves our claim about the orthogonal complement. By the same argument, $\mathbf{E}_{A_{\bar\psi_R}}\mathbb{D}^\wedge_{\bar\psi}$ is the orthogonal complement of $\mathbf{S}_\psi\mathbf{E}_{A_\psi}\mathbb{D}_\psi$.

We take the orthogonal complement of (4.2.13) and obtain the filtration

$$\mathbf{S}_{\bar\psi}\mathbf{E}_{A_{\bar\psi}}\mathbb{D}^\wedge_{\bar\psi} \subset \mathbb{D}^{\wedge,1}_{\bar\psi} \subset \mathbf{E}_{A_{\bar\psi}}\mathbb{D}^\wedge_{\bar\psi},$$

and conclude that the factor modules are locally free of rank 1.

Now let ψ be banal. We have to prove that $\mathbf{E}_{A_\psi}\mathbb{D}^\wedge_\psi \subset \mathbb{D}^{\wedge,1}_\psi$. The right hand side is the orthogonal complement of $\mathbb{D}^1_{\bar\psi} = \mathbf{E}_{A_{\bar\psi}}\mathbb{D}_{\bar\psi}$. Therefore we have to prove that

$$\langle\, \mathbf{E}_{A_{\bar\psi}} x,\ \mathbf{E}_{A_\psi}\hat{x}\rangle = 0, \quad \text{for } x \in \mathbb{D}_{\bar\psi},\ \hat{x} \in \mathbb{D}^\wedge_\psi.$$

Using (4.2.14), we find for the right hand side

$$\langle\, x,\ \mathbf{E}_{B_\psi}\mathbf{E}_{A_\psi}\hat{x}\rangle = \langle\, x,\ \mathbf{E}_\psi \hat{x}\rangle = 0.$$

□

Before proving the analogue of Corollary 4.2.8 in the case when K/F is ramified, we further analyze in this case the Eisenstein conditions.

Proposition 4.2.9. *Let r be special and K/F ramified. Let R be a O_E-algebra such that p is nilpotent in R. Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over R. Since $E' = E$, the decomposition (4.2.2) is defined for $S = R$. Then the Eisenstein condition* $(\mathrm{EC}_{\psi_0,r})$ *holds iff the following conditions are satisfied.*

(1) The R-module $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0} \subset \mathcal{L}_{\psi_0}$ is a direct summand which is locally free of rank 2.

(2) The action of $\iota(\pi)$ on $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ coincides with the action of $\varphi_0(\pi) \in R$, i.e., the action of the image of π by the homomorphism $O_F \xrightarrow{\varphi_0} O_E \longrightarrow R$.

Furthermore, a CM-pair $(\mathcal{P}, \iota)$ which satisfies (EC_r) *also satisfies the Kottwitz condition* (KC_r) *if and only if*

$$\mathrm{Tr}_R(\iota(\Pi) \mid \mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) = 0. \tag{4.2.16}$$

Proof. For the proof we may pass from R to an R-algebra S which is endowed with a $O_{\tilde{E}}$-algebra stucture. We continue with the notations of (4.2.5). The first assertion of the proposition is then an immediate consequence of Proposition 4.2.5. By Proposition 4.2.6, we need only to consider $(\mathrm{KC}_{\psi_0,r})$. The condition reads

$$\det_S(a \mid \mathcal{L}_{\psi_0}) = \varphi_{0,S}(a) \cdot \bar\varphi_{0,S}(a) \cdot \prod_{\varphi \in A_{\psi_0}} \varphi_S(a)^2, \quad a \in O_K \otimes_{O_{K^t},\psi_0} S.$$

In this case $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ is a locally free S-module of rank 2. By Proposition 4.2.5, it is enough to show

$$\det_S(a \mid \mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) = \varphi_{0,S}(a) \cdot \bar\varphi_{0,S}(a). \tag{4.2.17}$$

In this case, the Eisenstein condition says that

$$(T - \varphi_{0,S}(\Pi \otimes 1))(T - \bar{\varphi}_{0,S}(\Pi \otimes 1)) = T^2 + \psi_0(\pi) \tag{4.2.18}$$

annihilates $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$, cf (2.2.16). Note that the action of T^2 on $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ is by definition the action of $\iota(\Pi^2) = -\iota(\pi)$. Therefore the action of O_F on $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ via ι coincides with the action via $O_F \xrightarrow{\varphi_0} O_{\tilde{E}} \longrightarrow S$. The polynomial (4.2.18) is the characteristic polynomial of $\iota(\Pi)$ acting on the S-module $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$. This follows from the trace condition of the proposition. Therefore the desired equation (4.2.17) is a consequence of Lemma 4.2.10 below.

Conversely, assume that the Kottwitz condition holds. By Proposition 4.2.5, this implies

$$\det{}_S(a \mid \mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) \cdot \prod_{\varphi \in A_{\psi_0}} \varphi_S(a)^2 = \varphi_{0,S}(a) \cdot \bar{\varphi}_{0,S}(a) \cdot \prod_{\varphi \in A_{\psi_0}} \varphi_S(a)^2, \quad a \in O_K \otimes S.$$

We already remarked right after Definition 4.2.1 that φ_S is a non-zero divisor in the ring of polynomial functions. Therefore we conclude

$$\det{}_S(a \mid \mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}) = \varphi_{0,S}(a) \cdot \bar{\varphi}_{0,S}(a), \qquad \text{for all } a \in O_K \otimes S.$$

This implies that the characteristic polynomial of $\iota(\Pi)$ acting on $\mathbf{E}_{A_{\psi_0}}\mathcal{L}_{\psi_0}$ is the polynomial (4.2.18). Therefore the trace is 0. □

We state the needed Lemma without proof.

Lemma 4.2.10. *Let S be a ring. Let L be a locally free S-module of rank 2. Let $\alpha : L \longrightarrow L$ be an endomorphism with characteristic polynomial*

$$\det{}_S(T\mathrm{id}_L - \alpha \mid L) = T^2 - s_1 T + s_2.$$

Then for all $\lambda, \mu \in S$

$$\det{}_S(\mu\mathrm{id}_L - \lambda\alpha) = \mu^2 - s_1\mu\lambda + s_2\lambda^2.$$

Assume that the characteristic polynomial splits

$$T^2 - s_1 T + s_2 = (T - \rho_1)(T - \rho_2).$$

Consider L as $S[T]$-module, and let $\phi_i : S[T] \longrightarrow S$ be the S-algebra homomorphism such that $\phi_i(T) = \rho_i$. Then for each polynomial $a \in S[T]$

$$\det{}_S(a \mid L) = \phi_1(a) \cdot \phi_2(a).$$

□

Remark 4.2.11. Let $A \subset B$ be R-modules. Then we write $A \overset{c}{\subset} B$ if the factor module B/A is a finitely generated projective R-module of rank c.

Let $(\mathcal{P}, \iota)$ as in Proposition 4.2.9 such that $(\mathrm{EC}_{\psi_0,r})$ is satisfied. We write $\mathbb{D} = P/I_R P$. Let $\bar{Q}_{\psi_0}$ the kernel of $\mathbb{D}_{\psi_0} \longrightarrow \mathcal{L}_{\psi_0}$. By Lemma 3.1.15, $\mathbb{D}_{\psi_0}$ is a free $O_K \otimes_{O_{F^t},\psi_0} R$-module of rank 2. We obtain that

$$\mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0} \overset{2(e-1)}{\subset} \mathbb{D}_{\psi_0}.$$

On the other hand, the condition (1) of Proposition 4.2.9 says

$$\bar{Q}_{\psi_0} \overset{2}{\subset} \mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0} + \bar{Q}_{\psi_0} \overset{2(e-1)}{\subset} \mathbb{D}_{\psi_0}.$$

This implies $\bar{Q}_{\psi_0} \subset \mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0}$. Therefore we may reformulate the two conditions in Proposition 4.2.9 in one line:

$$\mathbf{S}_{\psi_0}\mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0} \overset{2}{\subset} \bar{Q}_{\psi_0} \overset{2}{\subset} \mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0}. \tag{4.2.19}$$

Corollary 4.2.12. *Let K/F be ramified. Let $(\mathcal{P}, \iota)$ be a CM-pair over an O_E-algebra R such that p is nilpotent in R. If (X, ι) satisfies the Eisenstein condition (EC_r), then the conjugate dual $(\mathcal{P}^\wedge, \iota^\wedge)$ also satisfies (EC_r).*

Proof. We use the notation of the last remark. The banal ψ are treated as in the unramified case. We need only to check that the conjugate dual satisfies $(\mathrm{EC}_{\psi_0,r})$. The orthogonal complement of $\mathbf{E}_{A_{\psi_0}}\mathbb{D}_{\psi_0}$ in $\mathrm{Hom}_R(\mathbb{D}_{\psi_0}, R)$ is $\mathbf{S}_{\psi_0}\mathbf{E}_{A_{\psi_0}} \mathrm{Hom}_R(\mathbb{D}_{\psi_0}, R)$, where in the last formula we use the action via $\iota^\vee$. We obtain the result by taking the orthogonal complement of (4.2.19). □

To end this subsection, we check that the Kottwitz condition (KC_r) is preserved under passage to the conjugate dual.

Proposition 4.2.13. *Let $K/F, r$ be arbitrary. Let $(\mathcal{P}, \iota)$ be a CM-pair which satisfies* (KC_r). *Then the conjugate dual $(\mathcal{P}^\wedge, \iota^\wedge)$ satisfies* (KC_r).

Proof. We may assume that R is endowed with the structure of an $O_{\tilde{E}}$-algebra. We use the notation of the proof of Corollary 4.2.8. In particular $\mathbb{D}_{\mathcal{P},R} = \mathbb{D}$ and $\mathbb{D}_{\mathcal{P}^\wedge,R} = \mathbb{D}^\wedge$, and we write $\mathcal{L}$ and $\mathcal{L}^\wedge$ for the Lie algebras of $\mathcal{P}$ and $\mathcal{P}^\wedge$. We have to show that for each R-algebra S and for each $\psi \in \Psi$

$$\det{}_S(a \mid \mathcal{L}^\wedge_{S,\psi}) = \prod_{\varphi|\psi} \varphi_S(a)^{r_\varphi}, \quad \text{for all } a \in O_K \otimes_{O_{K^t},\psi_R} S.$$

To show this, we may replace $\mathcal{P}$ by its base change $\mathcal{P}_S$. Therefore it is enough to consider the case $S = R$. Since $\mathbb{D}_\psi$ is locally on $\operatorname{Spec} R$ a free $O_K \otimes_{O_{K^t},\psi_R} R$-module of rank 2, we find

$$\det(a \mid \mathbb{D}_\psi) = \prod_{\varphi \in \Phi_\psi} \varphi_R(a)^2, \quad \text{for } a \in O_K \otimes_{O_{K^t},\psi_R} R.$$

Since $\mathcal{L}_\psi = \mathbb{D}_\psi / \mathbb{D}^1_\psi$ we find

$$\det(\bar{a} \mid \mathbb{D}^1_{\bar{\psi}}) = \prod_{\bar{\varphi}|\bar{\psi}} \bar{\varphi}_R(\bar{a})^{(2-r_{\bar{\varphi}})} = \prod_{\varphi \in \Phi_\psi} \varphi_R(a)^{r_\varphi}.$$

The perfect pairing (4.2.8) induces a perfect pairing

$$\mathbb{D}^1_{\bar{\psi}} \times \mathcal{L}^\wedge_{\psi} \longrightarrow R.$$

Therefore we obtain

$$\det(a \mid \mathcal{L}^\wedge_\psi) = \det(\bar{a} \mid \mathbb{D}^1_{\bar{\psi}}) = \prod_{\varphi \in \Phi_\psi} \varphi_R(a)^{r_\varphi}.$$

Therefore $(\mathrm{KC}_{\bar{\psi},r})$ for $\mathcal{P}$ implies $(\mathrm{KC}_{\psi,r})$ for $\mathcal{P}^\wedge$. □

4.3. **The pre-contracting functor.** Let $(\mathcal{P}, \iota)$ be a CM-pair of type r over an $O_{E'}$-algebra R such that p is nilpotent in R. We assume that $(\mathcal{P}, \iota)$ satisfies (EC_r). In other words $(\mathcal{P}, \iota) \in \mathfrak{dP}_{r,R}$, cf. Definition 4.1.1. We will define a functor that associates to $(\mathcal{P}, \iota)$ a new display $\mathcal{P}'$ of the same height with an action

$$\iota' : O_K \longrightarrow \operatorname{End} \mathcal{P}'.$$

In the case where r is banal, the display $\mathcal{P}'$ will be étale; in the case where r is special, the restriction of the action ι' to O_F will be strict with respect to $\varphi_{0,R} : O_F \longrightarrow O_E \longrightarrow R$. We will call this functor the *pre-contracting functor*.

Let us first restrict our attention to the case where K/F is a field extension. The case $K = F \times F$ will be treated separately because it needs different notations, see p. 56, starting before eq. (4.3.22). Each $\psi : K^t \longrightarrow \bar{\mathbb{Q}}_p$ induces a homomorphism

$$\tilde{\psi} : O_{K^t} \longrightarrow W(O_{K^t}) \xrightarrow{W(\psi)} W(O_{\tilde{E}}). \tag{4.3.1}$$

For an $O_{E'}$-algebra R we deduce a homomorphism $\tilde{\psi}_R : O_{K^t} \longrightarrow W(R)$ that is equivariant with respect to the Frobenius homomorphisms on both sides. This induces decompositions

$$\begin{aligned} O_{K^t} \otimes_{\mathbb{Z}_p} W(R) &\cong \textstyle\prod_{\psi \in \Psi} W(R), \\ O_K \otimes_{\mathbb{Z}_p} W(R) &\cong \textstyle\prod_{\psi \in \Psi} O_K \otimes_{O_{K^t},\tilde{\psi}_R} W(R) \end{aligned} \tag{4.3.2}$$

which lift the decomposition (4.2.2). Let $\sigma \in \operatorname{Gal}(K^t/\mathbb{Q}_p)$ be the Frobenius automorphism. The operators F and V act via $W(R)$ on the right hand side of (4.3.2). On the left hand side this induces maps

$$O_K \otimes_{O_{K^t},\tilde{\psi}_R} W(R) \underset{V}{\overset{F}{\rightleftarrows}} O_K \otimes_{O_{K^t},\tilde{\psi}_R \circ \sigma} W(R),$$

cf. (3.4.4). Recall that

$$\mathbf{E}_{A_\psi}(T) = \prod_{\varphi \in A_\psi} (T - \varphi(\Pi)) \in O_{E'}[T].$$

We lift this to a polynomial with coefficients in $W(O_{E'})$ by taking the Teichmüller representatives of the roots,

$$\tilde{\mathbf{E}}_{A_\psi}(T) = \prod_{\varphi \in A_\psi} (T - [\varphi(\Pi)]) \in W(O_{E'})[T]. \tag{4.3.3}$$

The image of this polynomial by the homomorphism $W(O_{E'}) \longrightarrow W(R)$ is denoted by $\tilde{\mathbf{E}}_{A_\psi,R}(T)$. If we reduce with respect to $\mathbf{w}_0 : W(O_{E'}) \longrightarrow O_{E'}$, we obtain the polynomial $\mathbf{E}_{A_\psi}(T)$. We note that in the case where R is a $\kappa_{E'}$-algebra, we have

$$\tilde{\mathbf{E}}_{A_\psi,R}(T) = T^{a_\psi}. \tag{4.3.4}$$

We consider the ring homomorphism

$$\begin{aligned} W(O_{E'})[T] \longrightarrow W(R)[T] &\longrightarrow O_K \otimes_{O_{K^t},\tilde{\psi}_R} W(R). \\ T &\longmapsto \Pi \otimes 1 \end{aligned} \tag{4.3.5}$$

We denote by $\tilde{\mathbf{E}}_{A_\psi,R}(\Pi \otimes 1)$ the image of $\tilde{\mathbf{E}}_{A_\psi}(T)$ under (4.3.5).

Let now $(\mathcal{P}, \iota) \in \mathfrak{d}\mathfrak{P}_{r,R}$, where $\mathcal{P} = (P, Q, F, \dot{F})$. We obtain decompositions of the $O_K \otimes_{\mathbb{Z}_p} W(R)$-modules P and Q,

$$P = \oplus_{\psi \in \Psi} P_\psi, \quad Q = \oplus_{\psi \in \Psi} Q_\psi. \tag{4.3.6}$$

For $x \in P_\psi$, we write

$$\tilde{\mathbf{E}}_{A_\psi} x = \tilde{\mathbf{E}}_{A_\psi,R}(\Pi \otimes 1) x. \tag{4.3.7}$$

On the left hand side we consider P_ψ as a $W(O_{E'})[T]$-module via (4.3.5).

We give first the recipe for the construction of $\mathcal{P}'$ for any $R \in \mathrm{Nilp}_{O_{E'}}$. Then we will discuss the case of a perfect field. This special case is then used to prove that $\mathcal{P}'$ is indeed a display.

We begin with the case where r is banal (and K/F is a field extension). Let $(\mathcal{P}, \iota)$ as above. We define

$$P' = \oplus_\psi P'_\psi, \quad Q' = \oplus_\psi Q'_\psi$$

as follows: for all ψ we set

$$P'_\psi = Q'_\psi = P_\psi. \tag{4.3.8}$$

By the Eisenstein condition (2.2.13), we have $\tilde{\mathbf{E}}_{A_\psi} P_\psi \subset Q_\psi$. Then we may define

$$\begin{aligned} \dot{F}' &: Q'_\psi \longrightarrow P'_{\psi\sigma}, \quad \dot{F}'(x) = \dot{F}(\tilde{\mathbf{E}}_{A_\psi} x), \quad x \in P_\psi \\ F' &: P'_\psi \longrightarrow P_{\psi\sigma}, \quad F'(x) = F(\tilde{\mathbf{E}}_{A_\psi} x), \quad x \in P_\psi. \end{aligned} \tag{4.3.9}$$

We define $F' : P' \longrightarrow P'$ and $\dot{F}' : Q' \longrightarrow P'$ as the direct sum of the maps above. We have to prove that $\mathcal{P}' = (P', Q', F', \dot{F}')$ is a display. The only non-trivial property we have to check is that $\dot{F}' : P_\psi \longrightarrow P_{\psi\sigma}$ is an F-linear isomorphism. We postpone the verification, cf. p. 54, below (4.3.16).

We now define the pre-contracting functor in the case where r is special and K/F unramified. In this case we have $\psi_0 \neq \bar{\psi}_0$. If ψ is banal, i.e., if $\psi \notin \{\psi_0, \bar{\psi}_0\}$, we keep the definitions (4.3.8) and (4.3.9). We set $P'_{\psi_0} = P_{\psi_0}$ and we define Q'_{ψ_0} as the kernel of the following map,

$$P'_{\psi_0} = P_{\psi_0} \longrightarrow P_{\psi_0}/Q_{\psi_0} \xrightarrow{\mathbf{E}_{A_{\psi_0},R}} \mathbf{E}_{A_{\psi_0},R}(P_{\psi_0}/Q_{\psi_0}) \subset P_{\psi_0}/Q_{\psi_0}. \tag{4.3.10}$$

It follows from Proposition 4.2.5 that $\mathbf{E}_{A_{\psi_0},R}(P_{\psi_0}/Q_{\psi_0})$ is locally free of rank 1 and is a direct summand of P_{ψ_0}/Q_{ψ_0}. Therefore

$$P'_{\psi_0}/Q'_{\psi_0} \cong \mathbf{E}_{A_{\psi_0},R}(P_{\psi_0}/Q_{\psi_0}) \tag{4.3.11}$$

is locally free of rank 1 and, as remarked at the end of the proof of Proposition 4.2.6, an element $a \in O_K \otimes_{O_{K^t},\psi_0} R$ acts on (4.3.11) by multiplication with $\varphi_{0,R}(a)$. This makes sense because $\varphi_0 : O_K \longrightarrow O_{\tilde{E}}$ factors through $O_{E'} \subset O_{\tilde{E}}$. We define

$$\begin{aligned} F' &: P'_{\psi_0} \longrightarrow P'_{\psi_0\sigma}, \quad F'(x) = F(\tilde{\mathbf{E}}_{A_{\psi_0,R}} x), \\ \dot{F}' &: Q'_{\psi_0} \longrightarrow P'_{\psi_0\sigma}, \quad \dot{F}'(y) = \dot{F}(\tilde{\mathbf{E}}_{A_{\psi_0,R}} y). \end{aligned} \tag{4.3.12}$$

The last equation makes sense because, by definition, $\tilde{\mathbf{E}}_{A_{\psi_0,R}} Q'_{\psi_0} \subset Q_{\psi_0}$. The definitions of the modules $P'_{\bar{\psi}_0}, Q'_{\bar{\psi}_0}$ and the restrictions of F' and $\dot{F}'$ to these modules are defined by interchanging the roles of ψ_0 and $\bar{\psi}_0$. This completes the definition of

$$F' : \oplus_\psi P'_\psi \longrightarrow \oplus_\psi P'_\psi, \quad \dot{F}' : \oplus_\psi Q'_\psi \longrightarrow \oplus_\psi P'_\psi. \tag{4.3.13}$$

Again we postpone the verification that $\mathcal{P}'$ is a display, cf. below (4.3.17). The tangent space $\mathcal{L}' = P'/Q'$ is a locally free R-module of rank 2. It has a decomposition

$$\mathcal{L}' = \mathcal{L}'_{\psi_0} \oplus \mathcal{L}'_{\bar{\psi}_0},$$

where an element $a \in O_K \otimes_{O_{K^t},\psi_0} R$ acts on the first summand by multiplication with $\varphi_{0,R}(a)$ and an element $a \in O_K \otimes_{O_{K^t},\bar{\psi}_0} R$ acts on the second summand by multiplication with $\bar{\varphi}_{0,R}(a)$.

Next we define the pre-contracting functor in the case where r is special and K/F ramified. In this case we have $\psi_0 = \bar{\psi}_0$. For banal ψ, we keep the definitions (4.3.8) and (4.3.9). The R-module $\mathbf{E}_{A_{\psi_{0,R}}}(P_{\psi_0}/Q_{\psi_0}) \subset P_{\psi_0}/Q_{\psi_0}$ is a direct summand which is locally free of rank 2. We set $P'_{\psi_0} = P_{\psi_0}$ and we define Q'_{ψ_0} as the kernel of (4.3.10). We define F' and $\dot{F}'$ by (4.3.12). Then $P'_{\psi_0}/Q'_{\psi_0} \cong \mathbf{E}_{A_{\psi_{0,R}}}(P_{\psi_0}/Q_{\psi_0})$ is locally free and we define as before $\mathcal{P}' = (P', Q', F', \dot{F}')$, with its O_K-action ι'. It follows from Proposition 4.2.9 that the action of O_F on $\mathbf{E}_{A_{\psi_{0,R}}}(P_{\psi_0}/Q_{\psi_0})$ via ι coincides with the action of via φ_0, i.e., the action via ι' on $\mathcal{P}'$ is strict. That $\mathcal{P}'$ is a display is proved around (4.3.20).

Now we consider the case where $R = k$ is a perfect field in more detail. We know that P_ψ is a free module of rank 2 over the discrete valuation ring $O_K \otimes_{O_{K^t},\tilde{\psi}} W(k)$. Therefore $P_\psi/\Pi P_\psi$ is a k-vector space of dimension two. In the perfect field case, we have now also the operator V,

$$F : P_\psi \longrightarrow P_{\psi\sigma}, \quad V : P_{\psi\sigma} \longrightarrow P_\psi, \quad V(P_{\psi\sigma}) = Q_\psi. \tag{4.3.14}$$

We will see that in all cases the Eisenstein condition implies that $V(P_{\psi\sigma}) \subset \Pi^{a_\psi} P_\psi$. Therefore we may define operators F' and V':

$$F' = \Pi^{a_\psi} F : P_\psi \longrightarrow P_{\psi\sigma}, \quad V' = \Pi^{-a_\psi} V : P_{\psi\sigma} \longrightarrow P_\psi. \tag{4.3.15}$$

The Dieudonné module of the display $\mathcal{P}'$ in the sense of Proposition 3.1.9 will then be (P, F', V').

We begin with the case when r is banal. Now $\mathbf{E}_{A_\psi,k}(T) = T^{a_\psi}$ acts on P_ψ as multiplication by Π^{a_ψ}. By the Eisenstein condition (2.2.13), Π^{a_ψ}annihilates P_ψ/Q_ψ. This implies $\Pi^{a_\psi} P_\psi \subset V(P_{\psi\sigma})$. By the rank condition, the factor $P_\psi/V(P_{\psi\sigma})$ has length $2a_\psi$ as $O_K \otimes_{O_{K^t},\tilde{\psi}} W(k)$-module. Since the same is true for the factor module $P_\psi/\Pi^{a_\psi} P_\psi$, we obtain

$$\Pi^{a_\psi} P_\psi = V(P_{\psi\sigma}). \tag{4.3.16}$$

Therefore $\dot{F}' = \dot{F}\Pi^{a_\psi} = V^{-1}\Pi^{a_\psi} : P_\psi \longrightarrow P_{\psi\sigma}$ is bijective. This shows that $\mathcal{P}'$ is a display. We set

$$F' = \oplus_\psi F\Pi^{a_\psi}, \quad V' = \oplus_\psi \Pi^{-a_\psi} V.$$

Then (P, F', V') is the Dieudonné module associated to $\mathcal{P}'$.

We obtain from (4.3.16) in the ramified case that $V^{2f} P_\psi = \Pi^{2ef} P_\psi = p^f P_\psi$ for all ψ and in the unramified case that $V^{2f} P_\psi = \pi^{ef} P_\psi$. This implies that in both cases $\mathcal{P}$ is isoclinic of slope 1/2.

Now we can verify that $\mathcal{P}'$ is a display for r banal, for an arbitrary $O_{E'}$-algebra R. Let $(\mathcal{P}, \iota) \in \mathfrak{d}\mathfrak{P}_{r,R}$. We must show that $\dot{F} : P \longrightarrow P$ is a Frobenius-linear isomorphism. We may assume that P is a free $W(R)$-module. Let $\det \dot{F}$ be the determinant of the matrix of $\dot{F}$ with respect to any given basis of the $W(R)$-module P. We must verify that $\det \dot{F}$ is a unit in $W(R)$. We have shown that, for each homomorphism $R \longrightarrow k$ to a perfect field k, the image of $\det \dot{F}$ by $W(R) \longrightarrow W(k)$ is a unit in $W(k)$. In particular $\mathbf{w}_0(\det \dot{F}) \in R$ has a nonzero image under any homomorphism $R \longrightarrow k$. But then $\mathbf{w}_0(\det \dot{F})$ is a unit in R, and this implies that $\det \dot{F} \in W(R)$ is a unit. This finishes in the banal case the proof that $\mathcal{P}'$ is a display.

Next we consider the case when r is special and K/F unramified. By our conventions, $\Pi = \pi$ is the prime element of F. Let $R = k$ be a perfect field and let $\mathcal{P} = (P, F, V)$, regarded as a Dieudonné module. If $\psi \in \Psi$ is banal, we find as above that $VP_{\psi\sigma} = \pi^{a_\psi} P_\psi$. Now let $\psi \in \{\psi_0, \bar{\psi}_0\}$. Since $\pi^{a_\psi+1}$ annihilates $\mathrm{Lie}_\psi X$ by the Eisenstein condition (2.2.18), we obtain

$\pi^{a_\psi+1}P_\psi \subset VP_{\psi\sigma}$. We note that P_ψ is a $O_K \otimes_{O_{K^t},\tilde{\psi}} W(k)$-module of rank 2. Therefore the factor module of the last inclusion is, by the rank condition, a $O_K \otimes_{O_{K^t},\tilde{\psi}} W(k)$-module of length 1 and is therefore annihilated by π. This implies

$$\pi^{a_\psi+1}P_\psi \subset VP_{\psi\sigma} \subset \pi^{a_\psi}P_\psi.$$

In particular

$$P_\psi/\pi^{a_\psi}P_\psi \xrightarrow{\sim} \mathrm{Lie}_\psi X/\pi^{a_\psi}\,\mathrm{Lie}_\psi X \tag{4.3.17}$$

is an isomorphism, as claimed in the beginning of the proof of Proposition 4.2.5. By definition (4.3.10) we have $Q'_\psi = \pi^{-a_\psi}VP_{\psi\sigma}$. The map $\dot{F}' = \pi^{a_\psi}\dot{F} : Q'_\psi \longrightarrow P_{\psi\sigma}$ is therefore surjective. Since we know this fact also for banal ψ we conclude that $(P, Q', F', \dot{F}')$ is a display. The associated Dieudonné module is (P, F', V'), where

$$\begin{aligned} F'_\psi &= \pi^{a_\psi}F_\psi : P_\psi \longrightarrow P_{\psi\sigma}, \\ V'_{\psi\sigma} &= \pi^{-a_\psi}V_{\psi\sigma} : P_{\psi\sigma} \longrightarrow P_\psi. \end{aligned} \tag{4.3.18}$$

Now we return to an arbitrary O-algebra R such that p is nilpotent in R. We note that the definition of (P', Q') commutes with arbitrary base change because $Q'_{\psi_0}/I(R)P'_{\psi_0}$ is defined as the kernel of an epimorphism of projective R-modules,

$$P_{\psi_0}/I(R)P_{\psi_0} \longrightarrow \mathbf{E}_{A_{\psi_0}}(P_{\psi_0}/Q_{\psi_0}).$$

We choose a normal decomposition of (P', Q'),

$$P' = T' \oplus L',$$

together with the Frobenius-linear endomorphism $\Phi' : P' \longrightarrow P'$ of the $W(R)$-module P' such that the restriction of Φ' to T' is F' and the restriction to L' is $\dot{F}'$. We have to show that the determinant of Φ in a locally chosen basis is a unit. Since we know that this is true after any base change $R \longrightarrow k$ with k a perfect field, this follows as in the banal case.

We can determine the possible slopes of $\mathcal{P}$ when r is special and K/F unramified. Let $\mathcal{P} = (P, F, V)$ over the perfect field k. By (4.3.15) we have $(V')^{2f} = \pi^{-(ef-1)}V^f$. Let

$$P_{\mathbb{Q}} = \oplus_\lambda N(\lambda)$$

be the decomposition into isoclinic components. Fix $\lambda = r/s$. Then we find a $W(k)$-lattice $\Lambda \subset N(\lambda)$ such that $V^s\Lambda = p^r\Lambda$. From $V^{2fs}\Lambda = p^{2fr}\Lambda$ we obtain

$$(\pi^{ef-1}(V')^{2f})^s\Lambda = p^{2rf}\Lambda, \text{ i.e., } (V')^{2fs}\Lambda = \pi^{-efs}\pi^s p^{2rf}\Lambda.$$

We write the right hand side as $p^{-sf}p^{s/e}p^{2rf}\Lambda$. This shows that $N(\lambda) \subset P'_{\mathbb{Q}}$ is an isoclinic rational Dieudonné submodule of slope

$$\frac{-sf + (s/e) + 2fr}{2fs} = -\frac{1}{2} + \frac{1}{2d} + \lambda.$$

Let us apply the Ahsendorf functor to $\mathcal{P}'$. We obtain a $\mathcal{W}_{O_F}(k)$-Dieudonné module $(P_{\mathrm{c}}, F_{\mathrm{c}}, V_{\mathrm{c}})$ of height 4 and dimension 2. The slopes of P_{c} are by Proposition 3.3.17

$$d(\lambda - \frac{1}{2}) + \frac{1}{2}. \tag{4.3.19}$$

The action of $O_K \otimes_{O_F} W_{O_F}(k) \cong W_{O_F}(k) \times W_{O_F}(k)$ on P_{c} defines a decomposition $P_{\mathrm{c}} = P_{\mathrm{c},0} \oplus P_{\mathrm{c},1}$ such that $V_{\mathrm{c}}(P_{\mathrm{c},0}) \subset P_{\mathrm{c},1}$ and $V_{\mathrm{c}}(P_{\mathrm{c},0}) \subset P_{\mathrm{c},1}$. The $W_{O_F}(k)$-module $P_{\mathrm{c},0}$ with the semi-linear operator V_{c}^2 is of height 2 and dimension 2. Therefore the possible slopes of $(P_{\mathrm{c},0}, V_{\mathrm{c}}^2)$ are with multiplicities $(1,1)$ or $(0,2)$. We conclude that the slopes of $(P_{\mathrm{c}}, V_{\mathrm{c}})$ are with multiplicities $(1/2, 1/2, 1/2, 1/2)$ or $(0,0,1,1)$. From (4.3.19) we find that in the first case all slopes λ of $\mathcal{P}$ are $1/2$, while in the second case we obtain the two slopes $\lambda = 1/2 - 1/2d$ and $\lambda = 1/2 + 1/2d$.

Now we consider the case where r is special and K/F is ramified. As in the last case, it is enough to verify that $\mathcal{P}'$ is a display when $R = k$ is a perfect field. Recall that $a_\psi = e$ for ψ banal and that $a_{\psi_0} = e - 1$. As above we find $VP_{\psi\sigma} = \Pi^e P_\psi$ for ψ banal. By the Eisenstein condition (2.2.16), $\mathcal{L}_{\psi_0}$ is annihilated by Π^{e+1} and the k-vector space $\Pi^{e-1}\mathcal{L}_{\psi_0}$ has dimension at most 2. We consider the following filtration by subvector spaces,

$$\mathcal{L}_{\psi_0} \supset \Pi\mathcal{L}_{\psi_0} \supset \Pi^2\mathcal{L}_{\psi_0} \supset \ldots \supset \Pi^e\mathcal{L}_{\psi_0} \supset \Pi^{e+1}\mathcal{L}_{\psi_0} = 0.$$

We have $\dim_k \Pi^m\mathcal{L}_{\psi_0}/\Pi^{m+1}\mathcal{L}_{\psi_0} \leq 2$ for all $m \geq 0$ since $\mathcal{L}_{\psi_0}$ is a quotient of P_{ψ_0}, which is a free $O_K \otimes_{O_{K^t},\tilde{\psi}_0} W(k)$-module of rank 2. Therefore we find

$$\dim_k \mathcal{L}_{\psi_0} = \dim_k \mathcal{L}_{\psi_0}/\Pi^{e-1}\mathcal{L}_{\psi_0} + \dim_k \Pi^{e-1}\mathcal{L}_{\psi_0} \leq 2(e-1) + 2 = 2e = \dim_k \mathcal{L}_{\psi_0}.$$

We must have equality

$$\dim_k \mathcal{L}_{\psi_0}/\Pi^{e-1}\mathcal{L}_{\psi_0} = 2(e-1), \quad \dim_k \Pi^{e-1}\mathcal{L}_{\psi_0} = 2.$$

The first equation shows that the natural map

$$P_{\psi_0}/\Pi^{e-1}P_{\psi_0} \longrightarrow \mathcal{L}_{\psi_0}/\Pi^{e-1}\mathcal{L}_{\psi_0} \tag{4.3.20}$$

is an isomorphism of vector spaces, as asserted in the beginning of the proof of Proposition 4.2.5. Finally we have by definition $Q'_{\psi_0} = \Pi^{-e+1}Q_{\psi_0} = \Pi^{-e+1}VP_{\psi_0\sigma}$. Therefore

$$\dot{F}' = \Pi^{e-1}V^{-1} : Q'_{\psi_0} \longrightarrow P_{\psi_0\sigma}$$

is bijective. We conclude that $(P', Q', F', \dot{F}')$ is a display. The associated Dieudonné module is (P, F', V'), where

$$\begin{aligned} V' = \Pi^{-e}V : P_{\psi\sigma} &\longrightarrow P_{\psi}, \qquad \psi \neq \psi_0 \\ V' = \Pi^{-e+1}V : P_{\psi_0\sigma} &\longrightarrow P_{\psi_0}. \end{aligned} \tag{4.3.21}$$

As in the unramified case we conclude for an arbitrary $R \in \mathrm{Nilp}_{O_{E'}}$ that our definitions (4.3.8), (4.3.9), (4.3.12) give a display $\mathcal{P}' = (P', Q', F', \dot{F}')$.

In the case of a perfect field k, the slopes of $\mathcal{P}'$ are computed in the same way as in the unramified case. We have the equation $(V')^f = \Pi^{-(ef-1)}V^f$. Let $N(\lambda) \subset P_{\mathbb{Q}}$ be an isoclinic component. We find a lattice $\Lambda \subset N(\lambda)$ such that $V^s\Lambda = p^r\Lambda$. We obtain that

$$(V')^{sf}\Lambda = \Pi^{-efs}\Pi^s p^{rf}\Lambda.$$

Since Π^{2e} and p differ by a unit, this implies that $N(\lambda) \subset P'_{\mathbb{Q}}$ is isoclinic of slope

$$\frac{-(fs/2) + (s/2e) + rf}{sf} = -1/2 + 1/2d + \lambda = (\lambda - 1/2) + 1/2d.$$

If we apply the Ahsendorf functor, we obtain an $\mathcal{W}_{O_F}(k)$-Dieudonné module $(P_{\mathrm{c}}, F_{\mathrm{c}}, V_{\mathrm{c}})$ with slopes $d(\lambda - 1/2) + 1/2$. If we consider the $O_K \otimes_{O_F} W_{O_F}(k)$-module P_{c} with the semi-linear operator V_{c} the possible slopes with multiplicity are $(1,1)$ or $(0,2)$ because $(P_{\mathrm{c}}, V_{\mathrm{c}})$ is of height 2 and dimension 2. If we regard $(P_{\mathrm{c}}, V_{\mathrm{c}})$ over $W_{O_F}(k)$, the heights are multiplied with 2 and then the possible heights are $(1/2, 1/2, 1/2, 1/2)$ or $(0, 0, 1, 1)$. As in the unramified case we conclude that $\mathcal{P}$ is either isoclinic of slope $1/2$ or has exactly two slopes $1/2 - 1/2d$ and $1/2 + 1/2d$.

Finally we consider the case where r is banal and $K = F \times F$. We set $\Theta = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(F^t, \bar{\mathbb{Q}}_p)$. In this case σ will denote the Frobenius automorphism in $\mathrm{Gal}(F^t/F)$. If we compose $\theta \in \Theta$ with the first, resp. second, projection $K^t = F^t \times F^t \longrightarrow F^t$ we obtain $\theta_1, \theta_2 \in \Psi$. Via the first, resp. the second, projection, we obtain isomorphisms

$$(O_F \times O_F) \otimes_{(O_{F^t} \times O_{F^t}),\theta_i} O_{E'} \cong O_F \otimes_{O_{F^t},\theta} O_{E'}, \quad i = 1, 2.$$

This leads to the decomposition

$$O_K \otimes O_{E'} = \prod_{\psi \in \Psi} O_K \otimes_{O_{K^t},\psi} O_{E'} = \big(\prod_{\theta \in \Theta} O_F \otimes_{O_{F^t},\theta} O_{E'}\big) \times \big(\prod_{\theta \in \Theta} O_F \otimes_{O_{F^t},\theta} O_{E'}\big). \tag{4.3.22}$$

Assume that $\psi \in \Psi$ factors through $\theta \in \Theta$. We define $\tilde{\psi}$ as the composite

$$O_{K^t} = O_{F^t} \times O_{F^t} \xrightarrow{\text{proj.}} O_{F^t} \longrightarrow W(O_{F^t}) \xrightarrow{W(\theta)} W(O_{E'}). \tag{4.3.23}$$

The first map is the projection to the first or second factor according to ψ. We denote by $\tilde{\theta}$ the composite of the last two arrows in (4.3.23). We obtain the decomposition

$$\begin{aligned} O_K \otimes_{\mathbb{Z}} W(O_{E'}) &= \prod_{\psi \in \Psi} O_K \otimes_{O_{K^t}, \tilde{\psi}} W(O_{E'}) \\ &= \Big(\prod_{\theta \in \Theta} O_F \otimes_{O_{F^t}, \tilde{\theta}} W(O_{E'}) \Big) \times \Big(\prod_{\theta \in \Theta} O_F \otimes_{O_{F^t}, \tilde{\theta}} W(O_{E'}) \Big). \end{aligned} \tag{4.3.24}$$

On the right hand side, the first set of factors correspond to those ψ which factor over the first projection and the second set of factors correspond to those ψ which factor over the second projection.

We consider a CM-pair $(\mathcal{P}, \iota)$ over $R \in \mathrm{Nilp}_{O_{E'}}$ which satisfies the Eisenstein condition. By (4.3.22) we obtain a decomposition

$$P = P_1 \times P_2 = (\bigoplus_{\theta \in \Theta} P_{1,\theta}) \oplus (\bigoplus_{\theta \in \Theta} P_{2,\theta}). \tag{4.3.25}$$

This decomposition corresponds to the decomposition into displays $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ induced by the $O_F \times O_F$-action on $\mathcal{P}$. By the definition of a CM-pair (at the beginning of subsection 2.3), the displays $\mathcal{P}_1$ and $\mathcal{P}_2$ have both height $2d$.

The maps F and $\dot{F}$ of the display $\mathcal{P}$ induce maps

$$F : P_{i,\theta} \longrightarrow P_{i,\theta\sigma}, \quad \dot{F} : Q_{i,\theta} \longrightarrow P_{i,\theta\sigma}. \tag{4.3.26}$$

The polynomial $\tilde{\mathbf{E}}_{A_\psi} \in W(O_{E'})[T]$ is defined as before, cf. (4.3.3). For $i = 1, 2$ we define the displays $\mathcal{P}'_i = (P'_i, Q'_i, F'_i, \dot{F}'_i)$ as follows

$$P'_i = Q'_i = P_i, \quad \dot{F}'_i(x) = \dot{F}(\tilde{\mathbf{E}}_{A_{\theta_i}} x), \; F'_i(x) = F(\tilde{\mathbf{E}}_{A_{\theta_i}} x), \quad x \in P_{i\theta}.$$

Here, by the convention (4.3.7), $\tilde{\mathbf{E}}_{A_{\theta_i}}$ acts as the multiplication by $\tilde{\mathbf{E}}_{A_{\theta_i}, R}(\pi \otimes 1) \in O_F \otimes_{O_{F^t}, \tilde{\theta}_i} W(R)$. We set $\mathcal{P}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2$. As in the unramified banal case, the verification that $\mathcal{P}'$ is a display reduces to the case of a perfect field. However, when R is a $\kappa_{E'}$-algebra, then $\tilde{\mathbf{E}}_{A_{\theta_i}, R}(\pi \otimes 1) = \pi^{a_{\theta_i}} \otimes 1$. If $R = k$ is a perfect field, we consider the Dieudonné module (P_i, F_i, V_i) of $\mathcal{P}_i$. We have

$$V_i(P_{i,\theta\sigma}) = \pi^{a_{\theta_i}} P_{i,\theta}. \tag{4.3.27}$$

We define

$$F'_i = \pi^{a_{\theta_i}} F_i : P_{i,\theta} \longrightarrow P_{i,\theta\sigma}, \quad V'_i = \pi^{-a_{\theta_i}} V_i : P_{i,\theta\sigma} \longrightarrow P_{i,\theta}.$$

Then (P_i, F'_i, V'_i) is the Dieudonné module of $\mathcal{P}'_i$. Finally we determine the slopes of $\mathcal{P}$. If we iterate (4.3.27) we find

$$V^f P_i = \pi^{\sum_\theta a_{\theta_i}} P_i. \tag{4.3.28}$$

We set $g_i = \sum_\theta a_{\theta_i}$. Then $g_1 + g_2 = ef$ because $a_{\theta_1} + a_{\theta_2} = e$. We see that $\mathcal{P}_i$ is isoclinic of slope $\lambda_i = g_i/d$ and that $\lambda_1 + \lambda_2 = 1$.

We summarize the properties of our constructions.

Definition 4.3.1. Let $R \in \mathrm{Nilp}_{O_{E'}}$. We define categories $\mathfrak{d}\mathfrak{P}'_{r,R}$ and $\mathfrak{P}'_{r,R}$ as follows.

(1) If r is banal, then $\mathfrak{d}\mathfrak{P}'_{r,R}$ is the category of pairs $(\mathcal{P}', \iota')$, where $\mathcal{P}'$ is an étale display (i.e., $P' = Q'$) of height $4d$ and where ι' is an O_K-action. In the split case $O_K = O_F \times O_F$, we require in addition that in the induced decomposition $\mathcal{P}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2$ both factors have height $2d$.

(2) If r is special and K/F is unramified, then the category $\mathfrak{d}\mathfrak{P}'_{r,R}$ is the category of pairs $(\mathcal{P}', \iota')$, where $\mathcal{P}'$ is a display of height $4d$ and dimension 2 with an action $\iota' : O_K \longrightarrow \mathrm{End}\, \mathcal{P}'$ such that the action of ι' restricted to O_F is strict with respect to $\varphi_{0,R} : O_F \xrightarrow{\varphi_0} O_{E'} \longrightarrow R$ and such that $\mathrm{Lie}\, \mathcal{P}' = P'/Q'$ is locally on $\mathrm{Spec}\, R$ a free $O_K \otimes_{O_F, \varphi_{0,R}} R$-module of rank 1.

(3) If r is special and K/F is ramified, then the category $\mathfrak{d}\mathfrak{P}'_{r,R}$ is the category of pairs $(\mathcal{P}', \iota')$, where $\mathcal{P}'$ is a display of height $4d$ and dimension 2 with an action $\iota' : O_K \longrightarrow \mathrm{End}\, \mathcal{P}'$ such that the action of ι' restricted to O_F is strict with respect to $\varphi_{0,R} : O_F \xrightarrow{\varphi_0} O_{E'} \longrightarrow R$.

The category $\mathfrak{P}'_{r,R}$ is the category of formal p-divisible groups X' with an O_K-action ι' such that the associated display $(\mathcal{P}', \iota')$ is an object of $\mathfrak{dP}'_{r,R}$.

Let r be special. We call $(\mathcal{P}, \iota)$ *supersingular* if $(\mathcal{P}', \iota')$ satisfies the nilpotence condition. We denote the full subcategory of supersingular objects of $\mathfrak{dP}_{r,R}$ by $\mathfrak{dP}^{\mathrm{ss}}_{r,R}$.

Theorem 4.3.2. *Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements in R is nilpotent. The construction above defines the* pre-contracting functor[5]

$$\mathfrak{C}'_{r,R} : \mathfrak{dP}_{r,R} \longrightarrow \mathfrak{dP}'_{r,R}$$

which commutes with arbitrary base change with respect to R. Furthermore,

(i) *if r is banal, the functor $\mathfrak{C}'_{r,R}$ is an equivalence of categories.*

(ii) *if r is special and the ring R is reduced, the functor $\mathfrak{C}'_{r,R}$ is an equivalence of categories.*

(iii) *if r is special and R is arbitrary, $\mathfrak{C}'_{r,R}$ induces an equivalence of categories*

$$\mathfrak{C}'_{r,R} : \mathfrak{dP}^{\mathrm{ss}}_{r,R} \longrightarrow \mathfrak{dP}'^{\,\mathrm{nilp}}_{r,R},$$

where the right hand side is the full subcategory of nilpotent displays.

Let r be special and K/F be ramified. Let $(\mathcal{P}, \iota) \in \mathfrak{dP}_{r,R}$ and let $(\mathcal{P}', \iota')$ be its image by $\mathfrak{C}'_{r,R}$. Then $(\mathcal{P}, \iota)$ satisfies (KC_r) *if and only if*

$$\mathrm{Tr}_R(\iota'(\Pi) \mid P'/Q') = 0.$$

Before proving this, we state a Corollary which we already proved in the construction of $\mathfrak{C}'_{r,R}$ above.

Corollary 4.3.3. *Let $k \in \mathrm{Nilp}_{O_{E'}}$ be a perfect field. Let $\mathcal{P} \in \mathfrak{dP}_{r,k}$ and let $\mathcal{P}'$ be its image by the functor $\mathfrak{C}'_{r,R}$.*

(1) Let r be banal and K/F a field extension. Then the display $\mathcal{P}$ is isoclinic of slope $1/2$ and $\mathcal{P}'$ is étale.

(2) Let K/F be split (and then r is banal). Then $\mathcal{P}$ decomposes into $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$, where $\mathcal{P}_1$ is isoclinic of slope λ and $\mathcal{P}_2$ is isoclinic of slope $1-\lambda$. The number λ depends only on r. The display $\mathcal{P}'$ is étale.

(3) Let r be special. Then $\mathcal{P}$ is either isoclinic of slope $1/2$ (supersingular case) or it has the two slopes $1/2 - 1/2d$ and $1/2 + 1/2d$ with the same multiplicity. In the first case $\mathcal{P}'$ is isoclinic of slope $1/2d$. In the second case it has the two slopes $0, 1/d$ with the same multiplicity.

□

Proof. We still have to prove the claimed equivalences of categories. We begin with the case where R is reduced.

Let us consider first the case where r is special and K/F unramified. It is enough to invert the construction of the functor $\mathfrak{C}'_{r,R}$. For any ψ, $P_\psi/I(R)P_\psi$ is locally on $\mathrm{Spec}\, R$ a free $(O_K/pO_K) \otimes_{\kappa_K, \psi} R$-module of rank 2, cf. Lemma 3.1.15. Let $\mathcal{P}' = (P', Q', F', \dot{F}')$ be an object of $\mathfrak{dP}'_{r,R}$. We define as follows an object $\mathcal{P} = (P, Q, F, \dot{F})$ of $\mathfrak{dP}_{r,R}$ such that $\mathcal{P}'$ is the image of $\mathcal{P}$ by the functor $\mathfrak{C}'_{r,R}$. We set $P_\psi = P'_\psi$ for $\psi \in \Psi$, and for $\psi \notin \{\psi_0, \bar{\psi}_0\}$ we set

$$Q_\psi = \pi^{a_\psi} P_\psi + I(R)P_\psi. \tag{4.3.29}$$

Since $P_\psi = Q'_\psi$, we have $\dot{F}'(I(R)P_\psi) \subset W(R)F'P_\psi \subset pP_{\psi\sigma}$. By (4.3.29) we find $\dot{F}'(Q_\psi) \subset \pi^{a_\psi} P_{\psi\sigma}$. Since p is not a zero divisor in $W(R)$, the element $\pi \in O_K$ acts injectively on P_{ψ_0}. Therefore we may define

$$\dot{F} = \pi^{-a_\psi} \dot{F}' : Q_\psi \longrightarrow P_{\psi\sigma}, \quad F = \pi^{-a_\psi} F' : Q_\psi \longrightarrow P_{\psi\sigma}.$$

If $\psi \in \{\psi_0, \bar{\psi}_0\}$ we consider the split homomorphism of R-modules

$$\pi^{a_\psi} : P_\psi/I(R)P_\psi \longrightarrow P_\psi/I(R)P_\psi. \tag{4.3.30}$$

It is split because $P_\psi/I(R)P_\psi$ is a free $(O_K/pO_K) \otimes_{\kappa_K, \psi} R$-module. We set

$$Q_\psi = \pi^{a_\psi} Q'_\psi + I(R)P_\psi.$$

[5]Later we will also have a *contracting functor* $\mathfrak{C}_{r,R}$, which explains our notation.

If we apply $\dot{F}'$ to the last equation we obtain that $\dot{F}'(Q_\psi) \subset \pi^{a_\psi} P_{\psi\sigma}$. Indeed, because the action of O_F on $\mathcal{P}'$ is strict π annihilates P_ψ/Q'_ψ. We conclude that $F'(\pi P_\psi) \subset F'(Q'_\psi) = p\dot{F}'(Q_\psi) \subset pP_{\psi\sigma}$ and therefore $F'(P_\psi) \subset \pi^{e-1}P_\psi \subset \pi^{a_\psi}P_\psi$. This justifies the following definition:

$$\dot{F} := \pi^{-a_\psi}\dot{F}' : Q_\psi \longrightarrow P_{\psi\sigma}, \quad F := \pi^{-a_\psi}F' : P_\psi \longrightarrow P_{\psi\sigma}.$$

It is obvious that we obtain a display $\mathcal{P} = (P, Q, F, \dot{F})$. We need to verify that the condition (EC_r) is satisfied. We check the conditions (2) and (3) of Proposition 4.2.7. By definition of $\mathfrak{d}\mathfrak{P}'_{r,R}$, the R-module P'_ψ/Q'_ψ is annihilated by π. The kernel of (4.3.30) is $\pi^{e-a_\psi}P_\psi$ and therefore contained in $Q'_\psi/I(R)P_\psi$. The image of the last module by (4.3.30) is therefore a direct summand of $P_\psi/I(R)P_\psi$. This image is $Q_\psi/I(R)P_\psi$. Therefore condition (2) holds. Moreover, we obtain an isomorphism

$$P_\psi/Q'_\psi \xrightarrow{\sim} \pi^{a_\psi}P_\psi + I(R)P_\psi/Q_\psi.$$

In particular, the last module is locally free of rank 1 and the action of π on this module coincides with multiplication by $\varphi_0(\pi)$ if $\psi = \psi_0$, resp., by $\bar{\varphi}_0(\pi)$ if $\psi = \bar{\psi}_0$. Hence condition (3) holds.

In the split case the same arguments hold but we need only the easy part because ψ_0 and $\bar{\psi}_0$ don't exist.

Next we consider the case where r is special and K/F ramified. Again we reverse the construction of the functor $\mathfrak{C}'_{r,R}$. Let $(P', Q', F', \dot{F}')$ be an object of $\mathfrak{d}\mathfrak{P}'_{r,R}$. We associate to it as follows an object $(P, Q, F, \dot{F}) \in \mathfrak{d}\mathfrak{P}_{r,R}$. We set $P_\psi = P'_\psi$ for all $\psi \in \Psi$. Assume that $\psi \neq \psi_0$. We have $Q'_\psi = P'_\psi$ because the action of O_{F^t} is strict. We set

$$Q_\psi = \Pi^e P'_\psi + I(R)P'_\psi.$$

It follows from Lemma 3.1.15 that $P_\psi/I(R)P_\psi$ is locally on $\operatorname{Spec} R$ a free $(O_K/pO_K) \otimes_{\kappa_K,\psi} R$-module. Therefore P_ψ/Q_ψ is a locally free R-module. From $F'P_\psi = p\dot{F}'P_\psi$, we find that $\dot{F}'Q_\psi \subset \Pi^e P'_{\psi\sigma}$. Since R is reduced, the map $\Pi^e : P_{\psi\sigma} \longrightarrow \Pi^e P_{\psi\sigma}$ is bijective. Therefore we may define

$$\dot{F} := \Pi^{-e}\dot{F}' \colon Q_\psi \longrightarrow P_{\psi\sigma}.$$

It is clear that this map is a Frobenius-linear epimorphism. Next, we set

$$P_{\psi_0} = P'_{\psi_0}, \quad Q_{\psi_0} = \Pi^{e-1}Q'_{\psi_0} + I(R)P'_{\psi_0}.$$

Since the action of O_F on $\mathcal{P}'$ is strict, we find

$$\Pi^{e+1}P'_{\psi_0} \subset \Pi^2 P'_{\psi_0} \subset Q'_{\psi_0}. \tag{4.3.31}$$

We consider the split homomorphism of R-modules,

$$\Pi^{e-1} \colon P'_{\psi_0}/I(R)P'_{\psi_0} \longrightarrow P'_{\psi_0}/I(R)P'_{\psi_0}. \tag{4.3.32}$$

The kernel of this map is the image of Π^{e+1}. Therefore the kernel is contained in $Q'_{\psi_0}/I(R)P'_{\psi_0}$. This implies that the image of $Q'_{\psi_0}/I(R)P'_{\psi_0}$ under (4.3.32) is a direct summand of $P'_{\psi_0}/I(R)P'_{\psi_0}$. Hence the cokernel P_{ψ_0}/Q_{ψ_0} is a locally free R-module. We apply F' to (4.3.31) and obtain

$$\Pi^2 F'P'_{\psi_0} \subset F'Q'_{\psi_0} = p\dot{F}'Q'_{\psi_0} \subset pP'_{\psi_0\sigma}.$$

Using this, we get

$$\dot{F}'Q_{\psi_0} = \dot{F}'\big(\Pi^{e-1}Q'_{\psi_0} + I(R)P'_{\psi_0}\big) \subset \Pi^{e-1}P'_{\psi_0\sigma} + F'P'_{\psi_0} \subset \Pi^{e-1}P'_{\psi_0\sigma}.$$

It follows that the following definitions of maps $Q_{\psi_0} \longrightarrow P_{\psi_0}$, resp. $P_{\psi_0} \longrightarrow P_{\psi_0}$, make sense:

$$\dot{F} = (1/\Pi^{e-1})\dot{F}', \quad F = (1/\Pi^{e-1})F'.$$

Therefore we have defined $\mathcal{P} = (P, Q, F, \dot{F})$. It is clear that we obtain a display. We have to verify the condition (EC_r). Only $(\mathrm{EC}_{\psi_0,r})$ is not completely obvious. We prove the conditions of Proposition 4.2.9. By the R-module homomorphism (4.3.32), $P'_{\psi_0}/I(R)P'_{\psi_0}$ is mapped to the direct summand $(\Pi^{e-1}P'_{\psi_0} + I(R)P'_{\psi_0})/I(R)P'_{\psi_0}$, and $Q'_{\psi_0}/I(R)P'_{\psi_0}$ is mapped to the direct summand $Q_{\psi_0}/I(R)P'_{\psi_0}$. We obtain an isomorphism

$$P'_{\psi_0}/Q'_{\psi_0} \xrightarrow{\sim} (\Pi^{e-1}P_{\psi_0} + I(R)P_{\psi_0})/Q_{\psi_0}.$$

Therefore by the strictness of the O_F-action, the right hand side is a locally free R-module of rank 2 and $\iota(\pi)$ acts on the right hand side as $\varphi_0(\pi)$. These are exactly the conditions of Proposition 4.2.9. The rank condition is now obvious for $(P, Q, F, \dot{F})$.

Finally, in the case where r is banal, including the split case (and R is reduced), we can reverse the functor $\mathfrak{C}'_{r,R}$ using the arguments for banal ψ given above.

Now we consider assertion (iii) of Theorem 4.3.2 when R is not reduced. It follows from Corollary 4.3.3 (3) that $\mathcal{P}$ is isoclinic of slope $1/2$ because $\mathcal{P}'$ is nilpotent. Therefore we may apply Grothendieck-Messing for displays Corollary 3.1.14. We consider a surjective homomorphism $S \longrightarrow R$ of $O_{E'}$-algebras and assume that the kernel $\mathfrak{a}$ is endowed with a divided power structure.

We define the category $\mathfrak{dP}_{r,S/R}$ as the full subcategory of the category of pairs $(\mathcal{P}_1, \iota_1)$ where $\mathcal{P}_1$ is a $\mathcal{W}(S/R)$-display, cf. Example 3.1.3, and where

$$\iota_1 : O_K \longrightarrow \operatorname{End} \mathcal{P}_1$$

is an action such that the base change $(\mathcal{P}, \iota)$ of such a pair by the morphism of frames $\mathcal{W}(S/R) \longrightarrow \mathcal{W}(R)$ lies in the category $\mathfrak{dP}_{r,R}$. We also say that $(\mathcal{P}_1, \iota_1)$ is a lift of $(\mathcal{P}, \iota)$ to a relative display. By Theorem 3.1.12, the lift $(\mathcal{P}_1, \iota_1)$ is uniquely determined by $(\mathcal{P}, \iota)$ if $\mathcal{P}$ satisfies the nilpotence condition.

In the same way we define the category $\mathfrak{dP}'_{r,S/R}$. Then the functor $\mathfrak{C}'_{r,R}$ of Theorem 4.3.2 extends to a functor

$$\mathfrak{C}'_{r,S/R} : \mathfrak{dP}_{r,S/R} \longrightarrow \mathfrak{dP}'_{r,S/R}. \tag{4.3.33}$$

Indeed, the definition of $\mathfrak{C}'_{r,S/R}$ is essentially the same as that of $\mathfrak{C}'_{r,R}$. We indicate it in the case where K/F is unramified or split. By the O_K-action, we have for the relative display $\mathcal{P}_1$ a decomposition,

$$P_1 = \oplus_\psi P_{1,\psi}, \quad Q_1 = \oplus_\psi Q_{1,\psi}.$$

We are going to define a $\mathcal{W}(S/R)$-display $\mathcal{P}'_1 = (P'_1, Q'_1, F'_1, \dot{F}'_1)$. We set $P'_{1,\psi} = P_{1,\psi}$ for all $\psi \in \Psi$. Since $\mathcal{P}_1$ is a lifting of $\mathcal{P}$, we have a natural isomorphism

$$P_{1,\psi}/Q_{1,\psi} = P_\psi/Q_\psi. \tag{4.3.34}$$

If $\psi \notin \{\psi_0, \bar{\psi}_0\}$, we set $Q'_{1,\psi} = P'_{1,\psi}$. By the condition (EC_r) for $\mathcal{P}$ we conclude that $\tilde{\mathbf{E}}_{A_\psi} P_{1,\psi} \subset Q_{1,\psi}$. Therefore we can define

$$\begin{aligned} &F'_1 \colon P'_{1,\psi} \longrightarrow P_{1,\psi\sigma}, \quad F'_1(x) = F_1(\tilde{\mathbf{E}}_{A_\psi} x), \quad x \in P'_{1,\psi} \\ &\dot{F}'_1 \colon Q'_{1,\psi} \longrightarrow P'_{1,\psi\sigma}, \quad \dot{F}'_1(x) = \dot{F}_1(\tilde{\mathbf{E}}_{A_\psi} x), \quad x \in Q'_{1,\psi}. \end{aligned} \tag{4.3.35}$$

In the split case this decribes $\mathcal{P}'_1$ already completely. Now we consider the case $\psi \in \{\psi_0, \bar{\psi}_0\}$. Then we define $Q'_{1,\psi}$ as the kernel of the map

$$P_{1,\psi} \longrightarrow P_\psi/Q_\psi \xrightarrow{\mathbf{E}_{A_\psi}} P_\psi/Q_\psi.$$

This implies $\tilde{\mathbf{E}}_{A_\psi} Q'_{1,\psi} \subset Q_{1,\psi}$. Therefore we can define

$$\begin{aligned} &F'_1 \colon P'_{1,\psi} \longrightarrow P'_{1,\psi\sigma}, \quad F'_1(x) = F_1(\tilde{\mathbf{E}}_{A_\psi} x), \quad x \in P'_{1,\psi}, \\ &\dot{F}'_1 \colon Q'_{1,\psi} \longrightarrow P'_{1,\psi_0\sigma}, \quad \dot{F}'_1(y) = \dot{F}_1(\tilde{\mathbf{E}}_{A_\psi} y), \quad y \in Q'_{1,\psi}. \end{aligned} \tag{4.3.36}$$

We then define $\mathcal{P}'_1 = (P'_1, Q'_1, F'_1, \dot{F}'_1)$, where $P'_1 = \oplus P'_{1,\psi}$ and $Q'_1 = \oplus Q'_{1,\psi}$. In the ramified case the same definition holds with slight modifications.

The functor $\mathfrak{C}'_{r,S/R}$ defines a natural isomorphism

$$\mathbb{D}_{\mathcal{P}}(S) \cong \mathbb{D}_{\mathcal{P}'}(S). \tag{4.3.37}$$

This relates deformations of $\mathcal{P}$ and deformations of its image $\mathcal{P}'$ under $\mathfrak{C}_{r,R}$ since $\mathcal{P}$ and $\mathcal{P}'$ are nilpotent.

Let $(\mathcal{P}, \iota) \in \mathfrak{dP}^{\mathrm{ss}}_{r,R}$. It has a unique lift $\mathcal{P}_1 \in \mathfrak{dP}_{r,S/R}$. The image $\mathcal{P}'_1$ by the functor $\mathfrak{C}'_{r,S/R}$ is the unique lift of $\mathcal{P}'$ to an object of $\mathfrak{dP}'_{r,S/R}$, cf. Theorem 3.1.12.

Let us fix $\mathcal{P}$. Let $\mathcal{M}$ be the set of all isomorphism classes of deformations of $(\mathcal{P}, \iota)$ to an object in $\mathfrak{d}\mathfrak{P}_{r,S}$. Let $\mathcal{M}'$ be the set of isomorphism classes of deformations of $(\mathcal{P}', \iota')$ to an object of $\mathfrak{d}\mathfrak{P}'_{r,S/R}$. We claim that the functor $\mathfrak{C}'_{r,S}$ defines a bijection,

$$\mathfrak{C}'_{r,S} : \mathcal{M} \longrightarrow \mathcal{M}'. \tag{4.3.38}$$

We indicate this when K/F is unramified. Let $\bar{Q} \subset \mathbb{D}_{\mathcal{P}}(R) = P/I(R)P$ be the image of Q, i.e., the Hodge filtration. The set $\mathcal{M}$ is identified with the set of liftings of $\bar{Q}$ to a direct summand $\bar{Q}_1 \subset \mathbb{D}_{\mathcal{P}}(S) = P_1/I(S)P_1$ which is a $O_K \otimes_{\mathbb{Z}_p} S$-submodule and such that the factor module satisfies the Eisenstein condition. The O_K-action gives a decomposition $\bar{Q}_1 = \oplus \bar{Q}_{1,\psi}$. For ψ banal, we must have by Proposition 4.2.7 that

$$\tilde{\mathbf{E}}_{A_\psi} \mathbb{D}_{\mathcal{P}}(S)_\psi = \bar{Q}_{1,\psi}. \tag{4.3.39}$$

We note that the left hand side is a direct summand of $\mathbb{D}_{\mathcal{P}}(S)_\psi$ as an S-module. This follows from the fact that $P_{1,\psi}/I(S)P_{1,\psi}$ is a free module over $S[T]/\mathbf{E}_\psi S[T]$. Therefore, there is exactly one possibility to lift the ψ-component of the Hodge filtration. We consider now liftings of $\bar{Q}_\psi$ when ψ is not banal. In this case the Eisenstein condition implies that

$$\mathbf{S}_\psi \tilde{\mathbf{E}}_{A_\psi} \mathbb{D}_{\mathcal{P}}(S)_\psi \subset \bar{Q}_{1,\psi} \subset \tilde{\mathbf{E}}_{A_\psi} \mathbb{D}_{\mathcal{P}}(S)_\psi.$$

By the freeness of $\mathbb{D}_{\mathcal{P}}(S)_\psi$ just mentioned, the multiplication by $\tilde{\mathbf{E}}_{A_\psi}$ gives an isomorphism

$$\tilde{\mathbf{E}}_{A_\psi} : \mathbb{D}_{\mathcal{P}}(S)_\psi / \mathbf{S}_\psi \mathbb{D}_{\mathcal{P}}(S)_\psi \cong \tilde{\mathbf{E}}_{A_\psi} \mathbb{D}_{\mathcal{P}}(S)_\psi / \mathbf{S}_\psi \tilde{\mathbf{E}}_{A_\psi} \mathbb{D}_{\mathcal{P}}(S)_\psi.$$

This shows that it is the same thing to lift $\bar{Q}_\psi$ to a direct summand $\bar{Q}_{1,\psi} \subset \mathbb{D}_{\mathcal{P}}(S)_\psi$ such that the Eisenstein condition is satisfied or to lift $\tilde{\mathbf{E}}_{A_\psi}^{-1} \bar{Q}_\psi$ to a direct summand $\bar{Q}'_{1,\psi}$ such that $\mathbb{D}_{\mathcal{P}}(S)_\psi / \bar{Q}'_{1,\psi}$ is annihilated by $\mathbf{S}_\psi$. The last condition means that the action of O_F is strict with respect to φ_0, resp., $\bar{\varphi}_0$. In other words,

$$\bar{Q}'_1 = \bar{Q}_{1,\psi_0} \oplus \bar{Q}_{1,\bar{\psi}_0} \oplus (\oplus_{\psi \neq \psi_0 \bar{\psi}_0} \mathbb{D}_{\mathcal{P}}(S)_\psi)$$

is a lift of the Hodge filtration $\bar{Q}' \subset \mathbb{D}_{\mathcal{P}'}(R) = P/I(R)P$ to a Hodge filtration $\bar{Q}'_1 \subset P_1/I(S)P_1$ such that the action of O_F is strict, i.e., the Hodge filtration $\bar{Q}'_1$ defines a point of $\mathcal{M}'$. This shows that (4.3.38) is bijective because the functor $\mathfrak{C}'_{r,S}$ maps the Hodge filtration $\bar{Q}_{1,\psi}$ to $\mathbf{E}_{A_\psi}^{-1} \bar{Q}_{1,\psi}$ when ψ is special by the definition (4.3.10). We leave the ramified case to the reader.

Finally we prove assertion (i) of Theorem 4.3.2, i.e., we assume that r is banal. We begin with the case where K/F is a field extension. Then $\mathcal{P}$ is by Corollary 4.3.3 (1) of slope 1/2. By (4.3.39) there is a unique way to lift the Hodge filtration and therefore the Grothendieck-Messing criterion implies that there is a unique way to lift $\mathcal{P}$ to an object $\mathcal{P}_1 \in \mathfrak{d}\mathfrak{P}_{r,S/R}$. On the other hand $\mathcal{P}'$ is étale. Therefore it lifts obviously uniquely, and (i) follows. In the case where K/F is split the same argument applies if $\mathcal{P}$ is local. If not, we consider the decomposition $\mathcal{P} = \mathcal{P}_\alpha \oplus \mathcal{P}_\beta$ induced by $O_K = O_F \times O_F$. By Corollary 4.3.3 (2), in each geometric point of $\operatorname{Spec} R$ one of the factors of this decomposition is isoclinic of slope 0 and the other is isoclinic of slope 1. That P_α is étale means that the locally free module P_α/Q_α is zero. This is true on an open and closed subset of $\operatorname{Spec} R$. Therefore we may assume without loss of generality that P_α is étale. Then $\mathcal{P}_\alpha$ has a unique lift and $\mathcal{P}_\beta$ has a unique lift by Grothendieck-Messing. Since $\mathcal{P}'$ is étale it has also a unique lift. This completes the proof in the split case. □

Corollary 4.3.4. *Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements in R is nilpotent. We denote by $\mathfrak{P}^{\mathrm{ss}}_{r,R}$ the full subcategory of objects of $\mathfrak{P}_{r,R}$ whose displays lie in $\mathfrak{d}\mathfrak{P}^{\mathrm{ss}}_{r,R}$. Let $\mathfrak{P}'^{\mathrm{form}}_{r,R}$ be the full subcategory of $\mathfrak{P}'_{r,R}$ whose objects are formal p-divisible groups. Then $\mathfrak{C}'_{r,R}$ induces an equivalence of categories*

$$\mathfrak{C}'_{r,R} : \mathfrak{P}^{\mathrm{ss}}_{r,R} \longrightarrow \mathfrak{P}'^{\mathrm{form}}_{r,R}.$$

□

4.4. **The contracting functor in the case of a special CM-type.** In this subsection, r will denote a special CM-type. In this case, we will compose the functor $\mathfrak{C}'_{r,R}$ with the Ahsendorf functor.

Definition 4.4.1. Let r be special. Let $R \in \mathrm{Nilp}_{O_F}$. We denote by $\mathfrak{dR}_R$[6] the category of $\mathcal{W}_{O_F}(R)$-displays $\mathcal{P}_{\mathrm{c}}$ endowed with a homomorphism of O_F-algebras

$$\iota_{\mathrm{c}} : O_K \longrightarrow \mathrm{End}\,\mathcal{P}_{\mathrm{c}},$$

such that $\mathcal{P}_{\mathrm{c}}$ is of height 4 and dimension 2. In the case where K/F is unramified, we require moreover that $\mathrm{Lie}\,\mathcal{P}_{\mathrm{c}}$ is locally on $\mathrm{Spec}\,R$ a free $O_K \otimes_{O_F} R$-module of rank 1.

We note that in the ramified case, the $O_K \otimes_{O_F} R$-module $\mathrm{Lie}\,\mathcal{P}_{\mathrm{c}}$ is in general not locally free on $\mathrm{Spec}\,R$.

Definition 4.4.2. Let r be special. Let R be a $O_{E'}$-algebra. We regard R as a O_F-algebra via $\varphi_{0,R} : O_F \xrightarrow{\varphi_0} O_E \longrightarrow R$. The contracting functor

$$\mathfrak{C}_{r,R} : \mathfrak{dP}_{r,R} \longrightarrow \mathfrak{dR}_R$$

is the composition of $\mathfrak{C}'_{r,R}$ with the Ahsendorf functor $\mathfrak{A}_{O_F/\mathbb{Z}_p,R}$.

Theorem 4.4.3. *Let r be special. Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements of R is nilpotent. Then the functor $\mathfrak{C}_{r,R}$ induces an equivalence of categories*

$$\mathfrak{C}_{r,R} : \mathfrak{dP}^{\mathrm{ss}}_{r,R} \longrightarrow \mathfrak{dR}^{\mathrm{nilp}}_R.$$

Here $\mathfrak{dR}^{\mathrm{nilp}}_R$ denotes the full subcategory of nilpotent displays in $\mathfrak{dR}_R$.

Proof. This follows from Proposition 4.3.2 and Theorem 3.3.2. □

Remark 4.4.4. Let $R = k$ be a perfect field with an $O_{E'}$-algebra structure. Then the construction of the functor $\mathfrak{C}_{r,k}$ simplifies.

We begin with the unramified case. Let $\mathcal{P} = (P, F, V) \in \mathfrak{dP}_{r,k}$ viewed as a Dieudonné module. The display $\mathcal{P}' = (P, F', V')$ is described after (4.3.17). Applying the Ahsendorf functor to it, we obtain the image $\mathcal{P}_{\mathrm{c}} = (P_{\mathrm{c}}, F_{\mathrm{c}}, V_{\mathrm{c}})$ of $\mathcal{P}$ by the functor $\mathfrak{C}_{r,k}$. The P_0 of (3.3.33) is in our case $P_{\mathrm{c}} = P_{\psi_0} \oplus P_{\bar{\psi}_0}$ and $V_{\mathrm{c}} = (V')^f$ is the V_{a} of (3.3.33). We know that the restriction of V' to $P_{\psi\sigma}$ is

$$V' = \pi^{-a_\psi} V : P_{\psi\sigma} \longrightarrow P_\psi.$$

We conclude that $(V')^f : P_{\psi_0} \longrightarrow P_{\bar{\psi}_0}$ is equal to $\pi^{-g_{\bar{\psi}_0}} V^f$ where

$$\begin{aligned} g_{\bar{\psi}_0} &= a_{\bar{\psi}_0} + a_{\bar{\psi}_0\sigma} + \ldots + a_{\bar{\psi}_0\sigma^{f-1}} \\ &= a_{\psi_0\sigma^{-f}} + a_{\psi_0\sigma^{-(f-1)}} + \ldots + a_{\psi_0\sigma^{-1}}. \end{aligned} \tag{4.4.1}$$

In the same way $(V')^f : P_{\bar{\psi}_0} \longrightarrow P_{\psi_0}$ is equal to $\pi^{-g_{\psi_0}} V^f$ where

$$g_{\psi_0} = a_{\psi_0} + a_{\psi_0\sigma} + \ldots + a_{\psi_0\sigma^{f-1}}. \tag{4.4.2}$$

From (2.2.17) we obtain $g_{\psi_0} + g_{\bar{\psi}_0} = ef - 1$. In summary, $P_{\mathrm{c}} = P_{\psi_0} \oplus P_{\bar{\psi}_0}$ as a $W_{O_F}(k) = O_F \otimes_{O_{F^t},\tilde{\psi}_0} W(k)$-module, and V_{c} is given by the matrix

$$\begin{pmatrix} 0 & \pi^{-g_{\psi_0}} V^f \\ \pi^{-g_{\bar{\psi}_0}} V^f & 0 \end{pmatrix}. \tag{4.4.3}$$

Finally F_{c} is determined by the equation $F_{\mathrm{c}} V_{\mathrm{c}} = \pi$. For instance, $F_{\mathrm{c}} : P_{\bar{\psi}_0} \longrightarrow P_{\psi_0}$ is equal to $(\pi^{g_{\bar{\psi}_0}+1}/p^f) F^f$. We obtain a Dieudonné module $(P_{\mathrm{c}}, F_{\mathrm{c}}, V_{\mathrm{c}})$ with respect to the perfect frame

$$\mathcal{W}_{O_F}(k) = (O_F \otimes_{O_{F^t},\tilde{\psi}_0} W(k), \pi O_F \otimes_{O_{F^t},\tilde{\psi}_0} W(k), k, F^f, F^f\pi^{-1}). \tag{4.4.4}$$

In the ramified case we have $P_{\mathrm{c}} = P_{\psi_0}$ as a module over $W_{O_F}(k) = O_F \otimes_{O_{F^t},\tilde{\psi}_0} W(k)$. If we apply the Ahsendorf functor to $\mathcal{P}'$, we obtain by (4.3.21)

$$V_{\mathrm{c}} = \Pi^{-ef+1} V^f : P_{\mathrm{c}} \longrightarrow P_{\mathrm{c}}. \tag{4.4.5}$$

F_{c} is determined by the equation $F_{\mathrm{c}} V_{\mathrm{c}} = \pi$, i.e., $F_{\mathrm{c}} = -(\Pi^{ef+1}/p^f) F^f$. We obtain a Dieudonné module $(P_{\mathrm{c}}, F_{\mathrm{c}}, V_{\mathrm{c}})$ for the frame (4.4.4)

[6] The symbol $\mathfrak{R}$ is to remind us that this is a category of *relative displays*.

We next add polarizations to the picture. We set $\mathbf{t}(a) = \mathrm{Tr}_{F/\mathbb{Q}_p} \vartheta^{-1}a$ where ϑ is the different of $F/\mathbb{Q}_p$.

Proposition 4.4.5. *Let r be special. Let $R \in \mathrm{Nilp}_{O_{E'}}$. Let $(\mathcal{P}_1, \iota_1)$ and $(\mathcal{P}_2, \iota_2)$ be objects of $\mathfrak{d}\mathfrak{P}_{r,R}$. Let $(\mathcal{P}'_1, \iota'_1)$ and $(\mathcal{P}'_2, \iota'_2)$ be their images by the functor $\mathfrak{C}'_{r,R}$. Assume given a bilinear form of displays*

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_{m,R},$$

where $\mathcal{P}_m$ is the multiplicative display of $\mathcal{W}(R)$. Assume that β is anti-linear for the O_K-actions ι_1, resp. ι_2, i.e,

$$\beta(\iota_1(a)x_1, x_2) = \beta(x_1, \iota_2(\bar{a})x_2), \quad x_1 \in P_1,\ x_2 \in P_2,\ a \in O_K. \tag{4.4.6}$$

Define

$$\tilde{\beta} : P_1 \times P_2 \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(R)$$

by the equation

$$\mathbf{t}(\xi\tilde{\beta}(x_1, x_2)) = \beta(\xi x_1, x_2), \quad x_1 \in P_1,\ x_2 \in P_2,\ \xi \in O_F \otimes_{\mathbb{Z}_p} W(R).$$

Then $\tilde{\beta}$ is a O_F-bilinear form of displays,

$$\tilde{\beta} : \mathcal{P}'_1 \times \mathcal{P}'_2 \longrightarrow \mathcal{L}_R,$$

where $\mathcal{L}_R$ is the Lubin-Tate display associated to the local field F and the algebra structure $\varphi_0 : O_F \longrightarrow O_{E'} \longrightarrow R$, cf. Definition 3.4.3. Furthermore, $\tilde{\beta}$ is anti-linear for the O_K-actions ι_1, resp. ι_2.

Proof. To avoid a conflict with the present notations, we adapt some of the notation of section 3.4 to our situation. What was K in section 3.4 is now F. We set $L^t = \psi_0(F^t) \subset E'$. We write the polynomials of that section as follows:

$$\tilde{\mathbf{E}}_{F,\psi}(Z) = \prod_{\chi : F \longrightarrow \tilde{E},\ \chi_{|F^t} = \psi} (Z - [\chi(\pi)]) \in W(O_{L^t})[Z].$$

We stress that here ψ denotes an embedding of F^t into E', not as elsewhere in this section an embedding of K^t into E'. For $\psi = \psi_0$, we consider the decomposition $\tilde{\mathbf{E}}_{F,\psi_0}(Z) = (Z - [\varphi_0(\pi)]) \cdot \tilde{\mathbf{E}}_{F,0}(Z)$ in $W(\varphi_0(F))[Z]$. In particular all of these polynomials lie in $W(O_{E'})[Z]$.

Let M be an $O_F \otimes_{O_{F^t},\psi} W(R)$-module. Then we write by our convention

$$\tilde{\mathbf{E}}_{F,\psi} m = \tilde{\mathbf{E}}_{F,\psi}(\pi \otimes 1)m,$$

where $\tilde{\mathbf{E}}_{F,\psi}(\pi \otimes 1) \in O_F \otimes_{O_{F^t},\psi} W(R)$ is the evaluation at $\pi \otimes 1$ in this $W(O_{L^t})$-algebra.

We first consider the assertion of Proposition 4.4.5 in the ramified case. We have the decomposition $P_i = \oplus_\psi P_{i,\psi}$. By (4.4.6), we find $\beta(P_{1,\psi}, P_{2,\psi'}) = 0$ for $\psi \neq \psi'$. We consider the restrictions of our bilinear forms

$$\begin{aligned} \beta_\psi &: P_{1,\psi} \times P_{2,\psi} \longrightarrow W(R) \\ \tilde{\beta}_\psi &: P_{1\psi} \times P_{2,\psi} \longrightarrow O_F \otimes_{O_{F^t},\psi} W(R). \end{aligned}$$

Lemma 4.4.6. *Let K/F be ramified. Then:*

$$\begin{aligned} \tilde{\beta}_\psi(\tilde{\mathbf{E}}_{A_\psi} x_1, \tilde{\mathbf{E}}_{A_\psi} x_2) &= \tilde{\mathbf{E}}_{F,\psi} \tilde{\beta}_\psi(x_1, x_2), \qquad x_1 \in P_{1,\psi},\ x_2 \in P_{2,\psi},\ \psi \neq \psi_0 \\ \tilde{\beta}_{\psi_0}(\tilde{\mathbf{E}}_{A_{\psi_0}} x_1, \tilde{\mathbf{E}}_{A_{\psi_0}} x_2) &= \tilde{\mathbf{E}}_{F,0} \tilde{\beta}_\psi(x_1, x_2), \qquad x_1 \in P_{1,\psi_0},\ x_2 \in P_{2,\psi_0}. \end{aligned}$$

Proof. We can restrict ourselves to the case where R is a $O_{\tilde{E}}$-algebra. Then we obtain

$$\begin{aligned} \tilde{\beta}_\psi\big((\Pi \otimes 1 - 1 \otimes [\varphi(\Pi)])x_1, x_2\big) &= \tilde{\beta}_\psi\big(x_1, (-\Pi \otimes 1 - 1 \otimes [\varphi(\Pi)])x_2\big) \\ &= -\tilde{\beta}_\psi\big(x_1, (\Pi \otimes 1 - 1 \otimes [\bar{\varphi}(\Pi)])x_2\big). \end{aligned}$$

In the case were $\psi \neq \psi_0$ we deduce

$$\tilde{\beta}_\psi(\tilde{\mathbf{E}}_{A_\psi} x_1, x_2) = (-1)^e \tilde{\beta}_\psi(x_1, \tilde{\mathbf{E}}_{B_\psi} x_2).$$

We find

$$\begin{aligned}\tilde{\mathbf{E}}_{B_\psi}(\Pi\otimes 1)\cdot\tilde{\mathbf{E}}_{A_\psi}(\Pi\otimes 1) &= \prod_{\varphi\in A_\psi}(\Pi\otimes 1-1\otimes[\varphi(\Pi)])(\Pi\otimes 1-1\otimes[\bar{\varphi}(\Pi)])\\ &= \prod_{\varphi\in A_\psi}(\Pi\otimes 1-1\otimes[\varphi(\Pi)])(\Pi\otimes 1+1\otimes[\varphi(\Pi)])\\ &= \prod_{\varphi\in A_\psi}(-\pi\otimes 1+1\otimes[\varphi(\pi)]) = (-1)^e\tilde{\mathbf{E}}_{F,\psi}(\pi\otimes 1).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\tilde{\beta}_\psi(\tilde{\mathbf{E}}_{A_\psi}x_1,\tilde{\mathbf{E}}_{A_\psi}x_2) = (-1)^e\tilde{\beta}_\psi(x_1,\tilde{\mathbf{E}}_{B_\psi}\tilde{\mathbf{E}}_{A_\psi}x_2) = \tilde{\beta}_\psi(x_1,\tilde{\mathbf{E}}_{F,\psi}x_2)\\ = \tilde{\mathbf{E}}_{F,\psi}\tilde{\beta}_\psi(x_1,x_2),\end{aligned}$$

which finishes the proof for $\psi\neq\psi_0$.

We turn now to the case ψ_0. The polynomials $\tilde{\mathbf{E}}_{A_{\psi_0}}$, $\tilde{\mathbf{E}}_{B_{\psi_0}}$, and $\tilde{\mathbf{E}}_{F,0}$ are of degree $e-1$. The same computations yield for $x_1\in P_{1,\psi_0}$ and $x_2\in P_{2,\psi_0}$,

$$\begin{aligned}\tilde{\beta}_{\psi_0}(\tilde{\mathbf{E}}_{A_{\psi_0}}x_1,x_2) = (-1)^{e-1}\tilde{\beta}_{\psi_0}(x_1,\tilde{\mathbf{E}}_{B_{\psi_0}}x_2),\\ \tilde{\mathbf{E}}_{B_{\psi_0}}(\Pi\otimes 1)\cdot\tilde{\mathbf{E}}_{A_{\psi_0}}(\Pi\otimes 1) = (-1)^{e-1}\tilde{\mathbf{E}}_{F,0}(\pi\otimes 1).\end{aligned}$$

The assertion for ψ_0 follows as before. □

We continue with the proof of Proposition 4.4.5 in the ramified case. We begin by showing that

$$\tilde{\beta}_\psi(Q'_{1,\psi},Q'_{2,\psi})\subset Q_{\mathcal{L},\psi}. \tag{4.4.7}$$

This is trivial for $\psi\neq\psi_0$. Let $y_1\in Q'_{1,\psi_0}$ and $y_2\in Q'_{2,\psi_0}$. By Lemma 3.4.2, the inclusion (4.4.7) is equivalent with

$$\tilde{\mathbf{E}}_{F,0}\tilde{\beta}_{\psi_0}(y_1,y_2)\in O_F\otimes_{O_{F^t},\psi_0} I(R)$$

By Lemma 4.4.6 we find

$$\tilde{\mathbf{E}}_{F,0}\tilde{\beta}_{\psi_0}(y_1,y_2) = \tilde{\beta}_{\psi_0}(\tilde{\mathbf{E}}_{A_{\psi_0}}y_1,\tilde{\mathbf{E}}_{A_{\psi_0}}y_2).$$

The elements $u_1=\tilde{\mathbf{E}}_{A_{\psi_0}}y_1$, resp., $u_2=\tilde{\mathbf{E}}_{A_{\psi_0}}y_2$, lie, by the definition of Q'_{1,ψ_0}, in Q_{1,ψ_0}, resp., by the definition of Q'_{2,ψ_0}, in Q_{2,ψ_0}. But for arbitrary elements $u_1\in Q_{1,\psi_0}$ and $u_2\in Q_{2,\psi_0}$, we have $\beta(u_1,u_2)\in I(R)$. By the definition of $\tilde{\beta}_{\psi_0}$, we find

$$\mathrm{Tr}_{F/F^t}\,\vartheta^{-1}\xi\tilde{\beta}_{\psi_0}(u_1,u_2) = \beta(\xi u_1,u_2)\in I(R),$$

for all $\xi\in O_F\otimes_{O_{F^t},\psi_0} W(R)$. But this implies $\tilde{\beta}_{\psi_0}(u_1,u_2)\in O_F\otimes_{O_{F^t},\psi_0} I(R)$, as desired.

Finally we have to check for $y_1\in Q'_{1,\psi}$ and $y_2\in Q'_{2,\psi}$ that

$$\tilde{\beta}_\psi(\dot{F}'y_1,\dot{F}'y_2) = \dot{F}_{\mathcal{L}}\tilde{\beta}_\psi(y_1,y_2).$$

If $\psi\neq\psi_0$ we find for the left hand side

$$\tilde{\beta}_\psi\big(\dot{F}(\tilde{\mathbf{E}}_{A_\psi})y_1,\dot{F}(\tilde{\mathbf{E}}_{A_\psi})y_2\big) = {}^{\dot{F}}\big(\tilde{\mathbf{E}}_{F,\psi}\tilde{\beta}_\psi(y_1,y_2)\big) = \dot{F}_{\mathcal{L}}\big(\tilde{\beta}_\psi(y_1,y_2)\big).$$

For ψ_0 we obtain

$$\tilde{\beta}_{\psi_0}\big(\dot{F}(\tilde{\mathbf{E}}_{A_{\psi_0}})y_1,\dot{F}(\tilde{\mathbf{E}}_{A_{\psi_0}})y_2\big) = {}^{\dot{F}}\big(\tilde{\mathbf{E}}_{F,0}\tilde{\beta}_{\psi_0}(y_1,y_2)\big) = \dot{F}_{\mathcal{L}}\big(\tilde{\beta}_{\psi_0}(y_1,y_2)\big).$$

This ends the proof of Proposition 4.4.5 in the ramified case.

Now we consider the unramified case. We consider the decomposition (4.3.2). Let us denote by e_ψ the idempotents corresponding to this decomposition. The conjugation of K/F maps e_ψ to $e_{\bar{\psi}}$. We consider the corresponding decompositions $P_i=\oplus P_{i,\psi}$, for $i=1,2$. We obtain that $\beta(P_{1,\psi_1},P_{2,\psi_2})=0$ for $\psi_2\neq\bar{\psi_1}$, and

$$\tilde{\beta}(P_{1,\psi},P_{2,\bar{\psi}})\subset O_F\otimes_{O_{F^t},\tilde{\psi}} W(R).$$

We note that there are natural isomorphisms

$$O_K\otimes_{O_{K^t},\tilde{\psi}} W(R)\cong O_F\otimes_{O_{F^t},\tilde{\psi}} W(R)\cong O_K\otimes_{O_{K^t},\tilde{\bar{\psi}}} W(R). \tag{4.4.8}$$

Therefore $\tilde{\beta}$ induces an $O_F \otimes_{O_{F^t},\tilde{\psi}} W(R)$-bilinear form

$$\tilde{\beta}_\psi : P_{1,\psi} \times P_{2,\bar{\psi}} \longrightarrow O_F \otimes_{O_{F^t},\tilde{\psi}} W(R).$$

With the identification (4.4.8), we have $\tilde{\mathbf{E}}_{A_{\bar{\psi}}}(\pi \otimes 1) = \tilde{\mathbf{E}}_{B_\psi}(\pi \otimes 1) \in O_F \otimes_{O_{F^t},\tilde{\psi}} W(R)$. The analogue of Lemma 4.4.6 is

$$\begin{aligned} \tilde{\beta}_\psi(\tilde{\mathbf{E}}_{A_\psi} x_1, \tilde{\mathbf{E}}_{A_{\bar{\psi}}} x_2) &= \tilde{\mathbf{E}}_{F,\psi} \tilde{\beta}_\psi(x_1, x_2), \qquad x_1 \in P_{1,\psi},\ x_2 \in P_{2,\bar{\psi}},\ \psi \neq \psi_0, \bar{\psi}_0, \\ \tilde{\beta}_\psi(\tilde{\mathbf{E}}_{A_\psi} x_1, \tilde{\mathbf{E}}_{A_{\bar{\psi}}} x_2) &= \tilde{\mathbf{E}}_{F,0} \tilde{\beta}_\psi(x_1, x_2), \qquad x_1 \in P_{1,\psi},\ x_2 \in P_{2,\bar{\psi}},\ \psi = \psi_0, \bar{\psi}_0. \end{aligned} \tag{4.4.9}$$

Here we recall again the notation introduced in the beginning of the proof: to be very precise, the expression $\tilde{\mathbf{E}}_{F,\psi}$ should be written as $\tilde{\mathbf{E}}_{F,\psi_{|F^t}}$. These identities follow from the identities

$$\tilde{\mathbf{E}}_{A_\psi} \tilde{\mathbf{E}}_{B_\psi} = \begin{cases} \tilde{\mathbf{E}}_{F,\psi}, & \psi \neq \psi_0, \bar{\psi}_0, \\ \tilde{\mathbf{E}}_{F,0}, & \psi = \psi_0, \bar{\psi}_0. \end{cases}$$

We need to check

$$\tilde{\beta}_\psi(Q'_{1,\psi}, Q'_{2,\bar{\psi}}) \subset Q_{\mathcal{L},\psi}. \tag{4.4.10}$$

It suffices to consider the case $\psi = \psi_0$. By Lemma 3.4.2, the inclusion (4.4.10) is equivalent with

$$\tilde{\mathbf{E}}_{F,0} \tilde{\beta}_{\psi_0}(y_1, y_2) \in O_F \otimes_{O_{F^t},\psi_0} I(R), \quad y_1 \in Q'_{1,\psi_0},\ y_2 \in Q'_{2,\bar{\psi}_0}.$$

But, as in the ramified case, this is an immediate consequence of (4.4.9). Finally we have to check that for $y_1 \in Q'_{1,\psi}$ and $y_2 \in Q'_{2,\bar{\psi}}$

$$\tilde{\beta}_\psi(\dot{F}' y_1, \dot{F}' y_2) = \dot{F}_{\mathcal{L}} \tilde{\beta}_\psi(y_1, y_2).$$

For this we can repeat the last two formulas in the proof of the ramified case. □

Let $R \in \mathrm{Nilp}_{O_F}$. Let $(\mathcal{P}_1, \iota_1)$ be an object of $\mathfrak{dP}'_{r,R}$. We denote by $(\mathcal{P}_1^\Delta, \iota_1^\Delta)$ the conjugate Faltings dual. It is defined from the Faltings dual exactly as the conjugate dual from the dual.

Corollary 4.4.7. *Let r be special. Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements of R is nilpotent. We regard R as an O_F-algebra via φ_0. Let $(\mathcal{P}, \iota)$ be an object of $\mathfrak{dP}_{r,R}$ and let $(\mathcal{P}', \iota') \in \mathfrak{dP}'_{r,R}$ be its image under the pre-contracting functor $\mathfrak{C}'_{r,R}$. Then the image of the conjugate dual $(\mathcal{P}^\wedge, \iota^\wedge)$ under $\mathfrak{C}'_{r,R}$ is the Faltings dual $((\mathcal{P}')^\Delta, (\iota')^\Delta)$.*

With the notation of Proposition 4.4.5, assume that $\mathcal{P}_1^\wedge$ and $\mathcal{P}_2$ are in $\mathfrak{dP}^{\mathrm{ss}}_{r,R}$. Then the canonical map

$$\mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}_{m,R}) \longrightarrow \mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}'_1 \times \mathcal{P}'_2, \mathcal{L}_R)$$

is bijective. Here these sets of bilinear forms Bil *are meant as in Proposition 4.4.5.*

Proof. We apply Proposition 4.4.5 to the canonical bilinear form $\beta_{\mathrm{can}} : \mathcal{P} \times \mathcal{P}^\wedge \longrightarrow \mathcal{P}_{m,R}$ and obtain

$$\tilde{\beta}_{\mathrm{can}} : \mathcal{P}' \times (\mathcal{P}^\wedge)' \longrightarrow \mathcal{L}_R.$$

By Proposition 3.4.10, we obtain a morphism of displays

$$\varkappa : (\mathcal{P}^\wedge)' \longrightarrow (\mathcal{P}')^\Delta. \tag{4.4.11}$$

By definition, $\tilde{\beta}_{\mathrm{can}}$ is given by a perfect $O_F \otimes W(R)$-bilinear form

$$P \times P^* \longrightarrow O_F \otimes W(R).$$

(Recall that $P^* = \mathrm{Hom}_{W(R)}(P, W(R))$.) We obtain an isomorphism

$$P^* \xrightarrow{\sim} \mathrm{Hom}_{O_F \otimes W(R)}(P, O_F \otimes W(R)).$$

But this says exactly that the map which $\varkappa$ induces on the "P-components" of the displays (4.4.11) is an isomorphism. It is elementary to see that a morphism of displays $\varkappa : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ which induces a $W(R)$-module isomorphism $P_1 \longrightarrow P_2$ is an isomorphism of displays.

Finally we prove the bijectivity of the last map in the corollary. The left hand side is, by (3.2.5),

$$\mathrm{Hom}_{\mathfrak{dP}_{r,R}}(\mathcal{P}_1, (\mathcal{P}_2)^\wedge).$$

This group is, by (iii) of Theorem 4.3.2, equal to

$$\mathrm{Hom}_{\mathfrak{dP}'_{r,R}}(\mathcal{P}'_1, (\mathcal{P}_2^{\wedge})') \cong \mathrm{Hom}_{\mathfrak{dP}'_{r,R}}(\mathcal{P}'_1, (\mathcal{P}'_2)^{\Delta}) \cong \mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}'_1 \times \mathcal{P}'_2, \mathcal{L}_R).$$

□

We now combine the last corollary with Theorem 3.4.11.

Theorem 4.4.8. *Let r be special. Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements of R is nilpotent. Let $(\mathcal{P}_1, \iota_1)$ and $(\mathcal{P}_2, \iota_2)$ be objects of $\mathfrak{dP}_{r,R}$, with images $(\mathcal{P}'_1, \iota'_1)$ and $(\mathcal{P}'_2, \iota'_2)$ under the pre-contracting functor $\mathfrak{C}'_{r,R}$, cf. Proposition 4.3.2. Since the actions ι'_i restricted to O_F are strict with respect to $\varphi_0 : O_F \longrightarrow O_{E'} \longrightarrow R$, the Ahsendorf functor $\mathfrak{A}_{O_F/\mathbb{Z}_p,R}$ may be applied to them. For $i = 1, 2$, let $\mathcal{P}_{i,c} = \mathfrak{A}_{O_F/\mathbb{Z}_p,R}(\mathcal{P}_i)$, $i = 1, 2$, with its O_F-algebra homomorphism*

$$\iota_{i,c} : O_K \longrightarrow \mathrm{End}_{\mathcal{W}_{O_F}(R)} \mathcal{P}_{i,c}.$$

If $\mathcal{P}_1^{\wedge}$ and $\mathcal{P}_2$ are in $\mathfrak{dP}^{\mathrm{ss}}_{r,R}$, then the natural homomorphism

$$\mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}_{m,R}) \longrightarrow \mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}_{1,c} \times \mathcal{P}_{2,c}, \mathcal{P}_{m,\mathcal{W}_{O_F}(R)}(\pi^{ef}/p^f)).$$

is a bijection.

The twist $\mathcal{P}_{m,\mathcal{W}_{O_F}(R)}(\pi^{ef}/p^f)$ of the multiplicative display is defined in Example 3.1.6. More precisely, this is the twist by the image of (π^{ef}/p^f) under the canonical map $O_F \longrightarrow W_{O_F}(R)$.

Proof. This follows from Corollary 4.4.7 and Theorem 3.4.11. □

Remark 4.4.9. Let $\breve{E} \subset \hat{\bar{\mathbb{Q}}}_p$ be the completion of the maximal unramified extension of the reflex field E of r. We extend $\varphi_0 : O_F \longrightarrow O_{\breve{E}}$ to an embedding $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$. We denote by $\tau \in \mathrm{Gal}(\breve{F}/F)$ the Frobenius automorphism. We apply the definition of η_0 after Definition 3.4.12,

$$\tau(\eta_0)\eta_0^{-1} = \pi^e/p, \quad \eta_0 \in O_{\breve{F}}^{\times}. \tag{4.4.12}$$

Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$. Via $\breve{\varphi}_0$ we consider R as an $O_{\breve{F}}$-algebra. Therefore $\eta_{0,R}$ is defined, and multiplication by $\eta_{0,R}^f$ defines an isomorphism

$$\mathcal{P}_{m,\mathcal{W}_{O_F}(R)}(\pi^{ef}/p^f) \xrightarrow{\sim} \mathcal{P}_{m,\mathcal{W}_{O_F}(R)}, \tag{4.4.13}$$

cf. (3.4.21). Therefore, if $R \in \mathrm{Nilp}_{O_{\breve{E}}}$, we can ignore the twist by (π^e/p) in Theorem 4.4.8.

We recall the definition of polarized CM-pairs $\mathfrak{P}^{\mathrm{pol}}_{r,S}$, cf. Definition 4.1.2. We also introduce the analogous category of polarized objects of $\mathfrak{dR}_R$, as follows.

Definition 4.4.10. Let $R \in \mathrm{Nilp}_{O_F}$. We denote by $\mathfrak{dR}^{\mathrm{pol}}_R$ the category of triples $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$ where $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}) \in \mathfrak{dR}_R$ (cf. Definition 4.4.1) and where

$$\beta_{\mathrm{c}} : \mathcal{P}_{\mathrm{c}} \times \mathcal{P}_{\mathrm{c}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_{O_F}(R)}$$

is a polarization which is anti-linear for the O_K-action ι_{c}.

Let r be a special local CM-type with reflex field E. We regard an algebra $R \in \mathrm{Nilp}_{O_{\breve{E}}}$ as an $O_{\breve{F}}$-algebra via $\breve{\varphi}_0$. We now define the contracting functor for polarized CM-pairs,

$$\mathfrak{C}^{\mathrm{pol}}_{r,R} : \mathfrak{dP}^{\mathrm{pol}}_{r,R} \longrightarrow \mathfrak{dR}^{\mathrm{pol}}_R. \tag{4.4.14}$$

Let $(\mathcal{P}, \iota, \beta) \in \mathfrak{dP}^{\mathrm{pol}}_{r,R}$. We apply the contracting functor $\mathfrak{C}_{r,R}$ to $(\mathcal{P}, \iota)$ and obtain $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}) \in \mathfrak{dR}_R$, cf. Definition 4.4.2. By Theorem 4.4.8, the polarization $\beta : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}_{m,R}$ induces an alternating bilinear form

$$\tilde{\beta}_{\mathrm{c}} : \mathcal{P}_{\mathrm{c}} \times \mathcal{P}_{\mathrm{c}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_{O_F}(R)}(\pi^{ef}/p^f). \tag{4.4.15}$$

If we combine this with the chosen isomorphism (4.4.13), we obtain a polarization of the $\mathcal{W}_{O_F}(R)$-display $\mathcal{P}_{\mathrm{c}}$,

$$\beta_{\mathrm{c}} : \mathcal{P}_{\mathrm{c}} \times \mathcal{P}_{\mathrm{c}} \longrightarrow \mathcal{P}_{m,\mathcal{W}_{O_F}(R)}.$$

Then $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$ is defined to be the image of $(\mathcal{P}, \iota, \beta)$ by the functor $\mathfrak{C}^{\mathrm{pol}}_{r,R}$.

Theorem 4.4.11. *Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$ be such that the ideal of nilpotent elements is nilpotent. The contracting functor $\mathfrak{C}_{r,R}^{\mathrm{pol}}$ induces an equivalence of categories*

$$\mathfrak{C}_{r,R}^{\mathrm{pol}} : \mathfrak{d}\mathfrak{P}_{r,R}^{\mathrm{ss,pol}} \longrightarrow \mathfrak{d}\mathfrak{R}_{R}^{\mathrm{nilp,pol}}.$$

Let $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$ the image of $(\mathcal{P}, \iota, \beta)$ under the functor $\mathfrak{C}_{r,R}^{\mathrm{pol}}$. Then

$$\mathrm{height}_{O_F} \beta_{\mathrm{c}} = \frac{1}{f} \,\mathrm{height}\, \beta,$$

cf. Definition 3.2.5.

Here the index "ss" indicates the full subcategory of supersingular displays and the index "nilp" the full subcategory of nilpotent displays.

Proof. We use the notation of Proposition 4.4.5. We have a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \tilde{\alpha}\ } & \mathrm{Hom}_{O_F \otimes_{\mathbb{Z}_p} W(R)}(P, O_F \otimes_{\mathbb{Z}_p} W(R)) \\ & \searrow^{\alpha} & \downarrow \mathbf{t}_* \\ & & \mathrm{Hom}_{W(R)}(P, W(R)) \end{array}$$

Here the map $\tilde{\alpha}$ is induced by $\tilde{\beta}$ and the map α is induced by β. The vertical map is defined by $\mathbf{t}_*(\ell) = \mathbf{t} \circ \ell$ and is an isomorphism. The map α induces the isogeny $\mathcal{P} \longrightarrow \mathcal{P}^\vee$ associated to β and the map $\tilde{\alpha}$ induces the isogeny $\mathcal{P}' \longrightarrow (\mathcal{P}')^\Delta$. Therefore these isogenies have the same height. If we apply the Ahsendorf functor to the last isogeny we obtain the map $\mathcal{P}_{\mathrm{c}} \longrightarrow \mathcal{P}_{\mathrm{c}}^\vee(\pi^{ef}/p^f)$ which is associated to $\tilde{\beta}_{\mathrm{c}}$. By Proposition 3.3.17 we obtain

$$\mathrm{height}_{O_F} \beta_{\mathrm{c}} = \mathrm{height}_{O_F} \tilde{\beta}_{\mathrm{c}} = \frac{1}{f} \,\mathrm{height}\, \beta.$$

□

Remark 4.4.12. Let us explain how the bijection between bilinear forms of Theorem 4.4.8 simplifies when $R = k$ is a perfect field in $\mathrm{Nilp}_{O_{E'}}$. We take $\mathcal{P}_1 = \mathcal{P}_2$.

We consider the Dieudonné module (P, F, V) of $\mathcal{P}$. We consider $\beta : P \times P \longrightarrow W(k)$ as a bilinear form of Dieudonné modules. Here we mean by $W(k)$ the Dieudonné module $(W(k), F, V)$, cf. (3.2.2). We define

$$\tilde{\beta} : P \times P \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(k) \tag{4.4.16}$$

as in Proposition 4.4.5. We know that $\tilde{\beta}$ induces a bilinear form of displays $\mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{L}_k$. In terms of Dieudonné modules, this means that the following equation holds,

$$\tilde{\beta}(V' x_1, V' x_2) = V_{\mathcal{L}} \tilde{\beta}(x_1, x_2). \tag{4.4.17}$$

In terms of the decomposition (4.3.6), the operator V' is given by (4.3.15).

By (3.4.10), the Ahsendorf functor applied to $\mathcal{L}_k$ gives the $\mathcal{W}_{O_F}(k)$-Dieudonné module

$$(O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k), \frac{\pi^{ef}}{p^f} F^f, \frac{p^f}{\pi^{ef-1}} F^{-f}). \tag{4.4.18}$$

The bilinear form $\tilde{\beta}$ gives by restriction to $P_{\mathrm{c}} = P \otimes_{O_{F^t}, \tilde{\psi}_0} W(k) \subset P$ the $O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)$-bilinear form

$$\tilde{\beta}_{\mathrm{c}} : P_{\mathrm{c}} \times P_{\mathrm{c}} \longrightarrow O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k). \tag{4.4.19}$$

Because this is obtained by applying the Ahsendorf functor to (4.4.17), $\tilde{\beta}_{\mathrm{c}}$ is a bilinear form of $\mathcal{W}_{O_F}(k)$-Dieudonné modules if we equip the right hand side with the $\mathcal{W}_{O_F}(k)$-Dieudonné module structure (4.4.18). Therefore we obtain

$$\tilde{\beta}_{\mathrm{c}}(V_{\mathrm{c}} x_1, V_{\mathrm{c}} x_2) = \frac{p^f}{\pi^{ef-1}} \,{}^{F^{-f}} \tilde{\beta}_{\mathrm{c}}(x_1, x_2), \quad x_1, x_2 \in P_{\mathrm{c}}. \tag{4.4.20}$$

In the case where K/F is ramified, we have $P_{\mathrm{c}} = P_{\psi_0}$, and

$$V_{\mathrm{c}} = \Pi^{-ef+1} V^f : P_{\psi_0} \longrightarrow P_{\psi_0},$$

cf. (4.4.5). Note that (4.4.20) can be checked easily from these expressions.

In the case where K/F is unramified, we have $P_{\rm c} = P_{\psi_0} \oplus P_{\bar{\psi}_0}$, and $V_{\rm c}$ is the endomorphism of $P_{\rm c} = P_{\psi_0} \oplus P_{\bar{\psi}_0}$ given by the matrix

$$\begin{pmatrix} 0 & \pi^{-g_{\psi_0}} V^f \\ \pi^{-g_{\bar{\psi}_0}} V^f & 0 \end{pmatrix},$$

cf. (4.4.3). Before (4.4.8) we already remarked that $\tilde{\beta}(P_{\psi_0}, P_{\psi_0}) = 0 = \tilde{\beta}(P_{\bar{\psi}_0}, P_{\bar{\psi}_0})$. Again (4.4.20) can be checked directly on these descriptions of $V_{\rm c}$.

Now we assume moreover that k is a $O_{\breve{E}}$-algebra. We have the map (4.4.12),

$$O_{\breve{F}} \longrightarrow O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k) = W_{O_F}(k).$$

We consider the image $\eta_{0,k} \in O_F \otimes_{O_{F^t}, \psi_0} W(k)$ of η_0. We set

$$\beta_{\rm c} = \eta_{0,k}^f \tilde{\beta}_{\rm c} : P_{\rm c} \times P_{\rm c} \longrightarrow O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k). \tag{4.4.21}$$

Then we find

$$\begin{aligned} \beta_{\rm c}(V_{\rm c} x_1, V_{\rm c} x_2) &= \eta_{0,k}^f \tilde{\beta}_{\rm c}(V_{\rm c} x_1, V_{\rm c} x_2) = \eta_{0,k}^f \frac{p^f}{\pi^{ef-1}} \; {}^{F^{-f}} \tilde{\beta}_{\rm c}(x_1, x_2) \\ &= \eta_{0,k}^f \frac{p^f}{\pi^{ef-1}} \; {}^{F^{-f}} (\eta_{0,k}^{-f} \beta_{\rm c}(x_1, x_2)) = \pi \; {}^{F^{-f}} \beta_{\rm c}(x_1, x_2), \end{aligned}$$

since $\eta_{0,k}^f \; {}^{F^{-f}}(\eta_{0,k}^{-f}) = \pi^{ef}/p^f$. Indeed, the left hand side of the last identity is the image of $(\eta_0 \tau^{-1}(\eta_0^{-1}))^f = \pi^{ef}/p^f$.

This shows that $\beta_{\rm c}$ is a bilinear form of $\mathcal{W}_{O_F}(k)$-Dieudonné modules, if we consider on $O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)$ the $\mathcal{W}_{O_F}(k)$-Dieudonné module structure which corresponds to $\mathcal{P}_{m, \mathcal{W}_{O_F}(R)}$, namely

$$(O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k), F^f, \pi F^{-f}).$$

Remark 4.4.13. Let us discuss the height identity in Theorem 4.4.11 in a more direct way. We may assume that R is a perfect field. We may write the equation in the form

$$\operatorname{ord}_p \det_{W(k)} \beta = f \operatorname{ord}_\pi \det_{W_{O_F}(k)} \beta_{\rm c}. \tag{4.4.22}$$

On the left hand side the determinant is taken with respect to an arbitrary basis of the $W(k)$-module P. After we take ord_p, the result is independent of the choice of the basis. The right hand side of this equation does not change if we replace $\beta_{\rm c}$ by the form $\tilde{\beta}_{\rm c}$ of (4.4.19). We begin with the ramified case. The decomposition $P = \oplus P_\psi$ is orthogonal with respect to β. Let β_ψ be the restriction to P_ψ. Let ψ be banal. The map $\Pi^{-e} V : P_{\psi\sigma} \longrightarrow P_\psi$ is a F^{-1}-linear isomorphism. From the equation

$$\beta_\psi(\Pi^{-e} V x, \Pi^{-e} V y) = \beta_\psi(V(\pi^{-e} x), V y) = \; {}^{F^{-1}} \beta_{\psi\sigma}(\pi^{-e} p x, y)$$

we conclude that $\operatorname{ord}_p \det_{W(k)} \beta_\psi = \operatorname{ord}_p \det_{W(k)} \beta_{\psi\sigma}$. Therefore this value is independent of ψ. In particular we obtain

$$\operatorname{ord}_p \det_{W(k)} \beta = f \operatorname{ord}_p \det_{W(k)} \beta_{\psi_0} = f \operatorname{ord}_\pi \det_{W_{O_F}(k)} \tilde{\beta}_{\rm c}.$$

The last equation follows because

$$\tilde{\beta}_{\rm c} : P_{\psi_0} \times P_{\psi_0} \longrightarrow W_{O_F}(k) = O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)$$

is obtained from β_{ψ_0} by the equation

$$\operatorname{Tr}_{W_{O_F}(k)/W(k)}(\vartheta^{-1} a \tilde{\beta}_{\rm c}(x, y)) = \beta_{\psi_0}(ax, y), \quad x, y \in P_{\psi_0},\ a \in O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k),$$

and since the pairing

$$\operatorname{Tr}_{W_{O_F}(k)/W(k)}(\vartheta^{-1} a_1 a_2) : (O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)) \times (O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)) \longrightarrow W(k)$$

is perfect.

In the unramified case we write $\psi\sigma^f = \bar{\psi}$. The modules P_{ψ_1} and P_{ψ_2} are orthogonal for $\psi_1 \neq \bar{\psi}_2$. We denote by β_ψ the restriction of β to $P_\psi \times P_{\bar{\psi}}$. We define $\operatorname{ord}_p \det \beta_\psi$ by taking an arbitrary basis of P_ψ and an arbitrary basis of $P_{\bar{\psi}}$. Assume that ψ is banal; then $\bar{\psi}$ is also banal. We obtain two F^{-1}-linear isomorphisms

$$\pi^{-a_\psi} V : P_{\psi\sigma} \longrightarrow P_\psi, \quad \pi^{-a_{\bar{\psi}}} V : P_{\bar{\psi}\sigma} \longrightarrow P_{\bar{\psi}}.$$

We have

$$\beta_\psi(\pi^{-a_\psi} Vx, \pi^{-a_{\bar{\psi}}} Vy) = \beta_\psi(\pi^{-e} Vx, Vy) = {}^{F^{-1}}\beta_{\psi\sigma}(p\pi^{-e}x, y).$$

We conclude that $\mathrm{ord}_p \det_W \beta_\psi = \mathrm{ord}_p \det_W \beta_{\psi\sigma}$. Because β is alternating, β_{ψ_0} and $\beta_{\bar{\psi}_0}$ have the same order of determinant. We conclude that $h := \mathrm{ord}_p \det_W \beta_\psi$ is independent of $\psi \in \Psi$. We find $\mathrm{ord}_p \det_W \beta = 2fh$. The form $\tilde{\beta}_{\mathrm{c}}$ is obtained from the restriction β_{ψ_0} by the equation

$$\mathrm{Tr}_{O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)/W(k)}(a\vartheta^{-1}\tilde{\beta}_{\mathrm{c}}(x,y)) = \beta_{\psi_0}(ax, y), \quad x \in P_{\psi_0},\ y \in P_{\bar{\psi}_0},\ a \in O_F.$$

Therefore we obtain

$$2h = 2\,\mathrm{ord}_p \det_W \beta_{\psi_0} = \mathrm{ord}_\pi \det_{O_F \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)} \tilde{\beta}_{\mathrm{c}}.$$

4.5. **The contracting functor in the case of a banal CM-type.** In the banal case we will associate to an object of the category $\mathfrak{dP}'_{r,R}$ (cf. Definition 4.3.1) an étale sheaf on $\mathrm{Spec}\, R$. The construction does not use the Ahsendorf functor $\mathfrak{A}_{O_F/\mathbb{Z}_p}$, which is not useful here.

Definition 4.5.1. Let R be a ring. An *étale Frobenius module* is a pair (M, Θ), where M is a finitely generated $W(R)$-module which is locally on $\mathrm{Spec}\, R$ free and where $\Theta : M \longrightarrow M$ is a Frobenius linear isomorphism, i.e., $\Theta : \sigma^*(M) \longrightarrow M$ is an isomorphism.

The following proposition is a variant of a result of Drinfeld, comp. [10, Prop. 2.1]. It can also be proved using the theory of displays. When R is an algebraically closed field, the proposition is a theorem of Dieudonné.

Proposition 4.5.2. *Let R be a ring such that p is nilpotent in R. There is a functor $\mathfrak{A}$ from the category of étale Frobenius modules over R to the category of locally constant p-adic étale sheaves which are finitely generated and flat over $\mathbb{Z}_p$. The functor $\mathfrak{A}$ is an equivalence of categories which commutes with arbitrary base change. It is compatible with the tensor product of étale Frobenius modules, resp., of p-adic étale sheaves.*

Proof. We give a sketch of the proof which shows how this equivalence is constructed. By [33, Lem. 42] it follows that the category of étale Frobenius modules over R and R/pR are equivalent. Indeed, a Frobenius module lifts locally and by loc. cit. two liftings are canonically isomorphic. Therefore we may assume $pR = 0$. Let (M, Θ) be an étale Frobenius module. We set $M_n = W_n(R) \otimes_{W(R)} M$. Because $pR = 0$, the Frobenius F on $W(R)$ induces a Frobenius $F : W_n(R) \longrightarrow W_n(R)$. By base change we obtain a F-linear map $\Theta_n : M_n \longrightarrow M_n$. We define a functor $\mathfrak{A}_n$ on the category of R-algebras,

$$\mathfrak{A}_n(S) = \{x \in W_n(S) \otimes_{W_n(R)} M_n \mid \Theta_{n,S}(x) = x\}.$$

One can show that $\mathfrak{A}_n$ is representable by a finite étale scheme over $\mathrm{Spec}\, R$. Clearly $\mathbb{Z}/p^n\mathbb{Z} = W_n(\mathbb{F}_p)$ acts on $\mathfrak{A}_n$. We define the associated p-adic sheaf

$$\mathfrak{A}_{(M,\Theta)} = \varprojlim_n \mathfrak{A}_n.$$

Let W be the étale sheaf of Witt vectors. We have $W \otimes_{\mathbb{Z}_p} \mathfrak{A}_{(M,\Theta)} = M$ in the sense of étale sheaves, where the action of Θ corresponds on the left hand side to the action of $F \otimes \mathrm{id}$.

Finally, we show the compatibility with tensor products. If (M', Θ') is a second étale Frobenius module, we set $(N, \Xi) = (M \otimes_{W(R)} M', \Theta \otimes \Theta')$. We obtain a natural homomorphism

$$\mathfrak{A}_{(M,\Theta)} \otimes_{\mathbb{Z}_p} \mathfrak{A}_{(M',\Theta')} \longrightarrow \mathfrak{A}_{(N,\Xi)}. \tag{4.5.1}$$

To prove that this is an isomorphism, we may reduce by base change to the case where R is an algebraically closed field. Then the assertion is clear by the theorem of Dieudonné. □

Definition 4.5.3. Let r be banal. Let $R \in \mathrm{Nilp}_{O_{E'}}$. Let $\mathrm{Et}(O_K)_R$ be the category of locally constant p-adic étale sheaves G over $\mathrm{Spec}\, R$ which are $\mathbb{Z}_p$-flat with $\mathrm{rank}_{\mathbb{Z}_p} G = 4d$ and which are equipped with an action

$$\iota : O_K \longrightarrow \mathrm{End}_{\mathbb{Z}_p} G.$$

The contracting functor is the functor

$$\mathfrak{C}_{r,R}\colon \mathfrak{d}\mathfrak{P}_{r,R} \longrightarrow \mathrm{Et}(O_K)_R,$$

which is the composite of the pre-contracting functor $\mathfrak{C}'_{r,R}$ of Theorem 4.3.2 and the functor $\mathfrak{A}$ of Proposition 4.5.2, applied to the étale Frobenius module $(P', \dot{F}')$. The functor commutes with arbitrary base change $R \longrightarrow R'$.

Theorem 4.5.4. *Let r be banal. Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements of R is nilpotent. Then the contracting functor is an equivalence of categories,*

$$\mathfrak{C}_{r,R}\colon \mathfrak{d}\mathfrak{P}_{r,R} \longrightarrow \mathrm{Et}(O_K)_R.$$

Proof. Since the objects in $\mathfrak{d}\mathfrak{P}'_{r,R}$ are étale, this is simply a combination of Proposition 4.3.2 and Proposition 4.5.2 □

Remark 4.5.5. In the banal case there is a functor

$$\mathfrak{P}_{r,R} \longrightarrow \mathfrak{d}\mathfrak{P}_{r,R} \tag{4.5.2}$$

from p-divisible groups to displays which is an equivalence of categories. Indeed, in the case when K/F is a field extension, the displays of objects in $\mathfrak{d}\mathfrak{P}_{r,R}$ are by Corollary 4.3.3 isoclinic of constant slope $1/2$ and therefore nilpotent. Therefore they are displays of formal p-divisble groups, cf. Theorem 3.1.11. In the split case $O_K = O_F \times O_F$ we have a corresponding decomposition of a display $\mathcal{P} \in \mathfrak{d}\mathfrak{P}_{r,R}$: $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$. In the case where $\mathcal{P}$ is nilpotent we can argue as before. If not, one of the summands is étale and the other is isoclinic of slope 1, cf. Corollary 4.3.3. But we have an equivalence between étale p-divisible groups over R and étale displays over R, which is easily defined by the $\mathfrak{A}$-functor. Therefore we conclude also in this case that the equivalence (4.5.2) exists.

We now add polarizations to the picture.

Lemma 4.5.6. *Let r be banal. Let $R \in \mathrm{Nilp}_{O_{E'}}$. Let $(\mathcal{P}_1, \iota_1)$ and $(\mathcal{P}_2, \iota_2)$ be in $\mathfrak{d}\mathfrak{P}_{r,R}$. Let*

$$\bar{\beta} : P_1/I(R)P_1 \times P_2/I(R)P_2 \longrightarrow R$$

be an R-bilinear form such that

$$\bar{\beta}(\iota_1(a)x_1, x_2) = \bar{\beta}(x_1, \iota_2(\bar{a})x_2), \quad a \in O_K.$$

Then the restriction of $\bar{\beta}$ to $Q_1/I(R)P_1 \times Q_2/I(R)P_2$ is zero.

Proof. We first consider the case where K/F is ramified. We consider $P_{i,\psi}/I(R)P_{i,\psi}$ as an $O_K \otimes_{O_{F^t},\psi} R$-module for $i = 1, 2$. Because of the isomorphism (4.2.6), it suffices to show that

$$\bar{\beta}(\mathbf{E}_{A_\psi} x_1, \mathbf{E}_{A_\psi} x_2) = 0, \quad x_1 \in (P_1/I(R)P_1)_\psi,\ x_2 \in (P_2/I(R)P_2)_\psi.$$

We consider $\mathbf{E}_{A_\psi}(\Pi \otimes 1) \in O_K \otimes_{O_{F^t},\psi} O_{E'}$. The image of this element by the conjugation of K/F is $(-1)^e \mathbf{E}_{B_\psi}(\Pi \otimes 1)$, cf. the proof of Lemma 4.4.6. Therefore we find

$$\bar{\beta}(\mathbf{E}_{A_\psi} x_1, \mathbf{E}_{A_\psi} x_2) = (-1)^e \bar{\beta}(x_1, \mathbf{E}_{B_\psi}\mathbf{E}_{A_\psi} x_2) = (-1)^e \bar{\beta}(x_1, \mathbf{E}_\psi x_2) = 0.$$

Now we assume that K/F is unramified. Then the condition on $\bar{\beta}$ implies that $P_{1,\psi_1}/I(R)P_{1,\psi_1}$ and $P_{2,\psi_2}/I(R)P_{2,\psi_2}$ are orthogonal with respect to $\bar{\beta}$ if $\psi_1 \neq \bar{\psi}_2$. Again by the isomorphism (4.2.6), it suffices to show that

$$\bar{\beta}(\mathbf{E}_{A_\psi} x_1, \mathbf{E}_{A_{\bar{\psi}}} x_2) = 0, \quad x_1 \in (P_1/I(R)P_1)_\psi,\ x_2 \in (P_2/I(R)P_2)_{\bar{\psi}}.$$

In this case the conjugation of K/F maps $\mathbf{E}_{A_\psi}$ to $\mathbf{E}_{B_{\bar{\psi}}}$. Therefore the last equation follows from

$$\mathbf{E}_{B_{\bar{\psi}}}(\pi \otimes 1)\mathbf{E}_{A_{\bar{\psi}}}(\pi \otimes 1) = \mathbf{E}_{\bar{\psi}}(\pi \otimes 1) = 0.$$

Exactly the same argument applies to the split case. □

Lemma 4.5.7. *In the situation of the last lemma, assume that R is a reduced ring. Let $\beta\colon P_1 \times P_2 \longrightarrow W(R)$ be a $W(R)$-bilinear form such that β is anti-linear for the O_K-actions ι_1, resp. ι_2, and such that*

$$\beta(F_1 x_1, F_2 x_2) = p\ {}^F\beta(x_1, x_2), \quad x_1 \in P_1, x_2 \in P_2.$$

Then β induces a bilinear form of displays

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_{m,R}.$$

Proof. We must verify that $\beta(Q_1, Q_2) \subset I(R)$ and that

$$\beta(\dot{F}_1 y_1, \dot{F}_2 y_2) = {}^{\dot{F}}\beta(y_1, y_2), \quad y_1 \in Q_1,\ y_2 \in Q_2.$$

The inclusion is a consequence of Lemma 4.5.6. To verify the last equation, we may multiply it by p^2 because p is not a zero divisor in $W(R)$. But then it follows from the assumptions on β. □

Definition 4.5.8. Let $\rho \in O_F \otimes_{\mathbb{Z}_p} W(R)$ be a unit. We define $O_F(\rho)$ as the p-adic étale sheaf associated by Proposition 4.5.2 to the étale Frobenius module $(O_F \otimes_{\mathbb{Z}_p} W(R), \Theta_\rho)$, where

$$\Theta_\rho(a \otimes \xi) = \rho \cdot (a \otimes {}^F\xi), \quad a \in O_F,\ \xi \in W(R). \tag{4.5.3}$$

When $\rho = 1$ we obtain the constant p-adic étale sheaf $O_F = O_F(1)$.

Let $\rho = \pi^e/p$. Let $R \in \mathrm{Nilp}_{O_{E'}}$ and let $(\mathcal{P}_i, \iota_i) \in \mathfrak{P}_{r,R}$ for $i = 1, 2$. We will associate to a bilinear form of displays

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_{m,R} \tag{4.5.4}$$

which is anti-linear for the O_K-actions ι_1, resp. ι_2, a bilinear form of p-adic étale sheaves which is anti-linear for the O_K-actions on $C_{\mathcal{P}_1} = \mathfrak{C}_{r,R}(\mathcal{P}_1)$, resp. $C_{\mathcal{P}_2} = \mathfrak{C}_{r,R}(\mathcal{P}_2)$,

$$\phi : C_{\mathcal{P}_1} \times C_{\mathcal{P}_2} \longrightarrow O_F(\rho). \tag{4.5.5}$$

For the construction we may assume that R is a $\kappa_{E'}$-algebra because étale sheaves are insensitive to nilpotent elements.

Let first K/F be ramified. Then we find for $x_1 \in P_1$ and $x_2 \in P_2$ that

$$\begin{aligned}\beta(\dot{F}'_1 x_1, \dot{F}'_2 x_2) = \beta(\dot{F}_1 \Pi^e x_1, \dot{F}_2 \Pi^e x_2) &= {}^{\dot{F}}\beta(\Pi^e x_1, \Pi^e x_2) \\ &= {}^{\dot{F}}\beta(\pi^e x_1, x_2) = {}^F\beta(\frac{\pi^e}{p} x_1, x_2).\end{aligned} \tag{4.5.6}$$

We used the equation

$$\dot{F}'_1 x_1 = \dot{F}_1 \Pi^e x_1, \tag{4.5.7}$$

which follows from (4.3.4) and (4.3.9). Recall the function $\mathbf{t}(a) = \mathrm{Tr}_{O_F/\mathbb{Z}_p} \vartheta^{-1} a$, for $a \in O_F$, where $\vartheta \in O_F$ is the different of $F/\mathbb{Q}_p$, cf. p. 63. We define $\tilde{\beta} : P_1 \times P_2 \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(R)$ by the equation

$$\mathbf{t}(\xi \tilde{\beta}(x_1, x_2)) = \beta(\xi x_1, x_2), \quad \xi \in O_F \otimes_{\mathbb{Z}_p} W(R). \tag{4.5.8}$$

Then $\tilde{\beta}$ is a bilinear form of $O_F \otimes_{\mathbb{Z}_p} W(R)$-modules. We conclude from (4.5.6) that

$$\tilde{\beta}(\dot{F}' x_1, \dot{F}' x_2) = \frac{\pi^e}{p} \cdot {}^F\tilde{\beta}(x_1, x_2). \tag{4.5.9}$$

Hence $\tilde{\beta}$ is a bilinear form of Frobenius modules. Since the functor $\mathfrak{A}$ of Proposition 4.5.2 commutes with tensor products, it induces a bilinear form (4.5.5).

Now we consider the case where K/F is an unramified field extension. For each $O_{E'}$-algebra R we have the decomposition

$$O_K \otimes_{\mathbb{Z}_p} W(R) = \prod_{\psi \in \Psi} O_K \otimes_{O_{K^t}, \tilde{\psi}} W(R), \tag{4.5.10}$$

which is induced by (4.3.2). The conjugation of K/F acts on $O_K \otimes_{\mathbb{Z}_p} W(R)$ via the first factor. We denote this by $\eta \mapsto \bar{\eta}$. We denote by ${}^F\eta$ the action of the Frobenius via the second factor. On the right hand side of (4.5.10) these actions become

$$\begin{aligned} O_K \otimes_{O_{K^t},\tilde{\psi}} W(O_{E'}) &\longrightarrow O_K \otimes_{O_{K^t},\tilde{\psi}\sigma^f} W(O_{E'}) \\ a \otimes \xi &\longmapsto \quad \bar{a} \otimes \xi, \\ O_K \otimes_{O_{K^t},\tilde{\psi}} W(R) &\longrightarrow O_K \otimes_{O_{K^t},\sigma\tilde{\psi}} W(R) \\ a \otimes \xi &\longmapsto \quad a \otimes {}^F\xi. \end{aligned} \tag{4.5.11}$$

Here σ denotes the Frobenius automorphism of $\mathrm{Gal}(K^t/\mathbb{Q}_p)$. Looking at the right hand side of (4.5.10), we define

$$\pi_r := (\pi^{a_\psi} \otimes 1)_{\psi \in \Psi} \in O_K \otimes_{\mathbb{Z}_p} W(O_{E'}). \tag{4.5.12}$$

It follows that

$$\pi_r \bar{\pi}_r = \pi^e \otimes 1.$$

Let $(\mathcal{P}, \iota) \in \mathfrak{P}_{r,R}$. We note that, since R is a $\kappa_{E'}$-algebra, the definition of $(\mathcal{P}', \iota') = \mathfrak{C}'_{r,R}(\mathcal{P}, \iota)$ in (4.3.9) takes the form

$$\dot{F}'(x) = \dot{F}(\pi_r x), \quad F'(x) = F(\pi_r x). \tag{4.5.13}$$

Now let us start with a bilinear form

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_m$$

which is anti-linear for the O_K-actions ι_1, resp. ι_2. We find

$$\begin{aligned} \beta(\dot{F}'_1 x_1, \dot{F}'_2 x_2) = \beta(\dot{F}_1 \pi_r x_1, \dot{F}_2 \pi_r x_2) &= {}^{\dot{F}}\beta(\pi_r x_1, \pi_r x_2) \\ &= {}^{\dot{F}}\beta(\pi^e x_1, x_2) = {}^F\beta(\frac{\pi^e}{p} x_1, x_2). \end{aligned} \tag{4.5.14}$$

As before, β defines the $O_F \otimes_{\mathbb{Z}_p} W(R)$-bilinear form

$$\tilde{\beta} : P_1 \times P_2 \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(R),$$

which by (4.5.14) satisfies

$$\tilde{\beta}(\dot{F}'_1 x_1, \dot{F}'_2 x_2) = \frac{\pi^e}{p} \cdot {}^F\tilde{\beta}(x_1, x_2). \tag{4.5.15}$$

Applying, as before, the $\mathfrak{A}$-functor to $\mathcal{P}'$, we obtain the desired bilinear form (4.5.5).

Finally we consider the split case. In this case we consider in the decomposition (4.3.22) the element

$$\pi_r = \pi_{r,1} \times \pi_{r,2} = \big((\pi^{a_{\theta_1}} \otimes 1)_{\theta \in \Theta}\big) \times \big((\pi^{a_{\theta_2}} \otimes 1)_{\theta \in \Theta}\big) \tag{4.5.16}$$

of (4.3.22). The conjugation acts on $O_K \otimes_{\mathbb{Z}_p} W(O_{E'})$ via the first factor. On the right hand side of (4.3.22), the conjugation just interchanges the two factors in parentheses. This shows that $\pi_r \bar{\pi}_r = \pi^e \otimes 1$. Now starting with a bilinear form[7] (4.5.4), the formulas (4.5.14), (4.5.15) from the unramified case continue to hold, and this finishes the construction in the split case.

Proposition 4.5.9. *Let r be banal. Let $R \in \mathrm{Nilp}_{O_{E'}}$ be such that the ideal of nilpotent elements of R is nilpotent. Let $(\mathcal{P}_1, \iota_1)$ and $(\mathcal{P}_2, \iota_2)$ be objects of $\mathfrak{d}\mathfrak{P}_{r,R}$. The construction above, which associates to a bilinear form of displays (4.5.4) a bilinear form of p-adic étale sheaves (4.5.5) is a bijection,*

$$\mathrm{Bil}_{O_K\text{-anti-linear}}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}_{m,R}) \longrightarrow \mathrm{Bil}_{O_K\text{-anti-linear}}(C_{\mathcal{P}_1} \times C_{\mathcal{P}_2}, O_F(\rho)).$$

Proof. We reduce the question to the case where R is reduced. Indeed, let $S \longrightarrow R$ be a pd-thickening in the category $\mathrm{Nilp}_{O_{E'}}$. Assume that $(\mathcal{P}_i, \iota_i) \in \mathfrak{d}\mathfrak{P}_{r,S}$ for $i = 1, 2$. It follows from Proposition 3.2.4 and Lemma 4.5.6 that any bilinear form

$$\bar{\beta} : \mathcal{P}_{1,R} \times \mathcal{P}_{2,R} \longrightarrow \mathcal{P}_{m,R}$$

[7] One should not confuse the notation $\mathcal{P}_1$ and $\mathcal{P}_2$ with the decomposition (4.3.25) which continues to exist, e.g., $\mathcal{P}_1 = \mathcal{P}_{1,1} \oplus \mathcal{P}_{1,2}$.

with the properties of (4.5.4) lifts uniquely to a bilinear form

$$\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_{m,S}.$$

Since bilinear forms of étale sheaves have the same property, we can assume that R is reduced.

We begin with the ramified case. For $i = 1, 2$, let $(\mathcal{P}_i, \iota_i) \in \mathfrak{d}\mathfrak{P}_{r,R}$, which correspond to $(P'_i, \dot{F}'_i)$ under the pre-contracting functor $\mathfrak{C}'_{r,R}$, cf. Theorem 4.3.2, and to $C_{\mathcal{P}_i}$ under the contraction functor $\mathfrak{C}_{r,R}$. We start with a bilinear form of p-adic sheaves

$$\tilde{\phi} : C_{\mathcal{P}_1} \times C_{\mathcal{P}_2} \longrightarrow O_F(\frac{\pi^e}{p})$$

with the properties of (4.5.5). We have to construct a bilinear form of displays (4.5.4) which induces ϕ. By Proposition 4.5.2, ϕ comes from a bilinear form of étale Frobenius modules

$$\tilde{\beta} : P'_1 \times P'_2 \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(R)$$

which satisfies

$$\tilde{\beta}(\dot{F}'_1 x_1, \dot{F}'_2 x_2) = \frac{\pi^e}{p}\ {}^F\tilde{\beta}(x_1, x_2). \tag{4.5.17}$$

After applying $\mathbf{t}$ we obtain a bilinear form β which satisfies

$$\beta(\dot{F}'_1 x_1, \dot{F}'_2 x_2) = {}^F\beta(\frac{\pi^e}{p} x_1, x_2). \tag{4.5.18}$$

By (4.5.7) we may write

$$F_i = p\dot{F}_i = \dot{F}'_i \frac{p}{\Pi^e},$$

because multiplication by p is injective on P_i. We deduce from (4.5.18)

$$\beta(F_1 x_1, F_2 x_2) = \beta(\dot{F}'_1 \frac{p}{\Pi^e} x_1, \dot{F}'_2 \frac{p}{\Pi^e} x_2) = {}^F\beta(\frac{\pi^e}{p}\frac{p}{\Pi^e} x_1, \frac{p}{\Pi^e} x_2) = p\ {}^F\beta(x_1, x_2).$$

By Lemma 4.5.7, it follows that β is a bilinear form of displays $\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_m$. This proves the ramified case.

Now let K/F be an unramified field extension. We begin with a bilinear form of p-adic étale sheaves (4.5.5) as before. This induces a bilinear form of étale Frobenius modules $\tilde{\beta} : P'_1 \times P'_2 \longrightarrow O_F \otimes_{\mathbb{Z}_p} W(R)$ which satisfies (4.5.17). Using (4.5.13), we rewrite this as

$$\tilde{\beta}(\dot{F}_1 \pi_r x_1, \dot{F}_2 \pi_r x_2) = \frac{\pi^e}{p}\ {}^F\tilde{\beta}(x_1, x_2).$$

We multiply this equation with p^2 and find for the left hand side

$$\tilde{\beta}(F_1 \pi_r x_1, F_2 \pi_r x_2) = \tilde{\beta}(\ {}^F\pi_r F_1 x_1,\ {}^F\pi_r F_2 x_2) = \tilde{\beta}(\ {}^F(\bar{\pi}_r \pi_r) F_1 x_1, F_2 x_2) = \pi^e \tilde{\beta}(F_1 x_1, F_2 x_2).$$

If we compare this to the right hand side multiplied with p^2, we obtain

$$\tilde{\beta}(F_1 x_1, F_2 x_2) = p\ {}^F\tilde{\beta}(x_1, x_2).$$

Setting now $\beta = \mathbf{t} \circ \tilde{\beta}$, the assumptions of Lemma 4.5.7 are satisfied. Therefore β induces a bilinear form of displays $\beta : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_m$.

In the split case the argument is the same using the π_r which appeared in this context. $\square$

On $\operatorname{Spec} \bar{\kappa}_E$ we can choose a trivialization of the twisted constant étale sheaf,

$$O_F(\pi^e/p) \xrightarrow{\sim} O_F, \tag{4.5.19}$$

as follows. Choose $\eta \in O_F \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_E)$ such that ${}^F\eta\eta^{-1} = \pi^e/p$ (this is equivalent to the choice of η_0 in (4.4.12)). Then the multiplication by η

$$\eta : (O_F \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_E), (\pi^e/p) \otimes F) \longrightarrow (O_F \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_E), 1 \otimes F)$$

is an isomorphism of étale Frobenius modules, which induces (4.5.19) under the $\mathfrak{A}$-functor into $\operatorname{Et}(O_K)_{\bar{\kappa}_E}$, cf. Proposition 4.5.2.

Definition 4.5.10. Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$. Let $\mathrm{Et}(O_K)_R^{\mathrm{pol}}$ be the category of p-adic étale sheaves $(G, \iota) \in \mathrm{Et}(O_K)_R$, equipped with a O_F-linear alternating form

$$\phi : G \times G \longrightarrow O_F, \tag{4.5.20}$$

which is anti-linear for the O_K-action.

Using the trivialization (4.5.19), and applying Proposition 4.5.9, we now obtain the contracting functor with polarizations which is a functor from $\mathfrak{d}\mathfrak{P}_{r,R}^{\mathrm{pol}}$ to $\mathrm{Et}(O_K)_R^{\mathrm{pol}}$.

Theorem 4.5.11. *Let r be banal. Let $R \in \mathrm{Nilp}\, O_{\breve{E}}$ be such that the ideal of nilpotent elements in R is nilpotent. Then the contracting functor $\mathfrak{C}_{r,R}^{\mathrm{pol}}$ is an equivalence of categories,*

$$\mathfrak{C}_{r,R}^{\mathrm{pol}} \colon \mathfrak{d}\mathfrak{P}_{r,R}^{\mathrm{pol}} \longrightarrow \mathrm{Et}(O_K)_R^{\mathrm{pol}}.$$

□

Remark 4.5.12. In the split case, let $C_{\mathcal{P}} = \mathfrak{C}_{r,R}^{\mathrm{pol}}(\mathcal{P})$. Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ be the decomposition induced by $O_K = O_F \times O_F$. This induces a decomposition $C_{\mathcal{P}} = C_{\mathcal{P},1} \times C_{\mathcal{P},2}$, where $C_{\mathcal{P},i}$ is the étale sheaf associated to the Frobenius module $(P_i, \dot{F}\pi_{r,i})$, $i = 1, 2$. Here the elements $\pi_{r,i}$ are defined in (4.5.16). The subsheaves $C_{\mathcal{P},i}$ of $C_{\mathcal{P}}$ are isotropic with respect to ϕ as in (4.5.20), and hence ϕ corresponds to an O_F-bilinear form

$$\phi : C_{\mathcal{P},1} \times C_{\mathcal{P},2} \longrightarrow O_F.$$

Remark 4.5.13. Let $k \in \mathrm{Nilp}_{O_{E'}}$ be an algebraically closed field. Let $(\mathcal{P}, \beta) \in \mathfrak{d}\mathfrak{P}_{r,k}^{\mathrm{pol}}$. We will give a description of $\mathfrak{C}_{r,R}^{\mathrm{pol}}(\mathcal{P}, \beta) = (C_{\mathcal{P}}, \phi)$. We write $\mathcal{P} = (P, F, V)$ as a Dieudonné module. The image of $\mathcal{P}$ under the contracting functor, a sheaf $C_{\mathcal{P}}$ on $\mathrm{Spec}\, k$, is simply an O_K-module.

Assume that K/F is ramified. From the definition of the pre-contracting functor (cf. Theorem 4.3.2) and the $\mathfrak{A}$-functor we have

$$C_{\mathcal{P}} = \{x \in P \mid V^{-1}\Pi^e x = x\}.$$

To describe this further, with its bilinear form of displays, we extend the bilinear form β to

$$\tilde{\beta} : P \times P \longrightarrow O_F \otimes W(k),$$

cf. (4.5.8). The decomposition $P = \oplus_{\psi \in \Psi} P_\psi$ is orthogonal with respect to $\tilde{\beta}$ and, by restriction, we obtain for every ψ

$$\tilde{\beta}_\psi : P_\psi \times P_\psi \longrightarrow O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k) \subset O_F \otimes W(k).$$

Let $x_\psi, x'_\psi \in P_\psi$. Since β is a polarization, we obtain

$$\tilde{\beta}_{\psi\sigma}(V^{-1}\Pi^e x_\psi, V^{-1}\Pi^e x'_\psi) = \frac{\pi^e}{p}\ {}^F\tilde{\beta}_\psi(x_\psi, x'_\psi). \tag{4.5.21}$$

The action of F on the right hand side is defined by (4.5.11). Fix $\psi_a \in \Psi$. The projection $x \mapsto x_{\psi_a}$ is an isomorphism

$$C_{\mathcal{P}} \cong \{x_{\psi_a} \in P_{\psi_a} \mid V^{-f}\Pi^{ef} x_{\psi_a} = x_{\psi_a}\}. \tag{4.5.22}$$

In particular, we see that $C_{\mathcal{P}}$ is indeed a free O_K-module of rank 2. For $x, x' \in C_{\mathcal{P}}$ we obtain from (4.5.21)

$$\tilde{\beta}_{\psi\sigma}(x_{\psi\sigma}, x'_{\psi\sigma}) = \frac{\pi^e}{p}\ {}^F\tilde{\beta}_\psi(x_\psi, x'_\psi).$$

Since $\psi_a = \psi_a \sigma^f$, we obtain

$$\tilde{\beta}_{\psi_a}(x_{\psi_a}, x'_{\psi_a}) = \left(\frac{\pi^e}{p}\right)^f\ {}^{F^f}\tilde{\beta}_{\psi_a}(x_{\psi_a}, x'_{\psi_a}).$$

In the same way we may interpret the sheaf $O_F(\pi^e/p)$: the projection

$$O_F \otimes_{\mathbb{Z}_p} W(k) \longrightarrow O_F \otimes_{O_{F^t}, \tilde{\psi}_a} W(k)$$

defines an isomorphism

$$O_F(\frac{\pi^e}{p}) \cong \{a_{\psi_a} \in O_F \otimes_{O_{F^t}, \tilde{\psi}_a} W(k) \mid a_{\psi_a} = (\frac{\pi^e}{p})^f\ {}^{F^f} a_{\psi_a}\}. \tag{4.5.23}$$

For the last equation we may write $\eta a_{\psi_a} = {}^{F^f}(\eta a_{\psi_a})$ (cf. (4.5.19) for η) or, equivalently, $\eta a_{\psi_a} \in O_F$. Therefore, using the expression (4.5.22) for $C_{\mathcal{P}}$, the restriction of $\tilde{\beta}$ to $C_{\mathcal{P}}$ multiplied by η gives the desired bilinear form

$$\begin{aligned} \phi\colon C_{\mathcal{P}} \times C_{\mathcal{P}} &\longrightarrow O_F \\ (x_{\psi_a}, x'_{\psi_a}) &\longmapsto \eta\tilde{\beta}_{\psi_a}(x_{\psi_a}, x'_{\psi_a}). \end{aligned} \tag{4.5.24}$$

Now let K/F be unramified. In this case, in the decomposition $P = \oplus_{\psi\in\Psi} P_\psi$, the summands P_{ψ_1} and P_{ψ_2} are orthogonal, unless $\psi_1 = \bar{\psi}_2$. The O_K-module $C_{\mathcal{P}}$ is, in this case, given by

$$C_{\mathcal{P}} = \{x = (x_\psi) \in P \mid V^{-1}\pi_r x_\psi = x_{\psi\sigma}\}, \tag{4.5.25}$$

where we recall the element π_r from (4.5.12). After fixing ψ_a, we can write

$$C_{\mathcal{P}} = \{(x_{\psi_a}, x_{\bar{\psi}_a}) \in P_{\psi_a} \oplus P_{\bar{\psi}_a} \mid V^{-f}\pi^g x_{\psi_a} = x_{\bar{\psi}_a}, V^{-f}\pi^{\bar{g}} x_{\bar{\psi}_a} = x_{\psi_a}\}, \tag{4.5.26}$$

where $g = a_{\psi_a} + a_{\psi_a\sigma} + \cdots + a_{\psi_a\sigma^{f-1}}$ and $\bar{g} = a_{\bar{\psi}_a} + a_{\bar{\psi}_a\sigma} + \cdots + a_{\bar{\psi}_a\sigma^{f-1}}$. Using the expression (4.5.26) for $C_{\mathcal{P}}$, we may write

$$\begin{aligned} \phi:\quad C_{\mathcal{P}} \times C_{\mathcal{P}} &\longrightarrow O_F(\rho) \\ (x_{\psi_a} + x_{\bar{\psi}_a}, y_{\psi_a} + y_{\bar{\psi}_a}) &\longmapsto \eta\tilde{\beta}_{\psi_a}(x_{\psi_a}, y_{\bar{\psi}_a}) + \eta\tilde{\beta}_{\bar{\psi}_a}(x_{\bar{\psi}_a}, y_{\psi_a}). \end{aligned} \tag{4.5.27}$$

We have for arbitrary elements $x_\psi \in P_\psi$ and $y_{\bar{\psi}} \in P_{\bar{\psi}}$ that

$$\tilde{\beta}_{\bar{\psi}}(V^{-f}\pi^g x_\psi, V^{-f}\pi^{\bar{g}} y_{\bar{\psi}}) = \frac{\pi^{ef}}{p^f}\ {}^{F^f}\tilde{\beta}_\psi(x_\psi, y_{\bar{\psi}}).$$

If $x = x_{\psi_a} + x_{\bar{\psi}_a}$ and $y = y_{\psi_a} + y_{\bar{\psi}_a}$ in $C_{\mathcal{P}}$ the last formula becomes

$$\tilde{\beta}_{\bar{\psi}}(x_{\bar{\psi}}, y_\psi) = \frac{\pi^{ef}}{p^f}\ {}^{F^f}\tilde{\beta}_\psi(x_\psi, y_{\bar{\psi}}).$$

By the formula (4.5.27) for ϕ we obtain

$$\phi(x,y) = {}^{F^f}\phi(x,y).$$

This shows again that $\phi(x,y) \in O_F$, cf. (4.5.23).

Finally let $K = F \times F$. We use the notation of the last Remark. We obtain

$$C_{\mathcal{P}_i} = \{x \in P_i \mid Vx = \pi_{r,i} x\ \},$$

where we recall the element $\pi_{r,i}$ from (4.5.15). We have the decomposition

$$C_{\mathcal{P}} = C_{\mathcal{P}_1} \oplus C_{\mathcal{P}_2}$$

If we fix $\theta_0 \in \Theta = \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(F^t, \bar{\mathbb{Q}}_p)$, the natural projection $P_i \to P_{i,\theta_0}$ defines an isomorphism

$$C_{\mathcal{P}_i} = \{x \in P_{i,\theta_0} \mid V^f x = \pi^{g_i} x\ \}, \quad \text{for } i = 1, 2, \tag{4.5.28}$$

where $g_i = \sum_\theta a_{\theta_i}$, cf. (4.3.28). The bilinear form $\tilde{\beta}$ induces by restriction

$$\tilde{\beta}_{\theta_0} : P_{1,\theta_0} \times P_{2,\theta_0} \to O_F \otimes_{O_{F^t},\tilde{\theta}_0} W(k).$$

In the notation of (4.5.28) we obtain

$$\begin{aligned} \phi : C_{\mathcal{P}_1} \times C_{\mathcal{P}_2} &\quad\to\quad O_F, \\ (x_1, x_2) &\quad\longmapsto\quad \eta\tilde{\beta}_{\theta_0}(x_1, x_2). \end{aligned}$$

This determines ϕ on $C_{\mathcal{P}}$ since the subspaces $C_{\mathcal{P}_i}$ for $i = 1, 2$ are isotropic, cf. Remark 4.5.12.

Proposition 4.5.14. *Let r be banal. Let $k \in \mathrm{Nilp}_{O_{E'}}$ be an algebraically closed field. Let $(\mathcal{P}, \iota, \beta)$ and $(\mathcal{P}^+, \iota^+, \beta^+)$ be two objects in $\mathfrak{dP}^{\mathrm{pol}}_{r,k}$.*

(i) *If K/F is split, then there exists a quasi-isogeny*

$$(\mathcal{P}, \iota, \beta) \to (\mathcal{P}^+, \iota^+, \beta^+). \tag{4.5.29}$$

(ii) *Let K/F be a field extension. Then there exists a quasi-isogeny (4.5.29) iff* $\mathrm{inv}(\mathcal{P}, \iota, \beta) = \mathrm{inv}(\mathcal{P}^+, \iota^+, \beta^+)$*, cf. (2.4.7).*

(iii) *Let K/F be an unramified field extension. If β is a polarization of height $2fh$ with $h \in \{0, 1\}$ then* $\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = (-1)^h$*. For a given h, there exists $(\mathcal{P}, \iota, \beta)$ with these properties.*

Proof. To prove the first assertion, we may apply the polarized contraction functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$ of Theorem 4.5.11. We choose an arbitrary isomorphism α_1 of the F-vector spaces $C_{\mathcal{P}_1} \otimes \mathbb{Q}$ and $C_{\mathcal{P}_1^+} \otimes \mathbb{Q}$. Since ϕ, resp. ϕ^+, define dualities of these spaces with $C_{\mathcal{P}_2} \otimes \mathbb{Q}$, resp. $C_{\mathcal{P}_2^+} \otimes \mathbb{Q}$, we can extend α_1 to an isomorphism $\alpha : (C_{\mathcal{P}}, \phi) \otimes \mathbb{Q} \to (C_{\mathcal{P}^+}, \phi^+) \otimes \mathbb{Q}$.

If K/F is a field extension, we conclude by Proposition 8.3.6 that the equality $\mathrm{inv}(\mathcal{P}, \iota, \beta) = \mathrm{inv}(\mathcal{P}^+, \iota^+, \beta^+)$ is equivalent to the equality $\mathrm{inv}((C_{\mathcal{P}}, \iota, \phi) \otimes \mathbb{Q}) = \mathrm{inv}((C_{\mathcal{P}^+}, \iota^+, \phi^+) \otimes \mathbb{Q})$. Therefore, by Definition 8.1.1, these anti-hermitian K-vector spaces are isomorphic, which proves our assertion.

Finally we prove the last assertion. We consider the bilinear form $\beta_\psi : P_\psi \times P_{\bar{\psi}} \to W(k)$. If we choose a $W(k)$-basis of P_ψ and $P_{\bar{\psi}}$, we can speak of $\mathrm{ord}_p \det_{W(k)} \beta_\psi$. This number is independent of ψ and equals h. Let $\beta(\psi)$ be the restriction of β to $P_\psi \oplus P_{\bar{\psi}}$. We obtain $\mathrm{ord}_p \det_{W(k)} \beta(\psi) = 2h$. Recall $\tilde{\beta}$, cf. (4.5.8). Let $\tilde{\beta}(\psi)$ the restriction of $\tilde{\beta}$,

$$\tilde{\beta}(\psi) : (P_\psi \oplus P_{\bar{\psi}}) \times (P_\psi \oplus P_{\bar{\psi}}) \to O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k).$$

Then we have

$$\mathrm{ord}_p \det_{W(k)} \beta(\psi) = \mathrm{ord}_\pi \det_{O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)} \tilde{\beta}(\psi). \tag{4.5.30}$$

Indeed, the function $\mathbf{t}(a) = \mathrm{Tr}_{O_F/O_{F^t}} \vartheta^{-1} a$, for $a \in O_F$, where $\vartheta \in O_F$ is the different of $F/\mathbb{Q}_p$, defines for an arbitrary $O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)$-module U an isomorphism of $O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)$-modules,

$$\mathrm{Hom}_{O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)}(U, O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)) \xrightarrow{\sim} \mathrm{Hom}_{W(k)}(U, W(k)), \quad \tilde{\alpha} \longmapsto \tilde{\alpha} \circ \mathbf{t}. \tag{4.5.31}$$

We apply this to $U = P_\psi \oplus P_{\bar{\psi}}$. If we regard $\tilde{\beta}$ as a homomorphism of U to the left hand side of (4.5.31) and β as a homomorphism from U to the right hand side, they correspond to each other. Therefore the cokernels of these two homomorphisms are isomorphic and have the same length. This shows (4.5.30). By Remark 4.5.13, we have for each ψ an isomorphism $C_{\mathcal{P}} \otimes_{O_F} (O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k)) \cong \mathcal{P}$. Since ϕ coincides with the restriction of $\tilde{\beta}(\psi)$ up to a unit, we conclude that $\mathrm{ord}_\pi \det_{O_F} \phi = 2h$. By Lemma 8.1.2, we have $\mathrm{inv}(C_{\mathcal{P}}, \iota, \phi) = (-1)^h$. By Proposition 8.3.6 we are done. $\square$

5. The alternative moduli problem revisited

In this section we give another proof of the main result of [17] which gives an alternative interpretation of the Drinfeld moduli space of special formal O_D-modules in the case of a quaternion division algebra D over a p-adic local field F. We also prove a refinement concerning descent data. The original proof was already simplified by Kirch [14], but the argument here is different and is based on the theory of displays.

5.1. **Special formal O_D-modules.** We fix the finite extension F of $\mathbb{Q}_p$, with uniformizer π and residue field κ_F. Let R be an O_F-algebra. Let (X, ι) be a p-divisible group over R with a strict action $\iota : O_F \longrightarrow \mathrm{End}\, X$. A relative polarization of X is a relative polarization of the display of X. Here, by a relative polarization we mean one with respect to O_F, cf. Definition 3.4.12.

If $R = k$ is a perfect field, we may work with the associated $\mathcal{W}_{O_F}(k)$-Dieudonné module (M, F, V) of X. It is obtained from the display of X by the Ahsendorf functor $\mathfrak{A}_{O_F/\mathbb{Z}_p, R}$, cf. Remark 3.3.16. In this language, a relative polarization is a $\mathcal{W}_{O_F}(k)$-alternating pairing

$$\psi : M \times M \longrightarrow \mathcal{W}_{O_F}(k),$$

such that

$$\psi(Fx, Fy) = \pi\ {}^F\psi(x, y), \quad x, y \in M, \tag{5.1.1}$$

cf. (3.2.2). If the bilinear form ψ is perfect, we will say that the polarization is principal.

We denote by D the quaternion division algebra with center F. Let $F' \subset D$ be a quadratic unramified extension of F. Let $O_D \subset D$ be the ring of integers. Recall that a *special formal O_D-module* X over R is a p-divisible group X over R of height $[D : \mathbb{Q}_p]$ with an action $\iota : O_D \longrightarrow \mathrm{End}\, X$ such that the restriction of ι to O_F is strict and such that $\mathrm{Lie}\, X$ is locally on $\mathrm{Spec}\, R$ a free $O_{F'} \otimes_{O_F} R$-algebra, cf. [9]. One can check that this condition is independent of the choice of F'.

Proposition 5.1.1. *Let k be an algebraically closed field of characteristic p with an O_F-algebra structure $O_F \longrightarrow k$. Let F' be an unramified quadratic extension F' of F. We denote by F and V the Frobenius and the Verschiebung acting on $W_{O_F}(k)$.*

Let (M, F, V) be a $\mathcal{W}_{O_F}(k)$-Dieudonné module (see Definition 3.1.8) of height 4 and dimension 2 which is endowed with an O_F-algebra homomorphism

$$\iota : O_{F'} \longrightarrow \mathrm{End}(M, F, V).$$

Assume that ι makes M/VM into a free module of rank 1 over $\kappa_{F'} \otimes_{\kappa_F} k$. Then there exists a principal relative polarization ψ on (M, F, V) such that

$$\psi(\iota(u)x, y) = \psi(x, \iota(u)y), \quad \text{for } u \in O_{F'},\ x, y \in M \tag{5.1.2}$$

Any other relative polarization ϕ of (X, ι) with the property (5.1.2) (with ψ replaced by ϕ) is of the form

$$\phi(x, y) = \psi(\iota(c)x, y)$$

for some element $c \in O_{F'}$.

Proof. We choose an embedding $O_{F'} \longrightarrow W_{O_F}(k)$. We set, for $i \in \mathbb{Z}/2\mathbb{Z}$,

$$M_i = \{x \in M \mid \iota(u)x = {}^{F^i}ux, \text{ for } u \in O_{F'}\}.$$

We have the decomposition

$$M = M_0 \oplus M_1. \tag{5.1.3}$$

The operators F and V are of degree 1. The k-vector spaces M_0/VM_1 and M_1/VM_0 are by assumption both of rank 1.

If ψ is a bilinear form with the properties (5.1.2), then the decomposition (5.1.3) is orthogonal. We choose alternating perfect forms $\tilde{\psi}_0$ resp. $\tilde{\psi}_1$ on the free $W_{O_F}(k)$-modules M_0 resp. M_1 of rank 2. These forms are unique up to a unit in $W_{O_F}(k)$. By this uniqueness we find an equation of the form

$${}^{F^{-2}}\tilde{\psi}_0(F^2x_0, F^2x_0') = \xi\pi^2\tilde{\psi}_0(x_0, x_0'), \quad \xi \in W_{O_F}(k), \text{ for all } x_0, x_0' \in M_0. \tag{5.1.4}$$

By assumption we have $\mathrm{ord}_\pi \det(F^2|M_0) = 2$. Comparing the determinants on both sides of (5.1.4), we conclude that ξ is a unit. Since k is algebraically closed we may write

$$\xi = {}^{F^{-2}}\eta\eta^{-1}.$$

Replacing $\tilde{\psi}_0$ by $\psi_0 := \eta\tilde{\psi}_0$ we may assume that we have $\xi = 1$ in equation (5.1.4).

With the same argument as before we find an equation

$${}^{F^{-1}}\psi_0(Fx_1, Fx_1') = \xi_1\pi\tilde{\psi}_1(x_1, x_1'), \quad \xi_1 \in W_{O_F}(k), \text{ for all } x_1, x_1' \in M_1.$$

Comparing the determinants we see that $\xi_1 \in W_{O_F}(k)$ is a unit. We set $\psi_1 = \xi_1\tilde{\psi}_1$ and $\psi = \psi_0 \oplus \psi_1$ (orthogonal sum). Then ψ satisfies (5.1.1). To prove (5.1.2) it suffices to show that

$$\psi_i(\iota(u)x_i, x_i') = \psi_i(x_i, \iota(u)x_i') \quad \text{for } i = 0, 1.$$

This is trivial from the definition of (M_i, ψ_i).

If we have a second ϕ satisfying (5.1.2), we find $c \in W_{O_F}(k)$ such that

$$\phi_0 = c\psi_0.$$

Since both sides of this equation satisfy (5.1.4) with $\xi = 1$ we obtain ${}^{F^2}c = c$. Therefore we have $c \in O_{F'} \subset W_{O_F}(k)$. We obtain that $\phi(x, y) = \psi(\iota(c)x, y)$. □

Corollary 5.1.2. *Let (N, ι) be a second $\mathcal{W}_{O_F}(k)$-Dieudonné module of height 4 and dimension 2 with an action of $O_{F'}$ such that N/VN is a free $\kappa_{F'} \otimes_{\kappa_F} k$-module of rank 1. Let $\rho : N \otimes \mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$ be a quasi-isogeny of height 0. Then*

$$\psi(\rho(z), \rho(w)), \quad z, w \in N \tag{5.1.5}$$

is a perfect bilinear form on N.

Proof. Let ψ_N be a perfect alternating form on N given by Proposition 5.1.1, and let ψ_{N_0} be its restriction to N_0. This form differs from the form (5.1.5) restricted to N_0 by a factor in $\zeta \in F'$. Since ρ has height 0 we conclude that ζ is a unit. □

Let K/F be a ramified quadratic extension of F generated by a prime element $\Pi \in K$ such that $\Pi^2 = -\pi$. Let $\tau \in \mathrm{Gal}(F'/F)$ be the Frobenius automorphism. Let

$$O_D = O_{F'}[\Pi],$$

such that the following relations hold:

$$\Pi u = \tau(u)\Pi, \quad \Pi^2 = -\pi, \quad u \in O_{F'}.$$

Then O_D is the maximal order in the quaternion division algebra over F.

We have $O_K = O_F[\Pi] \subset O_D$. We consider on O_D the involution:

$$d = u + v\Pi \longmapsto d' = u - \Pi v, \quad u, v \in O_{F'}. \tag{5.1.6}$$

It is trivial on $O_{F'}$ and induces the conjugation of O_K over O_F.

Proposition 5.1.3. *Let k be an algebraically closed field of characteristic p which is endowed with an algebra structure $O_F \longrightarrow k$. Let X be a special formal O_D-module over k. Let (M, F, V) be the $\mathcal{W}_{O_F}(k)$-Dieudonné module of X. Then there exists a principal relative polarization*

$$\psi : M \times M \longrightarrow W_{O_F}(k),$$

on X such that

$$\psi(\iota(d)x_1, x_2) = \psi(x_1, \iota(d')x_2). \tag{5.1.7}$$

Any other polarization with the property (5.1.7) is of the form $u\psi$, with $u \in O_F$.

Proof. We take ψ as in Proposition 5.1.1. Then we consider the alternating bilinear form

$$\psi_1(x, y) = \psi(\iota(\Pi)x, \iota(\Pi)y), \quad x, y \in M.$$

Then ψ_1 is by the uniqueness part of Proposition 5.1.1 of the form

$$\psi_1(x, y) = \psi(\iota(c)x, y), \quad c \in O_{F'}.$$

If we apply the last equations to $\pi^2\psi(x, y) = \psi(\iota(\Pi)x, \iota(\Pi)y)$, we obtain $c\tau(c) = \pi^2$. Therefore c is divisible by π. We write $c = a\pi$ for some unit $a \in O_{F'}$ with $a\tau(a) = 1$. We write $a = u\tau(u)^{-1}$ by Hilbert 90 and consider the form

$$\psi_2(x, y) = \psi(\iota(u)x, y).$$

Then we have

$$\begin{aligned}\psi_2(\iota(\Pi)x, \iota(\Pi)y) &= \psi(\iota(u)\iota(\Pi)x, \iota(\Pi)y) = \psi(\iota(\Pi)\iota(\tau(u))x, \iota(\Pi)y) = \\ &= \psi(\iota(c)\iota(\tau(u))x, y) = \psi(\iota(\pi u)x, y) = \pi\psi_2(x, y).\end{aligned}$$

Therefore ψ_2 satisfies the requirements (5.1.7). The uniqueness assertion is proved as before. $\square$

Definition 5.1.4. With the notation of Proposition 5.1.3 we call ψ a *Drinfeld polarization* on the special formal O_D-module X over k.

Proposition 5.1.5. *Let $\mathbb{X}$ be a special formal O_D-module over $\bar{\kappa}_F$. Let R be a $O_{\breve{F}}$-algebra such that p is nilpotent in R. Let (X, ι) be a special formal O_D-module over R such that there exists a quasi-isogeny of formal O_D-modules*

$$\mathbb{X} \otimes_{\bar{\kappa}_F} R/\pi R \longrightarrow X \otimes_R R/\pi R. \tag{5.1.8}$$

Then there is a principal relative polarization ψ on X which induces on O_D the involution $d \mapsto d'$ and which is up to a factor in $F^\times$ compatible with a Drinfeld polarization on $\mathbb{X}$ by the quasi-isogeny (5.1.8).

Assume that R is noetherian and that $\operatorname{Spec} R$ *is connected. Then any other relative polarization on X which induces the involution $d \mapsto d'$ is of the form $f\psi$ for some $f \in O_F$.*

Proof. We recall some generalities from [9] which are formulated there for Cartier modules. We fix an embedding $O_{F'} \longrightarrow O_{\breve{F}}$. From this we obtain homomorphisms $O_{F'} \longrightarrow R$ and $\lambda : O_{F'} \longrightarrow W_{O_F}(O_{F'}) \longrightarrow W_{O_F}(R)$. Let $\bar{\lambda}$ be the composite with the conjugation of F'/F.

Let $\mathcal{P}$ be the $\mathcal{W}_{O_F}(R)$-display of X. The action of O_D on $\mathcal{P}$ is also denoted by ι. We have the decompositions

$$P = P_0 \oplus P_1, \quad Q = Q_0 \oplus Q_1 \tag{5.1.9}$$

such that for $a \in O_{F'} \subset O_D$ the action of $\iota(a)$ on P_0 is multiplication by $\lambda(a)$ and the action on P_1 is multiplication by $\bar{\lambda}(a)$ and $Q_i = Q \cap P_i$. We regard (5.1.9) as a $\mathbb{Z}/2\mathbb{Z}$-grading. Then $F, \Pi, \dot{F}$ are all homogenous of degree 1,

$$F : P_i \longrightarrow P_{i+1}, \quad \Pi : P_i \longrightarrow P_{i+1}, \quad \dot{F} : Q_i \longrightarrow P_{i+1}. \tag{5.1.10}$$

To show the existence of ψ, we begin with the case where R is reduced and $\operatorname{Spec} R$ is connected. We remark that the set invariants of $F : W_{O_F}(R) \longrightarrow W_{O_F}(R)$ is $O_F \subset W_{O_F}(R)$.

We assume that there exists a critical index i for X. We may suppose that $i = 0$, i.e., the homomorphism

$$\Pi : P_0/Q_0 \longrightarrow P_1/Q_1$$

is zero. We consider the composite $\Phi : P_0 \xrightarrow{\Pi} Q_1 \xrightarrow{\dot{F}} P_0$. We claim that Φ is a Frobenius linear isomorphism. It is enough to show that $\det \Phi \in W_{O_F}(R)$ is a unit. By base change we may assume that $R = k$ is a perfect field. Since $i = 0$ is critical we find $\Pi P_0 \subset Q_1 = VP_0$. Since P_1/VP_0 and $P_1/\Pi P_0$ are k-vector spaces of dimension 1 we obtain $\Pi P_0 = VP_0$. Therefore $V^{-1}\Pi = \Phi$ is bijective and therefore a Frobenius linear isomorphism.

For each $n \in \mathbb{N}$ we consider the functor on the category of R-algebras,

$$U_0(n) : S \longmapsto (P_0 \otimes_{W_{O_F}(R)} W_{O_F,n}(S))^{\Phi}, \tag{5.1.11}$$

where the RHS denotes invariants of the Frobenius-linearly extended Φ. This functor is representable by a scheme which is finite and étale over $\operatorname{Spec} R$. Moreover the existence of the quasi-isogeny (5.1.8) implies that this scheme is constant. The sheaf (5.1.11) is therefore with its natural O_F-module structure isomorphic to the constant O_F-module $(O_F/\pi^n O_F)^2$. We set $U_0 = \operatorname{proj\,lim} U_0(n)$ and obtain an isomorphism functorial in R-algebras S,

$$W_{O_F}(S) \otimes_{W_{O_F}(R)} P_0 \cong W_{O_F}(S) \otimes_{O_F} U_0.$$

We choose a perfect alternating pairing $\alpha : U_0 \times U_0 \longrightarrow O_F$ and extend it by base change to a perfect bilinear pairing

$$\psi_0 : P_0 \times P_0 \longrightarrow W_{O_F}(R).$$

We extend this bilinear form to a form

$$\psi : (P \otimes \mathbb{Q}) \times (P \otimes \mathbb{Q}) \longrightarrow W_{O_F}(R) \otimes \mathbb{Q},$$

as follows. We note that $\pi \in W_{O_F}(R) \otimes \mathbb{Q}$ is a unit. Therefore we may define a bilinear alternating form on the $W_{O_F}(R) \otimes \mathbb{Q}$-module $P_1 \otimes \mathbb{Q}$ by the equation

$$\psi_1(x_1, x_1') = \frac{1}{\pi}\psi_0(\Pi x_1, \Pi x_1'), \quad x_1, x_1' \in P_1 \otimes \mathbb{Q}.$$

We define ψ on $P \otimes \mathbb{Q}$ as the orthogonal sum of ψ_0 and ψ_1. Clearly ψ induces on O_D the involution $d \mapsto d'$. We show the equation

$$\psi(\dot{F}x, \dot{F}x') = \frac{1}{\pi}\ {}^{F}\psi(x, x') = {}^{\dot{F}}\psi(x, x'), \quad x, x' \in P \otimes \mathbb{Q}. \tag{5.1.12}$$

Since ψ_0 is the linear extension of α, we find the equation

$$\psi_0(\dot{F}\Pi x_0, \dot{F}\Pi x_0') = {}^{F}\psi_0(x_0, x_0'), \quad x_0, x_0' \in P_0.$$

We note that $\dot{F}$ extends to an endomorphism of $P \otimes \mathbb{Q}$. This extension commutes with the action of the field F. From the definition of ψ_1, we obtain

$$\pi\psi_1(\dot{F}x_0, \dot{F}x_0') = \psi_0(\Pi\dot{F}x_0, \Pi\dot{F}x_0') = {}^{F}\psi_0(x_0, x_0'), \quad x_0, x_0' \in P_0 \otimes \mathbb{Q}.$$

If we take $x_0, x_0' \in Q_0$ we have $\psi_0(x_0, x_0') \in I_{O_F}(R)$ because $Q_0/I_{O_F}(R)P_0$ has rank 1 and ψ_0 is alternating. We deduce

$$\psi_1(\dot{F}x_0, \dot{F}x_0') = \frac{1}{\pi}\ {}^{F}\psi_0(x_0, x_0') = {}^{\dot{F}}\psi_0(x_0, x_0') \in W_{O_F}(R), \quad x_0, x_0' \in Q_0.$$

This equation shows that ψ_1 induces a pairing

$$\psi_1 : P_1 \times P_1 \longrightarrow W_{O_F}(R).$$

This pairing is perfect. Indeed, for the verification we may assume that R is a perfect field. Then we already have a perfect pairing by Proposition 5.1.3, which agrees with the form ψ here up to a unit constant.

To complete the verification of (5.1.12), we need to show that

$$\psi_0(\dot{F}x_1, \dot{F}x_1') = {}^{\dot{F}}\psi_1(x_1, x_1'), \quad x_1, x_1' \in Q_1.$$

We may verify this in $P \otimes \mathbb{Q}$. We may assume that $x_1 = \Pi x_0$, $x_1' = \Pi x_0'$. Then we find

$$\begin{aligned}\psi_0(\dot{F}x_1, \dot{F}x_1') &= \psi_0(\dot{F}\Pi x_0, \dot{F}\Pi x_0') = {}^{F}\psi_0(x_0, x_0') = \frac{1}{\pi^2}\, {}^{F}\psi_0(\Pi x_1, \Pi x_1') \\ &= \frac{1}{\pi}\, {}^{F}\psi_1(x_1, x_1') = {}^{\dot{F}}\psi_1(x_1, x_1'),\end{aligned}$$

as desired.

We now drop the assumption that R is reduced but we continue to assume that the map

$$\Pi : P_0/Q_0 \longrightarrow P_1/Q_1 \tag{5.1.13}$$

is zero. It suffices to show that, if $R' \longrightarrow R$ is a *pd*-thickening, then a polarization ψ on $\mathcal{P}$ lifts to $\mathcal{P}'$. Here $\mathcal{P}$ and $\mathcal{P}'$ are the displays of X_R and $X_{R'}$. We apply the Grothendieck-Messing lifting theorem for displays [1], cf. the end of subsection 3.1. Hence it suffices to see that $Q'/I(R')P'$ is totally isotropic for the pairing on $P'/I(R')P' \times P'/I(R')P'$ induced by ψ. But $Q'/I(R')P' = (Q'/I(R')P')_0 \oplus (Q'/I(R')P')_1$ as an orthogonal direct sum, and both summands are free of rank one over R', locally on $\operatorname{Spec} R'$, whence the assertion.

For a general $O_{\breve{F}}$-algebra R such that p is nilpotent in R we consider the closed subscheme $\operatorname{Spec} R/\mathfrak{a}_0$ which represents the property that (5.1.13) is zero. In the same way $\operatorname{Spec} R/\mathfrak{a}_1$ is defined from the map $\Pi : P_1/Q_1 \longrightarrow P_0/Q_0$. We consider the exact sequence

$$0 \longrightarrow R/(\mathfrak{a}_0 \cap \mathfrak{a}_1) \longrightarrow R/\mathfrak{a}_0 \times R/\mathfrak{a}_1 \longrightarrow R/(\mathfrak{a}_0 + \mathfrak{a}_1) \longrightarrow 0.$$

Since we know the existence of ψ for $R/\mathfrak{a}_0$ and $R/\mathfrak{a}_1$, a gluing argument shows the existence of ψ for $R/(\mathfrak{a}_0 \cap \mathfrak{a}_1)$. Over a field one of the indices 0 or 1 is critical. This implies the $V(\mathfrak{a}_0) \cup V(\mathfrak{a}_1) = \operatorname{Spec} R$, i.e. the ideal $\mathfrak{a}_0 \cap \mathfrak{a}_1$ is nilpotent. We can as above apply the Grothendieck-Messing criterion to $R \longrightarrow R/(\mathfrak{a}_0 \cap \mathfrak{a}_1)$ to obtain the existence of ψ for R.

For the last assertion we may assume by rigidity that R is an algebraically closed field. Let ψ' be a second polarization. We may assume that the index 0 is critical. Then ψ' induces by restriction an alternating pairing $U_0 \times U_0 \longrightarrow O_F$. This differs from α above by a factor $f \in O_F$. □

With the notation of the last Proposition we may regard ψ as an isomorphism $X \longrightarrow X^{\nabla}$, cf. (3.4.22). By duality O_D^{opp} acts on X^{∇}. If we compose this with the isomorphism $O_D \longrightarrow O_D^{\mathrm{opp}}$ defined by the involution (5.1.6), we obtain an action of O_D on X^{∇}. This gives a special formal O_D-module which we denote by X^{Δ}. The polarization ψ may be regarded as an isomorphism of special formal O_D-modules

$$\lambda : X \longrightarrow X^{\Delta}. \tag{5.1.14}$$

Corollary 5.1.6. *Let $\psi_{\mathbb{X}}$ be a Drinfeld polarization on $\mathbb{X}$. Let*

$$\rho \colon \mathbb{X} \otimes_{\bar{\kappa}_F} R/\pi R \longrightarrow X \otimes_R R/\pi R$$

be a quasi-isogeny of height 0. Then the relative quasi-polarization $\bar{\psi}$ on $X \otimes_R R/\pi R$ induced by $\psi_{\mathbb{X}}$ is a relative principal polarization that lifts to a relative polarization ψ on X.

Proof. We take a perfect relative polarization ψ on X which exists by Proposition 5.1.3. Then the polarization induced from $\psi_{\mathbb{X}}$ must be of the form $f\bar{\psi}$ for some $f \in F^{\times}$. Since ρ is of height 0, we conclude the f is a unit. Therefore the induced involution is perfect. □

Let us recall the Drinfeld moduli functor $\mathcal{M}_{\mathrm{Dr}}$ on the category of schemes S over $\operatorname{Spf} O_{\breve{F}}$. We will use the notation $\overline{S} = S \otimes_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \overline{\kappa}_F$. We fix a special formal O_D-module $(\mathbb{Y}, \iota_{\mathbb{Y}})$ over the $O_{\breve{F}}$-algebra $\overline{\kappa}_F$. We call $\mathbb{Y}$ a *framing object.* By [9] there is a quasi-isogeny of height 0 between any two choices. For a scheme $S \longrightarrow \operatorname{Spf} O_{\breve{F}}$, a point of $\mathcal{M}_{\mathrm{Dr}}(S)$ consists of the following data up to isomorphism:

(1) A special formal O_D-module (Y, ι) over S.

(2) A quasi-isogeny of O_D-modules of height 0

$$\rho : Y \times_S \overline{S} \longrightarrow \mathbb{Y} \times_{\operatorname{Spec} \bar{\kappa}_F} \overline{S}. \tag{5.1.15}$$

The functor is representable by the p-adic formal $O_{\breve{F}}$-scheme $\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{F}}$.

We define the functor $\mathcal{M}_{\mathrm{Dr}}(i)$ by replacing in (2) height 0 by the condition $\mathrm{height}_{O_F} \rho = 2i$. We set

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} = \coprod_{i \in \mathbb{Z}} \mathcal{M}_{\mathrm{Dr}}(i).$$

Let (Y, ι) be a special formal O_D-module. Let $u \in D^\times$. Then we define a new special formal O_D-module (Y^u, ι^u) by setting

$$Y^u = Y, \quad \iota^u(d) = \iota(u^{-1} d u), \quad \text{for } d \in O_D.$$

The multiplication $\iota(u) : (Y^u, \iota^u) \longrightarrow (Y, \iota)$ is a quasi-isogeny of special formal O_D-modules. We obtain for each $i \in \mathbb{Z}$ an isomorphism of functors

$$u : \mathcal{M}_{\mathrm{Dr}}(i) \xrightarrow{\sim} \mathcal{M}_{\mathrm{Dr}}(i + \mathrm{ord}_D\, u), \quad (Y, \rho) \longmapsto (Y^u, \iota_{\mathbb{Y}}(u)\rho^u), \tag{5.1.16}$$

This defines an action of $D^\times$ on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$. If $u \in O_D^\times$, the multiplication by $\iota(u)$ defines an isomorphism $\iota(u) : (Y^u, \iota_{\mathbb{Y}}(u)\rho^u) \to (Y, \rho)$. Therefore the action of $D^\times$ factors through $\mathrm{ord}_D : D^\times \to \mathbb{Z}$. We will call this action the *translation*.

We endow $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ with a Weil descent datum relative to $O_{\breve{F}}/O_F$. Let $\tau \in \mathrm{Gal}(\breve{F}/F)$ be the Frobenius automorphism. Let $\varepsilon : O_{\breve{F}} \longrightarrow R$ be an algebra in $\mathrm{Nilp}_{O_{\breve{F}}}$. We denote by $R_{[\tau]}$ the ring R with the new $O_{\breve{F}}$-algebra structure $\varepsilon \circ \tau$. The Frobenius τ induces $\bar{\tau} : \bar{\kappa}_F \longrightarrow \bar{\kappa}_F$. We have the Frobenius morphism

$$F_{\mathbb{Y},\tau} : \mathbb{Y} \longrightarrow \tau_* \mathbb{Y}. \tag{5.1.17}$$

For a $\bar{\kappa}_F$-algebra $\varepsilon : \bar{\kappa}_F \longrightarrow \bar{R}$, we set $\phi(r) = r^{p^f}$ for $r \in \bar{R}$. This defines a $\bar{\kappa}_F$-algebra homomorphism $\bar{R} \longrightarrow \bar{R}_{[\tau]}$. If we apply the functor $\mathbb{Y}$ we obtain (5.1.17). We will define a morphism

$$\omega_{\mathcal{M}_{\mathrm{Dr}}} : \mathcal{M}_{\mathrm{Dr}}(i)(R) \longrightarrow \mathcal{M}_{\mathrm{Dr}}(i+1)(R_{[\tau]}). \tag{5.1.18}$$

Let $(Y, \rho) \in \mathcal{M}_{\mathrm{Dr}}(i)(R)$. We define ρ' as the composite

$$Y_{R \otimes_{O_{\breve{F}}} \bar{\kappa}_F} \xrightarrow{\rho} \varepsilon_* \mathbb{Y} \xrightarrow{\varepsilon_* F_{\mathbb{Y},\tau}} \varepsilon_* \tau_* \mathbb{Y}.$$

The image of (Y, ρ) under (5.1.18) is by definition (Y, ρ'). Since $\mathrm{height}_{O_F} F_{\mathbb{Y},\tau} = 2$, we obtain that $\mathrm{height}_{O_F} \rho' = 2i + 2$. From (5.1.18) we obtain a Weil descent datum

$$\omega_{\mathcal{M}_{\mathrm{Dr}}} : \tilde{\mathcal{M}}_{\mathrm{Dr}}(R) \longrightarrow \tilde{\mathcal{M}}_{\mathrm{Dr}}(R_{[\tau]}) \tag{5.1.19}$$

on the functor $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ (compare [27]). We introduce the notation

$$\tilde{\mathcal{M}}_{\mathrm{Dr}}^{(\tau)} = \tilde{\mathcal{M}}_{\mathrm{Dr}} \times_{\operatorname{Spf} O_{\breve{F}}, \operatorname{Spf} \tau} \operatorname{Spf} O_{\breve{F}}. \tag{5.1.20}$$

Then we have $\tilde{\mathcal{M}}_{\mathrm{Dr}}^{(\tau)}(R) = \tilde{\mathcal{M}}_{\mathrm{Dr}}(R_{[\tau]})$. We write (5.1.19) in the form

$$\omega_{\mathcal{M}_{\mathrm{Dr}}} : \tilde{\mathcal{M}}_{\mathrm{Dr}} \longrightarrow \tilde{\mathcal{M}}_{\mathrm{Dr}}^{(\tau)}. \tag{5.1.21}$$

The translation $\Pi : \mathcal{M}_{\mathrm{Dr}}(i) \to \mathcal{M}_{\mathrm{Dr}}(i+1)$ is an isomorphism. We use it to identify these functors. By Drinfeld's theorem we obtain an isomorphism

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} \cong (\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{F}}) \times \mathbb{Z}. \tag{5.1.22}$$

We denote by ω_τ the action of τ via the second factor on $\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{F}}$.

Proposition 5.1.7. *The Weil descent datum $\omega_{\mathcal{M}_{Dr}}$ induces on the right hand side of (5.1.22) the Weil descent datum*

$$\omega_{\mathcal{M}_{Dr}} : \ (\xi, i) \longmapsto (\omega_\tau(\xi), i+1). \tag{5.1.23}$$

The translation functor is on the right hand side $(\xi, i) \mapsto (\xi, i+1)$.

Proof. Let $(Y,\rho) \in \mathcal{M}_{\mathrm{Dr}}(R)$. Composing $\omega_{\mathcal{M}_{\mathrm{Dr}}}$ with the translation we obtain a Weil-descent datum on $\mathcal{M}_{\mathrm{Dr}}$,

$$\alpha : \mathcal{M}_{\mathrm{Dr}}(R) \to \mathcal{M}_{\mathrm{Dr}}(R_\tau).$$

It associates to (Y,ρ) the point $(Y^{\Pi^{-1}},\rho_1)$, where ρ_1 is the composite

$$Y^{\Pi^{-1}}_{R\otimes_{O_{\breve F}}\bar\kappa_F} \xrightarrow{\rho} \varepsilon_*\mathbb{Y}^{\Pi^{-1}} \xrightarrow{\varepsilon_* F_{\mathbb{Y},\tau}} \varepsilon_*\tau_*\mathbb{Y}^{\Pi^{-1}} \xrightarrow{\iota(\Pi^{-1})} \varepsilon_*\tau_*\mathbb{Y}. \tag{5.1.24}$$

Our assertion says that Drinfeld's morphism $\mathcal{M}_{\mathrm{Dr}} \to \widehat{\Omega}_F$ fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dr}}(R) & \xrightarrow{\quad\alpha\quad} & \mathcal{M}_{\mathrm{Dr}}(R_{[\tau]}) \\ & \searrow \quad \swarrow & \\ & \widehat{\Omega}_F(R) & \end{array}$$

This is stated as an exercise in the proof of [27, Prop. 3.77], but we give the verification. We have to go back to Drinfeld's proof and therefore we use his notation. A point of $\widehat{\Omega}_F(R)$ is given by data (η,T,u,r) ([9], §2, Thm.). Drinfeld constructs the data (η,T,u) entirely from a graded Cartier module $M = \oplus M_i$. The Cartier modules M and $M^{\Pi^{-1}}_{[\tau]}$ are the same and the gradings are also the same because $M \mapsto M^{\Pi^{-1}}$ shifts the grading by 1 and $M \mapsto M_{[\tau]}$ shifts the grading by 1 in the opposite direction. Finally, we have to see that the rigidification r is not changed by the application of $\alpha_{\breve G}$. This can be checked on the geometric points of $\operatorname{Spec} R$. But over an algebraically closed field L, the rigidification is obtained as follows. We take the morphism of rational Dieudonné modules

$$N \to \mathbf{N}$$

induced by (5.1.15) for $S = \overline{S} = \operatorname{Spec} L$. Then r is obtained by taking the invariants by $V^{-1}\Pi$ on both sides. We see from the definition (5.1.24) that $\alpha_{\breve G}$ does not change r. □

Let $\mathrm{Aut}^o_D(\mathbb{Y})$ be the group of quasi-isogenies of $\mathbb{Y}$ which commute with the action of $\iota_{\mathbb{Y}}$. With the notation of (5.1.9), let $N = N_0 \oplus N_1 = P \otimes \mathbb{Q}$ be the rational Dieudonné module of $\mathbb{Y}$. The natural map

$$\mathrm{Aut}^o_D(\mathbb{Y}) \to \mathrm{GL}_F(N_0^{V^{-1}\Pi}) \cong \mathrm{GL}_2(F)$$

is an isomorphism. This group acts on the functor $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ as follows. For $g \in \mathrm{Aut}^o_D(\mathbb{Y})$ we define

$$g : \mathcal{M}_{\mathrm{Dr}}(i) \to \mathcal{M}_{\mathrm{Dr}}(i + \operatorname{ord}\det g), \quad (Y,\rho) \longmapsto (Y, g\rho).$$

The action commutes with the translation. Let J_{Dr} be the cokernel

$$\begin{array}{ccccc} \mathbb{Z} & \to & \mathrm{Aut}^o_D(\mathbb{Y}) \times \mathbb{Z} & \to J_{\mathrm{Dr}} \to 0. \\ i & \longmapsto & (\pi^i, -2i) & \end{array}$$

The second group acts on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$ such that the factor $\mathbb{Z}$ acts by translation. We obtain an action of J_{Dr} on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$. We introduce the groups $J^{*\mathrm{r}}$ and $J^{*\mathrm{ur}}$ as cokernels

$$\begin{array}{cccc} F^\times & \to & \mathrm{Aut}^o_D(\mathbb{Y}) \times K^\times & \to J^{*\mathrm{r}} \to 0, \\ f & \longmapsto & (f, f^{-1}) & \\ & & & \\ F^\times & \to & \mathrm{Aut}^o_D(\mathbb{Y}) \times (F')^\times & \to J^{*\mathrm{ur}} \to 0. \\ f & \longmapsto & (f, f^{-1}) & \end{array} \tag{5.1.25}$$

The homomorphisms $\operatorname{ord}_K : K^\times \to \mathbb{Z}$, resp. $2\operatorname{ord}_{F'} : (F')^\times \to \mathbb{Z}$, induce homomorphisms $J^{*\mathrm{r}} \to J_{\mathrm{Dr}}$, resp. $J^{*\mathrm{ur}} \to J_{\mathrm{Dr}}$. Therefore these groups act on $\tilde{\mathcal{M}}_{\mathrm{Dr}}$.

5.2. **The alternative theorem in the ramified case.** Let K be a ramified quadratic extension of F. We also assume that $p \neq 2$. We choose prime elements $\Pi \in O_K$ and $\pi \in O_F$ such that $\Pi^2 = -\pi$ as in section 2. With the notation before Proposition 5.1.3, we regard O_K as the subring of O_D.

For each $i \in \mathbb{Z}$ we define the functor $\mathcal{N}(i) = \mathcal{N}_{K/F,r}(i)$ on the category of schemes $S \longrightarrow \operatorname{Spf} O_{\breve{F}}$. We fix a special formal O_D-module $\mathbb{Y}$ over $\bar{\kappa}_F$ and we fix a Drinfeld polarization $\psi_{\mathbb{Y}}$. We denote by $\lambda_{\mathbb{Y}} : \mathbb{Y} \longrightarrow \mathbb{Y}^{\Delta}$ the isomorphism associated to $\psi_{\mathbb{Y}}$, cf. (5.1.14). We will consider p-divisible groups X on S with an action $\iota : O_K \longrightarrow \operatorname{End} X$ such that the restriction of this action to O_F is strict. By duality we obtain an action of O_K on the Faltings dual X^{∇}. If we compose this action with the conjugation of K/F we obtain $\iota^{\Delta} : O_K \longrightarrow \operatorname{End} X^{\nabla}$. We write $X^{\Delta} = (X^{\nabla}, \iota^{\Delta})$ and call this the *Faltings conjugate dual* of (X, ι).

Definition 5.2.1. A point of $\mathcal{N}(i)(S)$ consists of the following data:

(1) A formal p-divisible group X over S with an action
$$\iota : O_K \longrightarrow \operatorname{End} X,$$
such that the restriction of ι to O_F is a strict action.

(2) An isomorphism of O_K-modules $\lambda : X \longrightarrow X^{\Delta}$ which induces a relative polarization on X, cf. Corollary 3.4.13.

(3) A quasi-isogeny of O_K-modules
$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{Y} \times_{\operatorname{Spec} \bar{\kappa}_F} \overline{S}.$$

We require that the following conditions are satisfied.

a) ρ respects the O_K-actions. There is an element $u \in O_F^{\times}$ such that the following diagram of quasi-isogenies is commutative
$$\begin{array}{ccc} X \times_S \bar{S} & \xrightarrow{\ \rho\ } & \mathbb{Y} \times_{\operatorname{Spec} \bar{\kappa}_F} \bar{S} \\ {\scriptstyle u\pi^i\lambda}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathbb{Y}}} \\ X^{\Delta} \times_S \bar{S} & \xleftarrow[\ \rho^{\Delta}\ ]{} & \mathbb{Y}^{\Delta} \times_{\operatorname{Spec} \bar{\kappa}_F} \bar{S}. \end{array} \tag{5.2.1}$$

b)
$$\operatorname{Tr}(\iota(\Pi) \mid \operatorname{Lie} X) = 0. \tag{5.2.2}$$

Two such data $(X_1, \iota_1, \lambda_1, \rho_1)$ and $(X_2, \iota_2, \lambda_2, \rho_2)$ define the same point of $\mathcal{N}(i)(S)$ iff there is an isomorphism $\alpha : (X_1, \iota_1) \longrightarrow (X_2, \iota_2)$ which respects the polarizations up to a factor in $O_F^{\times}$ and such that α commutes with ρ_1 and ρ_2.

We note that changing λ by a factor in $O_F^{\times}$ does not alter the points of $\mathcal{N}(S)$. The existence of ρ implies that $\dim X = 2$ and that the O_F-height of X is 4. The condition b) implies the following Kottwitz condition for the characteristic polynomial,
$$\operatorname{char}(\iota(a) \mid \operatorname{Lie} X) = (T - a)(T - \bar{a}), \quad a \in O_K. \tag{5.2.3}$$
Clearly the functor $\mathcal{N}(i)$ does not depend on the choice of the Drinfeld polarization $\lambda_{\mathbb{Y}}$.

It follows from [27] that $\mathcal{N}(i)$ is representable by a formal scheme which is locally formally of finite type over $\operatorname{Spf} O_{\breve{F}}$.

Let $S = \operatorname{Spec} R$, $R \in \operatorname{Nilp}_{O_{\breve{F}}}$. Let $\mathcal{P}_X$ be the $\mathcal{W}_{O_F}(R)$-display associated to the p-divisible group X. The conjugate dual $\mathcal{W}_{O_F}(R)$-display $\mathcal{P}_X^{\Delta}$ is nilpotent. It corresponds to X^{Δ}. We denote by $\psi : \mathcal{P}_X \times \mathcal{P}_X \longrightarrow \mathcal{P}_{m, W_{O_F}(R)}$ the bilinear form of displays which corresponds to λ. We may reformulate the commutativity of the diagram (5.2.1) as follows: the quasi-polarization $\rho^* \psi_{\mathbb{Y}}$ coincides with $\pi^i \psi_{R/\pi R}$ of $(\mathcal{P}_X)_{R/\pi R}$ up to a factor in $O_F^{\times}$.

We obtain from (5.2.1) that
$$4i = 2 \operatorname{height}_{O_F} \rho.$$

As for the functors $\mathcal{M}_{\mathrm{Dr}}(i)$ we have functor isomorphisms
$$\Pi : \mathcal{N}(i) \xrightarrow{\sim} \mathcal{N}(i+1), \quad (X, \rho) \longmapsto (X, \iota(\Pi)\rho), \tag{5.2.4}$$

which we call the translations. Let $\tau \in \mathrm{Gal}(\breve{F}/F)$ be the Frobenius automorphism. Using the Frobenius $F_{\mathbb{Y},\tau} : \mathbb{Y} \longrightarrow \tau_*\mathbb{Y}$ we define

$$\omega_{\mathcal{N}} : \mathcal{N}(R)(i) \longrightarrow \mathcal{N}(i+1)(R_{[\tau]}). \tag{5.2.5}$$

exactly as $\omega_{\mathcal{M}_{\mathrm{Dr}}}$ in (5.1.18). This defines a Weil descent datum $\omega_{\mathcal{N}}$ relative to $O_{\breve{F}}/O_F$ on the functor

$$\tilde{\mathcal{N}} = \coprod_{i\in\mathbb{Z}} \mathcal{N}(i).$$

Lemma 5.2.2. *The action of the group $J^{*\mathrm{r}}$ from (5.1.25) on the O_K-module $\mathbb{Y}$ gives an isomorphism*

$$J^{*\mathrm{r}} \xrightarrow{\sim} J^{\cdot},$$

where

$$J^{\cdot} = \{\alpha \in \mathrm{Aut}^o_K(\mathbb{Y}) \mid \psi_{\mathbb{Y}}(\alpha(x),\alpha(y)) = \mu(\alpha)\psi_{\mathbb{Y}}(x,y),\ \textit{for some}\ \mu(\alpha) \in F^{\times},\ x,y \in P_{\mathbb{Y}} \otimes \mathbb{Q}\}.$$

□

The group $J^{*\mathrm{r}}$ acts on the functor $\tilde{\mathcal{N}}$ by

$$(Y,\iota,\rho) \longmapsto (Y,\iota,g\rho), \quad \text{for } g \in J^{*\mathrm{r}}.$$

We have a natural morphism of functors on $\mathrm{Nilp}_{O_{\breve{F}}}$

$$\mathcal{M}_{\mathrm{Dr}}(i) \longrightarrow \mathcal{N}(i). \tag{5.2.6}$$

This is defined as follows. Let $(Y,\iota,\rho) \in \mathcal{M}_{\mathrm{Dr}}(i)(S)$ be a point. Let ψ be a Drinfeld polarization on Y which is compatible with the quasi-isogeny ρ, cf. Proposition 5.1.5. It is uniquely determined up to a factor in $O_F^{\times}$. Locally on $\bar{S}$ we have $\rho^*\psi_{\mathbb{Y}} = f\psi$ for $f \in F$. Since $\mathrm{height}_{O_F}\,\rho = 2i$ we obtain $\mathrm{ord}_{\pi} f = i$. Therefore $(Y, \iota_{|O_K}, \psi, \rho) \in \mathcal{N}(i)(S)$.

The main result of [17] may now be formulated as follows. Note that in loc. cit. Weil descent data were not considered.

Theorem 5.2.3 ([17]). *Assume that $p \neq 2$. The functor morphisms* (5.2.6) *define a functor isomorphism*

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} \xrightarrow{\sim} \tilde{\mathcal{N}}$$

which commutes with the Weil descent data and the action of the group $J^{\cdot} = J^{*\mathrm{r}}$ on both sides. In particular it commutes with the translations.

It is clear that the morphism of functors $\tilde{\mathcal{M}}_{\mathrm{Dr}} \longrightarrow \tilde{\mathcal{N}}$, is compatible with the Weil descent data $\omega_{\mathcal{M}_{\mathrm{Dr}}}$ and $\omega_{\mathcal{N}}$ relative to $O_{\breve{F}}/O_F$ and with the translations (5.1.16) and (5.2.4). From this we see that it commutes also with the actions of $J^{*\mathrm{r}}$. We need to prove that it is an isomorphism. For the proof we need some preparations.

Let k be an algebraically closed field which is an $O_{\breve{F}}$-algebra. We consider a $W_{O_F}(k)$-Dieudonné module M of height 4 and dimension 2. We assume that an O_K-action $\iota : O_K \longrightarrow \mathrm{End}\,M$ on M is given such that the restriction to O_F is via $O_F \longrightarrow W_{O_F}(k)$.

Let

$$\psi : M \times M \longrightarrow W_{O_F}(k) \tag{5.2.7}$$

be a relative polarization, i.e., an alternating $W_{O_F}(k)$-bilinear form such that

$$\psi(Fx_1, Fx_2) = \pi\ {}^F\psi(x_1,x_2).$$

We require that

$$\psi(\iota(a)x, y) = \psi(x, \iota(\bar{a})y), \quad a \in O_K.$$

Proposition 5.2.4. *Let $\mathbb{M}$ be the $W_{O_F}(k)$-Dieudonné module of a special formal O_D-module with a Drinfeld polarization $\psi_{\mathbb{M}}$, cf. Definition 5.1.4. Let (M,ι,ψ) be as above and such that ψ is perfect. We assume that there exists an isomorphism of rational $W_{O_F}(k)$-Dieudonné modules $\rho : \mathbb{M}\otimes\mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$, such that ρ is a homomorphism of O_K-modules and such that ρ respects the polarizations $\psi_{\mathbb{M}}$ and ψ up to a factor in $F^{\times}$.*

Then there exists a unique O_D-module structure on M such that M becomes the Dieudonné module of a special formal O_D-module and such that ρ is a quasi-isogeny of O_D-modules.

Proof. We will write $ax := \iota(a)x$, for $a \in O_K$ and $x \in M$. For $a \in O_F$ this coincides by definition with the action via $O_F \longrightarrow W_{O_F}(k)$.

We define $\tilde{W} = O_K \otimes_{O_F} W_{O_F}(k)$. We extend the conjugation of K over F by linearity to $\tilde{W}$ over $W_{O_F}(k)$. We denote the traces of K/F and of $\tilde{W}/W_{O_F}(k)$ both by Tr. The Frobenius endomorphism of $W_{O_F}(k)$ extends O_K-linearly to $\tilde{W}$ and is denoted by F. It will be impossible to confuse this with the field F.

We define a hermitian form

$$h : M \times M \longrightarrow \tilde{W},$$

by requiring that

$$\operatorname{Tr} \xi \Pi^{-1} h(x, y) = \psi(x, \xi y) \quad \xi \in \tilde{W},\ x, y \in M.$$

Then h is $\tilde{W}$-linear in the second variable and hermitian,

$$h(x, y) = \overline{h(y, x)}.$$

The pairing h is perfect and satisfies the equation

$$h(Fx, Fy) = \pi\ {}^{F}h(x, y).$$

Since $N := M \otimes \mathbb{Q}$ is the rational Dieudonné module of a special formal O_D-module with its Drinfeld polarization, we have a decomposition

$$N = N_0 \oplus N_1, \tag{5.2.8}$$

which is orthogonal with respect to ψ (see (5.1.3)). One should note that N_0 and N_1 are not $\tilde{W}$-modules.

We note that for $n_0, n_0' \in N_0$, $n_1, n_1' \in N_1$ we have

$$\begin{gathered} h(n_0, n_0') = \tfrac{1}{2}\Pi\psi(n_0, n_0'), \quad h(n_1, n_1') = \tfrac{1}{2}\Pi\psi(n_1, n_1'), \\ h(n_0, n_1) = \tfrac{1}{2}\psi(n_0, \Pi n_1). \end{gathered} \tag{5.2.9}$$

Indeed, the equation

$$\operatorname{Tr} \frac{\Pi}{\Pi} h(n_0, n_0') = \psi(n_0, \Pi n_0') = 0$$

implies that $\Pi^{-1}h(n_0, n_0') \in W_{O_F}(k) \otimes \mathbb{Q}$. We obtain the first equation of (5.2.9):

$$2\Pi^{-1}h(n_0, n_0') = \operatorname{Tr} \Pi^{-1}h(n_0, n_0') = \psi(n_0, n_0').$$

The proof of the next equation is the same. We have

$$\operatorname{Tr} \Pi^{-1}h(n_0, n_1) = \psi(n_0, n_1) = 0.$$

This implies $h(n_0, n_1) \in W_{O_F}(k) \otimes \mathbb{Q}$. We obtain the last equation of (5.2.9):

$$2h(n_0, n_1) = \operatorname{Tr} h(n_0, n_1) = \psi(n_0, \Pi n_1).$$

In particular we see from (5.2.9) that an element $n_0 \in N_0$ is isotropic for h.

We call an element $x \in M$ primitive if it is not in ΠM. We find an element $x \in M \cap N_0$ such that $x \notin \pi M$. Assume that $x = \Pi y$ for some $y \in M$. Then $y \in M \cap N_1$. Then it is clear that y is a primitive element in M. Interchanging the role of the indices 0 and 1, we may assume that $x \in M \cap N_0$ is primitive.

Since the pairing h is perfect and x is isotropic for h, we find an element $y' \in M$, such that $h(x, y') = 1$. We can even choose y' to be isotropic for h. Indeed, we set $y = y' + \lambda x$ for some $\lambda \in \tilde{W}$. Then $h(x, y) = 1$. We compute:

$$h(y, y) = h(y', y') + h(y', \lambda x) + h(\lambda x, y') = h(y', y') + \lambda + \bar{\lambda}.$$

We choose $\lambda = -(1/2)h(y', y')$ (which is legitimate, as $p \neq 2$) and obtain $h(y, y) = 0$. According to (5.2.8) we write

$$y = y_0 + y_1, \quad y_0 \in N_0,\ y_1 \in N_1.$$

We write

$$1 = h(x, y) = h(x, y_0) + h(x, y_1).$$

We have $h(x, y_0) \in \Pi W_{O_F}(k) \otimes \mathbb{Q}$ and $h(x, y_1) \in W_{O_F}(k) \otimes \mathbb{Q}$ by the formulas (5.2.9). This implies $h(x, y_0) = 0$ and $h(x, y_1) = 1$. On the other hand, we find by (5.2.8)

$$0 = h(y, y) = h(y_0 + y_1, y_0 + y_1) = h(y_0, y_1) + h(y_1, y_0).$$

Since $h(y_0, y_1) \in W_{O_F}(k) \otimes \mathbb{Q}$, this implies $h(y_0, y_1) = 0$ and then $h(y, y_0) = 0$. The elements x and y generate N as a $\tilde{W} \otimes \mathbb{Q}$-vector space. Because we already proved that $h(x, y_0) = 0$, we conclude $y_0 = 0$. Therefore $y = y_1 \in M \cap N_1$. We obtain

$$M = \tilde{W}x + \tilde{W}y,$$

because h is unimodular on the right hand side. Then the elements $x, \Pi x, y, \Pi y$ are a basis of the $W_{O_F}(k)$-module M. We have $x, \Pi y \in M \cap N_0$ and $y, \Pi x \in M \cap N_1$ and therefore

$$M = (M \cap N_0) \oplus (M \cap N_1).$$

This shows that the O_D-module structure on N induces an O_D-module structure on M. □

Proof of Theorem 5.2.3. We consider the morphism (5.2.6) for $i = 0$ and denote it by

$$\mathcal{M}_{\mathrm{Dr}} \longrightarrow \mathcal{N}. \tag{5.2.10}$$

Clearly it is enough to show that this is an isomorphism. Proposition 5.2.4 shows that for any algebraically closed field k which is an $O_{\breve{F}}$-algebra, the induced map $\mathcal{M}_{\mathrm{Dr}}(k) \longrightarrow \mathcal{N}(k)$ is bijective.

We note that the morphism (5.2.10) is formally unramified. Indeed, let $S \longrightarrow R$ be a surjective morphism in $\mathrm{Nilp}_{O_{\breve{F}}}$ with nilpotent kernel. Let X be a p-divisible group over S with base change X_R over R. Then an O_D-module structure on X_R lifts by rigidity in at most one way to X. We consider the underlying topological spaces in (5.2.6) with their induced structure of reduced schemes. Then we obtain a formally unramified morphism of $\bar{\kappa}_F$-schemes

$$\mathcal{M}_{\mathrm{Dr,red}} \longrightarrow \mathcal{N}_{\mathrm{red}} \tag{5.2.11}$$

These schemes are locally of finite type over $\bar{\kappa}_F$ and have irreducible components which are proper over $\bar{\kappa}_F$, cf. [27, Prop. 2.32]. Moreover, the morphism is bijective on geometric points. Then the irreducible components of both schemes correspond bijectively to each other. We consider a point $x \in \mathcal{M}_{\mathrm{Dr}}(\bar{\kappa}_F)$. Let X be the union of all irreducible components which pass through x with the reduced scheme structure. Let $y \in \mathcal{N}(\bar{\kappa}_F)$ be the image of x and define $Y \subset \mathcal{N}$ in the same way as X. Then $X \longrightarrow Y$ is a finite morphism. If we remove all points in X resp. Y which belong to components not passing through x, resp. y, we obtain a finite morphism of open neighbourhoods $U \longrightarrow V$ of $x \in \mathcal{M}_{\mathrm{Dr,red}}$ and $y \in \mathcal{N}_{\mathrm{red}}$. Therefore (5.2.11) is a finite morphism of schemes locally of finite type over the algebraically closed field $\bar{\kappa}_F$. Since this morphism is unramified and bijective on geometric points, it is an isomorphism.

Lemma 5.2.5. *Let $S \longrightarrow \bar{\kappa}_F$ be a surjective morphism in $\mathrm{Nilp}_{O_{\breve{F}}}$ such that the kernel is nilpotent and endowed with divided powers. Then the map*

$$\mathcal{M}_{\mathrm{Dr}}(S) \longrightarrow \mathcal{N}(S)$$

is bijective.

Let us assume that the lemma is proved. Then we consider points x and y as above. We consider an open affine neighbourhood U of x. By the isomorphism (5.2.11) we regard U also as a neighbourhood of y. Let $n \in \mathbb{N}$. For a suitable ideal sheaf of definition $\mathcal{J}$ of $\mathcal{N}$, we have a homomorphism

$$(\mathcal{O}_{\mathcal{N}}/\mathcal{J})(U) \longrightarrow \mathcal{O}_{\mathcal{M}_{\mathrm{Dr}}}(U)/\pi^n \mathcal{O}_{\mathcal{M}_{\mathrm{Dr}}}(U).$$

This map is surjective modulo π by (5.2.11) and is therefore surjective. We note that by EGA0_I, Prop. 7.2.4 the ring $(\mathcal{O}_{\mathcal{N}}/\mathcal{J})(U)$ is π-adic. It follows that

$$\mathcal{O}_{\mathcal{N}}(U) \longrightarrow \mathcal{O}_{\mathcal{M}_{\mathrm{Dr}}}(U)$$

is surjective. Taking the inductive limit over U we obtain an epimorphism of local rings

$$\mathcal{O}_y \longrightarrow \mathcal{O}_x. \tag{5.2.12}$$

By [27, Thm. 2.16] this is a homomorphism of noetherian adic rings (comp. EGA I, Prop. 10.1.6). The ring $\mathcal{O}_x$ is, as a local ring of the scheme $\mathcal{M}_{\mathrm{Dr}}$, regular of dimension 2. Let $\mathfrak{m}_y$ and $\mathfrak{m}_x$ be the maximal ideals of the local rings. We remark that the squares of the ideals are open because the topologies are adic.

We apply Lemma 5.2.5 to $S = \mathcal{O}_y/\mathfrak{m}_y^2$. Then we obtain an oblique arrow which makes the following diagram commutative,

$$\begin{array}{ccc} \mathcal{O}_y & \longrightarrow & \mathcal{O}_x \\ \downarrow & \swarrow & \\ \mathcal{O}_y/\mathfrak{m}_y^2 . & & \end{array}$$

It follows that there is a surjective homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \mathfrak{m}_y/\mathfrak{m}_y^2$. The epimorphism of local rings also gives a surjection in the other direction. We conclude

$$\dim_{\bar{\kappa}_F} \mathfrak{m}_y/\mathfrak{m}_y^2 = 2.$$

Therefore $\mathcal{O}_y$ is a regular local ring of dimension 2, and the map (5.2.12) is an isomorphism. It follows that the map of sheaves

$$\mathcal{O}_{\mathcal{N}} \longrightarrow \mathcal{O}_{\mathcal{M}_{\mathrm{Dr}}}$$

is an isomorphism. Finally let J be the maximal ideal sheaf of definition of $\mathcal{O}_{\mathcal{N}}$. By the isomorphism (5.2.12) we obtain an isomorphism

$$\mathcal{O}_y/J\mathcal{O}_y \longrightarrow \mathcal{O}_x/\pi\mathcal{O}_x.$$

Therefore $J\mathcal{O}_y = \pi\mathcal{O}_y$. Therefore $J = \pi\mathcal{O}_{\mathcal{N}}$ is an ideal sheaf of definition. We obtain that (5.2.10) is an isomorphism of formal schemes.

It remains to prove Lemma 5.2.5. We denote by $\mathfrak{m}$ the kernel of $S \longrightarrow \bar{\kappa}_F$. Let $\xi : \operatorname{Spec} S \longrightarrow \mathcal{N}$ be a morphism. We show that it lifts uniquely to $\operatorname{Spec} S \longrightarrow \mathcal{M}_{\mathrm{Dr}}$. We denote by $y \in \mathcal{N}(\bar{\kappa}_F)$ the point induced by ξ. Let $x \in \mathcal{M}_{\mathrm{Dr}}(\bar{\kappa}_F)$ be the unique point over y.

We denote by $\mathcal{P}$ the O_F-display of the special formal O_D-module over $\bar{\kappa}_F$ which corresponds to x. We denote by $\tilde{\mathcal{P}}$ the unique $\mathcal{W}_{O_F}(S/\bar{\kappa}_F)$-display which lifts $\mathcal{P}$, cf. Theorem 3.1.12. We write $\tilde{\mathcal{P}} = (\tilde{P}, \hat{Q}, F, \dot{F})$. The O_D-action on $\mathcal{P}$ extends to $\tilde{\mathcal{P}}$. Therefore we have the decompositions

$$\tilde{P} = \tilde{P}_0 \oplus \tilde{P}_1, \quad \hat{Q} = \hat{Q}_0 \oplus \hat{Q}_1.$$

We consider only the most interesting case where Π acts trivially on $\operatorname{Lie} \mathcal{P}$, i.e., $\operatorname{Spec} \bar{\kappa}_F \longrightarrow \mathcal{M}_{\mathrm{Dr}}$ is a singular point of the special fibre, cf. [9]. In this case we obtain Frobenius-linear isomorphisms

$$\dot{F} \circ \Pi : \tilde{P}_0 \longrightarrow \hat{Q}_1 \longrightarrow \tilde{P}_0, \quad \dot{F} \circ \Pi : \tilde{P}_1 \longrightarrow \hat{Q}_0 \longrightarrow \tilde{P}_1.$$

We set $\tilde{U}_i = \{x \in \tilde{\mathcal{P}}_i \mid \dot{F} \circ \Pi(x) = x\}$. Then the canonical morphism $W_{O_F}(S) \otimes_{O_F} \tilde{U}_i \longrightarrow \tilde{P}_i$ is an isomorphism.

We can make the same construction with the display $\mathcal{P}$. Then we obtain $U_i \subset P_i$ such that the canonical O_F-module homomorphism $\tilde{U}_i \longrightarrow U_i$ is an isomorphism. Using our knowledge about $\mathcal{P}$ we find elements $\tilde{e}_i \in U_i$, for $i = 0, 1$ such that

$$\tilde{e}_0, \Pi\tilde{e}_1 \in \tilde{P}_0, \quad \tilde{e}_1, \Pi\tilde{e}_0 \in \tilde{P}_1,$$

are a basis of the $W_{O_F}(S)$-module $\tilde{P}$. The natural polarization ψ on $\mathcal{P}$ extends to a polarization $\tilde{\psi}$ on $\tilde{\mathcal{P}}$ which is given by the conditions

$$\tilde{\psi}(\tilde{e}_0, \Pi\tilde{e}_1) = 1 = \tilde{\psi}(\tilde{e}_1, \Pi\tilde{e}_0),$$

and such that the decomposition $\tilde{P} = \tilde{P}_0 \oplus \tilde{P}_1$ is orthogonal with respect to $\tilde{\psi}$.

We classify now the liftings of $\operatorname{Spec} \bar{\kappa}_F \longrightarrow \mathcal{N}$ to a point $\operatorname{Spec} S \longrightarrow \mathcal{N}$. We consider the Hodge filtration $L = Q/I_{O_F}(k)P \subset P/I_{O_F}(k)P$. Since we compute now all the time modulo the augmentation ideal $I_{O_F}(k) \subset W_{O_F}(k)$, resp., $I_{O_F}(S) \subset W_{O_F}(S)$, we continue to simply write $\tilde{e}_0$ when we mean the residue class in $\tilde{P}/I(S)\tilde{P}$. The k-vector space L has the basis $\Pi e_0, \Pi e_1$. Therefore a lifting of L to a direct summand $\tilde{L} \subset \tilde{P}/I(S)\tilde{P}$ has a unique basis of the form

$$f_0 = \Pi\tilde{e}_1 + \gamma\tilde{e}_0 + \delta\tilde{e}_1, \quad f_1 = \Pi\tilde{e}_0 + \alpha\tilde{e}_0 + \beta\tilde{e}_1,$$

because it is complementary to the module generated by $\tilde{e}_0, \tilde{e}_1$. Since we want a lifting of L we have $\alpha, \beta, \gamma, \delta \in \mathfrak{m}$. The lifting $\tilde{L}$ determines a lifting of the display to S. The form ψ lifts to a polarization of this display if and only if $\tilde{L}$ is isotropic under ψ. Therefore we must have

$$0 = \tilde{\psi}(\Pi\tilde{e}_0 + \alpha\tilde{e}_0 + \beta\tilde{e}_1, \Pi\tilde{e}_1 + \gamma\tilde{e}_0 + \delta\tilde{e}_1)$$

One obtains easily that the right hand side is

$$\tilde{\psi}(\Pi\tilde{e}_0, \delta\tilde{e}_1) + \tilde{\psi}(\alpha\tilde{e}_0, \Pi\tilde{e}_1) = -\delta + \alpha.$$

Since the lifting $\tilde{L}$ should define a point of $\mathcal{N}$, the condition 2) in the definition of points of $\mathcal{N}$ implies

$$0 = \mathrm{Tr}(\Pi \mid \tilde{P}/\tilde{L}) = \alpha + \delta.$$

Because $p \neq 2$ we obtain $\alpha = \delta = 0$. This implies that $\tilde{L} = (\tilde{L} \cap \tilde{P}_0) \oplus (\tilde{L} \cap \tilde{P}_1)$. This shows that the display over S defined by $\tilde{L}$ is the display of a special formal O_D-module. Therefore the liftings of $\mathrm{Spec}\,\bar{\kappa}_F \longrightarrow \mathcal{N}$ to $\mathcal{N}(S)$ correspond via (5.2.6) bijectively to the liftings of $\mathrm{Spec}\,\bar{\kappa}_F \longrightarrow \mathcal{M}_{\mathrm{Dr}}$ to a point of $\mathcal{M}(S)$. This proves Lemma 5.2.5 and Theorem 5.2.3.

□

The properties of Drinfeld's moduli scheme $\mathcal{M}_{\mathrm{Dr}}$ imply the following corollary, cf., e.g., [4].

Corollary 5.2.6. *The formal scheme $\mathcal{N}$ is π-adic and has semi-stable reduction. The special fiber $\mathcal{N} \otimes_{O_{\breve{F}}} \bar{\kappa}_F$ of $\mathcal{N}$ is a reduced scheme.* □

Finally we prove the uniqueness of the framing object, cf. (i) of subsection 2.5. We begin with this question in the category $\mathfrak{dR}_R^{\mathrm{pol}}$, cf. Definition 4.4.10.

Proposition 5.2.7. *Let r be special and let K/F be ramified. Let $k \in \mathrm{Nilp}_{O_F}$ be an algebraically closed field. Let $(\mathcal{P}_{c,1}, \iota_{c,1}, \beta_{c,1})$ and $(\mathcal{P}_{c,2}, \iota_{c,2}, \beta_{c,2})$ be two objects in $\mathfrak{dR}_k^{\mathrm{pol}}$. Assume that* $\mathrm{inv}(\mathcal{P}_{c,i}, \iota_{c,i}, \beta_{c,i})) = -1$ *for $i = 1, 2$. Then there exists a quasi-isogeny $\alpha : \mathcal{P}_{c,1} \longrightarrow \mathcal{P}_{c,2}$ which respects $\iota_{c,i}$ and $\beta_{c,i}$.*

If the forms $\beta_{c,i}$ are perfect, then the actions $\iota_{c,i}$ extend to actions $\tilde{\iota}_{c,i} : O_D \longrightarrow \mathrm{End}_{O_F} \mathcal{P}_{c,i}$ such that $\mathcal{P}_{c,i}$ becomes a special formal O_D-module with Drinfeld polarization β_i and such that α becomes a homomorphism of O_D-modules.

For the proof we need some preparations.

Lemma 5.2.8. *Let $K/F, r, k$ as in the last Proposition and let $(\mathcal{P}_c, \iota_c, \beta_c) \in \mathfrak{dR}_k^{\mathrm{pol}}$. Assume that* $\mathrm{inv}(\mathcal{P}_c, \iota_c, \beta_c) = -1$*, cf. Definition 8.3.1. Then the $\mathcal{W}_{O_F}(k)$-display $\mathcal{P}_c$ is isoclinic of slope* $1/2$.

Proof. The $K \otimes_{O_F} W_{O_F}(k)$-vector space $N = P_c \otimes \mathbb{Q}$ has dimension 2. The isoclinic decomposition of the $\mathcal{W}_{O_F}(k)$-isocrystal N is invariant under the action of O_K and has therefore at most two summands. We have to show that there is only one summand. If not, we have $N = N_0 \oplus N_1$, where N_0 is étale and N_1 is dual to N_0. The dimension of each N_i as a $K \otimes_{O_F} W_{O_F}(k)$-vector space is one. Therefore we find a generator $e_0 \in N_0$ such the $V_c e_0 = e_0$. We use the notation of before Definition 8.3.1. Let $e_1 \in N_1$ be the generator such that $\varkappa_c(e_0, e_1) = 1$. Let τ be the Frobenius acting via the second factor on $K \otimes_{O_F} W_{O_F}(k)$. From the equation

$$\varkappa_c(V_c e_0, V_c e_1) = \pi \varkappa_c(e_0, e_1)^{\tau^{-1}} = \pi,$$

we conclude that $V_c e_1 = \pi e_1$. Therefore $V_c(e_0 \wedge e_1) = \pi e_0 \wedge e_1$. This implies that the invariant of $(\mathcal{P}_c, \iota_c, \beta_c)$ is 1, which contradicts the assumption $(\mathcal{P}_c, \iota_c, \beta_c) = -1$. □

Let $(\mathcal{P}_c, \iota_c) \in \mathfrak{dR}_k$ be isoclinic of slope $1/2$. Then there is an $W_{O_F}(k)$-lattice $\Lambda \subset P_c \otimes \mathbb{Q}$ which is invariant by $\pi^{-1} V_c^2$. Then there is also a lattice invariant by the "square root" $\Pi^{-1} V_c$. One deduces that the invariants C of $\Pi^{-1} V_c$ acting on $P_c \otimes \mathbb{Q}$ form a K-vector space of dimension 2 and

$$P_c \otimes \mathbb{Q} = C \otimes_{O_F} W_{O_F}(k). \tag{5.2.13}$$

The anti-hermitian form $\varkappa_c$ associated to β_c by (8.3.1) induces the anti-hermitian form on the K-vector space C

$$\varkappa_c : C \times C \longrightarrow K. \tag{5.2.14}$$

Indeed, for $x, y \in C$ we find

$$\pi \varkappa_{\mathrm{c}}(x,y) = \varkappa_{\mathrm{c}}(\Pi x, \Pi y) = \varkappa_{\mathrm{c}}(V_{\mathrm{c}}x, V_{\mathrm{c}}y) = \pi\ {}^{F^{-1}}\varkappa_{\mathrm{c}}(x,y).$$

This shows that $\varkappa_{\mathrm{c}}(x,y) \in K \otimes_{O_F} W_{O_F}(k)$ is invariant by the Frobenius F acting on $W_{O_F}(k)$, and therefore this element is in K. The same argument shows that $\beta_{\mathrm{c}}(x,y) \in F$. The form $\varkappa_{\mathrm{c}}$ restricted to C is obtained from the restriction of β_{c} to C by the formula

$$\mathrm{Tr}_{K/F}(a\varkappa_{\mathrm{c}}(x,y)) = \beta_{\mathrm{c}}(ax,y), \quad x,y \in C,\ a \in K.$$

Lemma 5.2.9. *Let $(\mathcal{P}_{\mathrm{c},1}, \iota_{\mathrm{c},1}, \beta_{\mathrm{c},1})$ and $(\mathcal{P}_{\mathrm{c},2}, \iota_{\mathrm{c},2}, \beta_{\mathrm{c},2})$ be objects of $\mathfrak{dR}_k^{\mathrm{pol}}$ such that $\mathcal{P}_{\mathrm{c},1}$ and $\mathcal{P}_{\mathrm{c},2}$ are isoclinic of slope $1/2$. Then the canonical map*

$$\mathrm{Hom}\left((\mathcal{P}_{\mathrm{c},1}, \iota_{\mathrm{c},1}, \beta_{\mathrm{c},1}), (\mathcal{P}_{\mathrm{c},2}, \iota_{\mathrm{c},2}, \beta_{\mathrm{c},2})\right) \otimes \mathbb{Q} \longrightarrow \mathrm{Hom}_K\left((C_1, \beta_{\mathrm{c},1}), (C_2, \beta_{\mathrm{c},2})\right)$$

is an isomorphism.

Proof. This is an immediate consequence of the isomorphism (5.2.13) because the K-action, β_{c}, and V_{c} on $P_{\mathrm{c}} \otimes \mathbb{Q}$ can be recovered from the right hand side of the isomorphism. The map V_{c} is induced from $\Pi \otimes F^{-1}$ on the right hand side. □

Lemma 5.2.10. *There is the following relation between the invariants defined in Definition 8.3.1 and in Definition 8.1.1,*

$$\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota, \beta_{\mathrm{c}}) = -\mathrm{inv}(C, \beta_{\mathrm{c}}). \tag{5.2.15}$$

We remark here that (C, β_{c}) determines $(\mathcal{P}_{\mathrm{c}}, \iota, \beta_{\mathrm{c}})$ up to isogeny.

Proof. Let x_1, x_2 be a basis of the K-vector space C. Then the right hand side of (5.2.15) is given by the 2×2-determinant

$$\det(\varkappa_{\mathrm{c}}(x_i, x_j)).$$

By definition of C we have $V_{\mathrm{c}}x_i = \Pi x_i$. We conclude that $V_{\mathrm{c}}(x_1 \wedge x_2) = -\pi(x_1 \wedge x_2)$ in $\wedge_K^2 C$. From Lemma 8.3.3 we obtain that the determinant above gives $-\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota, \beta_{\mathrm{c}})$. □

Lemma 5.2.11. *Let $(\mathcal{P}_{\mathrm{sp}}, \iota_{\mathrm{sp}})$ the $\mathcal{W}_{O_F}(k)$-display of a special formal O_D-module. We denote by ψ a Drinfeld polarisation. Let ι'_{sp} be the restriction of ι_{sp} to $O_K \subset O_D$. Then $(\mathcal{P}_{\mathrm{sp}}, \iota'_{\mathrm{sp}}, \psi) \in \mathfrak{dR}_k^{\mathrm{pol}}$, and*

$$\mathrm{inv}(\mathcal{P}_{\mathrm{sp}}, \iota'_{\mathrm{sp}}, \psi) = -1.$$

Proof. We write $M = P_{\mathrm{sp}}$ and consider it as a $\mathcal{W}_{O_F}(k)$-Dieudonné module. Let $N = M \otimes \mathbb{Q}$. Then ψ is a relative polarization that satisfies (5.1.2). By the decomposition (5.1.3) (or (5.1.9)) we obtain a decomposition

$$N = N_0 \oplus N_1,$$

which is orthogonal with respect to ψ. As in the proof of Lemma 5.2.9, we consider the invariants $C_{\mathrm{sp}} = N^{V^{-1}\Pi}$. Because $V^{-1}\Pi$ is homogenous of degree zero, the decomposition of N induces $C_{\mathrm{sp}} = C_0 \oplus C_1$. Each C_i is a F-vector space of dimension 2. The restriction of ψ is a nondegenerate alternating pairing

$$\psi : C_{\mathrm{sp}} \times C_{\mathrm{sp}} \longrightarrow F$$

Let $\varkappa : C_{\mathrm{sp}} \times C_{\mathrm{sp}} \longrightarrow K$ be the anti-hermitian form associated to ψ as before Lemma 5.2.9. We choose a basis e_0, f_0 of the F-vector space C_0 such that $\psi(e_0, f_0) = 2$. We claim that

$$\varkappa(e_0, e_0) = \varkappa(f_0, f_0) = 0, \quad \varkappa(e_0, f_0) = 1. \tag{5.2.16}$$

Indeed, we write $\varkappa(e_0, e_0) = a + \Pi b$, $a, b \in F$. By definition of $\varkappa$ we find

$$\mathrm{Tr}_{K/F}(\varkappa(e_0, e_0)) = \psi(e_0, e_0) = 0, \quad \mathrm{Tr}_{K/F}(\Pi\varkappa(e_0, e_0)) = \psi(\Pi e_0, e_0) = 0.$$

The last equation follows because C_0 and C_1 are orthogonal. This implies $a = b = 0$. Clearly it is enough to verify the last equation of (5.2.16). Again we write $\varkappa(e_0, f_0) = a + \Pi b$, $a, b \in F$. Then we find

$$\mathrm{Tr}_{K/F}(\varkappa(e_0, f_0)) = \psi(e_0, f_0) = 2, \quad \mathrm{Tr}_{K/F}(\Pi\varkappa(e_0, f_0)) = \psi(\Pi e_0, f_0) = 0,$$

and therefore $a = 1$ and $b = 0$. Since e_0, f_0 is a basis of the K-vector space C_{sp}, the determinant

$$\det \begin{pmatrix} \varkappa(e_0, e_0) & \varkappa(e_0, f_0) \\ \varkappa(f_0, e_0) & \varkappa(f_0, f_0) \end{pmatrix} = 1$$

gives the invariant $1 = \mathrm{inv}(C_{\mathrm{sp}}, \psi) = -\mathrm{inv}(\mathcal{P}_{\mathrm{sp}}, \iota'_{\mathrm{sp}}, \psi)$ by the last Lemma. □

Proof. (of Proposition 5.2.7) By Lemma 5.2.8 we know that $\mathcal{P}_{\mathrm{c},i}$ is isoclinic of slope 1/2 for $i = 1, 2$. Therefore Lemma 5.2.9 is applicable. By Lemma 5.2.10, the associated K-vector spaces $(C_i, \beta_{\mathrm{c},i})$ have the same invariant 1 and are therefore isomorphic. Therefore we find the quasi-isogeny α by Lemma 5.2.9.

We use the notations of Lemma 5.2.11. By what we just proved we find a quasi-isogeny $(\mathcal{P}_{\mathrm{sp}}, \iota'_{\mathrm{sp}}, \psi) \longrightarrow (\mathcal{P}_{\mathrm{c},1}, \iota_{\mathrm{c},1}, \beta_{\mathrm{c},1})$. If $\beta_{\mathrm{c},1}$ is perfect, this quasi-isogeny extends by Proposition 5.2.4 to a quasi-isogeny of special formal O_D-modules and so does α. □

We can now prove the uniqueness of the framing object.

Proposition 5.2.12. *Let r be special and K/F ramified. Let k be an algebraically closed field in* Nilp_{O_E}. *Let $(\mathcal{P}, \iota, \beta) \in \mathfrak{P}^{\mathrm{pol}}_{r,k}$ be an object such that β is perfect, cf. Definition 4.1.2. Assume that* $\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = -1$. *Then $\mathcal{P}$ is isoclinic of slope* 1/2.

If moreover $(\mathcal{P}_1, \iota_1, \beta_1)$ is a second triple with the same properties, then there is a quasi-isogeny of height zero

$$\rho : (\mathcal{P}, \iota, \beta) \longrightarrow (\mathcal{P}_1, \iota_1, \beta_1),$$

such that there is an $f \in O_F^\times$ with

$$\beta_1(\rho(x), \rho(y)) = \beta(fx, y), \quad x, y \in P.$$

Proof. We apply the functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$ to $(\mathcal{P}, \iota, \beta)$ and obtain $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$, cf. (4.4.14). By the definition of this functor, β_{c} is perfect. We conclude from Proposition 8.3.2 that $\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}) = -1$. By Lemma 5.2.8, $\mathcal{P}_{\mathrm{c}}$ is isoclinic of slope (1/2). By Corollary 4.3.3 and Proposition 3.3.17, $\mathcal{P}$ is isoclinic of slope 1/2.

By Proposition 5.2.7, we find a quasi-isogeny $\alpha : (\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}) \longrightarrow (\mathcal{P}_{\mathrm{c},1}, \iota_{\mathrm{c},1}, \beta_{\mathrm{c},1})$ which we can make into a quasi-isogeny of special formal O_D-modules. The height of α is then a multiple of 2. Composing α with an endomorphism of the special formal O_D-module $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}})$, we can obtain a quasi-isogeny of height 0 of O_D-modules $\rho_{\mathrm{c}} : (\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}) \longrightarrow (\mathcal{P}_{\mathrm{c},1}, \iota_{\mathrm{c},1})$. Then ρ_{c} respects the Drinfeld polarizations β_{c} and $\beta_{\mathrm{c},1}$ up to a constant in $O_F^\times$. By Theorem 4.4.11, we obtain a quasi-isogeny of height zero as claimed in the proposition. □

Remark 5.2.13. We chose here the framing object for $\mathcal{N}$ as coming from the Drinfeld moduli problem. It can also be characterized in terms of the moduli problem $\mathcal{N}$, cf. [14]: it is a triple (X, ι, λ) consisting of a p-divisible strict formal O_F-module X over $\overline{\kappa}_F$, with an action ι of O_K satisfying the Kottwitz condition (5.2.3), and a perfect relative polarization λ such that the special automorphism group is isomorphic to $\mathrm{SL}_2(F)$, comp. [14, Prop. 3.2].

5.3. **The alternative theorem in the unramified case.** In this subsection K denotes an *unramified* quadratic extension of F. Let k be an algebraically closed field of characteristic p which is endowed with an O_F-algebra structure. We will sometimes write $F' = K$ if we refer to subsection 5.1. Let τ be the Frobenius of F'/F. We write $\tau(a) = \bar{a}$ for $a \in O_{F'}$.

Let M be the $\mathcal{W}_{O_F}(k)$-Dieudonné module of a special formal O_D-module over k, as in Proposition 5.1.3. In addition to ψ, we use another polarization of M,

$$\theta : M \times M \longrightarrow W_{O_F}(k).$$

This is an alternating bilinear form of M which satisfies

$$\begin{aligned} \theta(Fx_1, Fx_2) &= \pi\, {}^F\theta(x_1, x_2), \quad && x_1, x_2 \in M \\ \theta(\iota(a)x_1, x_2) &= \theta(x_1, \iota(\bar{a})x_2), && a \in O_{F'}, \\ \theta(\iota(\Pi)x_1, x_2) &= \theta(x_1, \iota(\Pi)x_2), && \\ \mathrm{ord}_\pi \det \theta &= 2. && \end{aligned} \tag{5.3.1}$$

The polarization θ is unique up to a constant in $O_F^\times$. It is constructed as follows: We choose an element $\delta \in O_{F'}^\times$, such that $\delta + \tau(\delta) = 0$. We set $\Pi_1 = \delta\Pi$. Then Π_1 is invariant under the involution (5.1.6) and therefore we have

$$\psi(\iota(\Pi_1)x, y) = \psi(x, \iota(\Pi_1)y).$$

We define

$$\theta(x,y) = \psi(\iota(\Pi_1)x, y). \tag{5.3.2}$$

We see that θ is alternating. It induces on D the involution given by

$$\Pi^\dagger = \Pi, \quad u^\dagger = \tau(u), \quad \text{for } u \in F'. \tag{5.3.3}$$

Conversely, assume that θ is a polarization with the properties (5.3.1). Let $\psi_1(x,y) = \theta(\Pi_1 x, y)$. Using $\Pi_1\Pi = -\Pi\Pi_1$ we see that ψ_1 satisfies the properties (5.1.7). By Proposition 5.1.3 this shows the uniqueness of θ with the properties above.

Let $(\mathbb{Y}, \iota_{\mathbb{Y}})$ be a special formal O_D-module over the O_F-algebra $\overline{\kappa}_F$ and such that $\iota(\Pi)$ acts as zero on $\operatorname{Lie}\mathbb{Y}$. We endow $\mathbb{Y}$ with the polarization $\theta_{\mathbb{Y}}$ defined above, cf. (5.3.2).

Definition 5.3.1. We define for each $i \in \mathbb{Z}$ the functor $\mathcal{N}(i) = \mathcal{N}_{K/F}(i)$ on the category $(\mathrm{Sch}/\operatorname{Spf} O_{\breve{F}})$. A point of $\mathcal{N}(i)(S)$ consists of the following data:

(1) A formal p-divisible group X over S with an action

$$\iota : O_K \longrightarrow \operatorname{End} X,$$

such that the restriction of ι to O_F is a strict action.

(2) A relative polarization θ on X such that the determinant of θ is π^2 up to a unit and such that θ induces on O_K the conjugation over O_F.

(3) A quasi-isogeny of O_K-modules

$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{Y} \times_{\operatorname{Spec} \bar{\kappa}_F} \overline{S}.$$

Here, if $S = \operatorname{Spec} R$, the condition in (2) means that the polarization of the corresponding O_F-display $\mathcal{P}$ of Y has determinant π^2, up to a unit in $W_{O_F}(R)$. We require that the following conditions are satisfied.

a) ρ respects O_K-actions. The relative quasi-polarization $\rho^*\theta_{\mathbb{Y}}$ differs from $\pi^i\theta$ by a factor in $O_F^\times$.

b) $\operatorname{Lie} X$ is locally on S a free $O_K \otimes_{O_F} \mathcal{O}_S$-module of rank 1.

We note that, as in the ramified case, the O_F-height of X is 4 and the dimension 2. The condition b) implies the following Kottwitz condition for the characteristic polynomial:

$$\operatorname{char}(\iota(a) \mid \operatorname{Lie} X) = (T-a)(T-\bar{a}), \quad a \in O_K.$$

It follows from [27] that $\mathcal{N}(i)$ is representable by a formal scheme which is locally formally of finite type over $\operatorname{Spec} O_{\breve{F}}$. The functor $\mathcal{N}(0)$ will be also denoted by $\mathcal{N}$.

We have a natural functor morphism

$$\mathcal{M}_{\mathrm{Dr}}(i) \longrightarrow \mathcal{N}(i). \tag{5.3.4}$$

Indeed, let $(Y, \rho) \in \mathcal{M}_{\mathrm{Dr}}(i)(R)$. Then we have the Drinfeld polarization ψ of Y and we define θ_Y by the formula (5.3.2). This gives a point of $\mathcal{N}(R)(i)$.

The diagram similiar to (5.2.1) shows that

$$\operatorname{height}_{O_F} \rho = 2i.$$

We will define a translation functor isomorphism

$$\Pi : \mathcal{N}(i) \xrightarrow{\sim} \mathcal{N}(i+1). \tag{5.3.5}$$

Let (Y, ι) be a special formal O_D-module over $R \in \mathrm{Nilp}_{O_F}$. We fix a Drinfeld polarization ψ. This is also a Drinfeld polarization for (Y^Π, ι^Π). For the polarizations θ and θ^Π derived by (5.3.2), we obtain $\theta^\Pi = -\theta$. We consider the morphism $\iota^\Pi : Y^\Pi \longrightarrow Y$. One easily checks that

$$\theta(\iota(\Pi)x, \iota(\Pi)y) = \pi\theta^\Pi(x,y).$$

This is an identity of bilinear forms on the $\mathcal{W}_{O_F}(R)$-display of Y.

If (X, ι) is a p-divisible O_K-module, we define the conjugate p-divisible O_K-module $(X^{\mathrm{c}}, \iota^{\mathrm{c}})$ by setting $X^{\mathrm{c}} = X$ and $\iota^{\mathrm{c}}(a) = \iota(\bar{a})$ for $a \in O_K$. For the special formal O_D-module Y we have

$$(Y^{\mathrm{c}}, (\iota_{|O_K})^{\mathrm{c}}) = (Y^{\Pi}, \iota^{\Pi}_{|O_K}).$$

Let $R \in \mathrm{Nilp}_{O_{\breve{F}}}$ and let $(X, \iota, \theta, \rho) \in \mathcal{N}(i)(R)$. We define

$$\rho^{\mathrm{c}} : X^{\mathrm{c}}_{\bar{R}} \xrightarrow{\rho} \mathbb{Y}^{\mathrm{c}}_{\bar{R}} = \mathbb{Y}^{\Pi}_{\bar{R}} \xrightarrow{\iota(\Pi)} \mathbb{Y}_{\bar{R}}.$$

We set $\theta^{\mathrm{c}} = -\theta$. Then $(X^{\mathrm{c}}, \iota^{\mathrm{c}}, \theta^{\mathrm{c}}, \rho^{\mathrm{c}}) \in \mathcal{N}(i+1)(R)$. This defines the translation functor morphism (5.3.5). It is clearly an isomorphism. With this definition, the functor morphism (5.3.4) commutes with the translations on source and target.

Let $\tau \in \mathrm{Gal}(\breve{F}/F)$ be the Frobenius automorphism. Using the Frobenius $F_{\mathbb{Y},\tau} : \mathbb{Y} \longrightarrow \tau_* \mathbb{Y}$, we obtain a morphism

$$\omega_{\mathcal{N}} : \mathcal{N}(i) \longrightarrow \mathcal{N}(i+1)^{(\tau)}$$

with the same definition as (5.1.18). This induces a Weil descent datum $\omega_{\mathcal{N}}$ on

$$\tilde{\mathcal{N}} = \coprod_{i \in \mathbb{Z}} \mathcal{N}(i).$$

Lemma 5.3.2. *The action of the group $J^{*\mathrm{ur}}$ (cf. (5.1.25) for F', which is now denoted by K) on the O_K-module $\mathbb{Y}$ gives an isomorphism*

$$J^{*\mathrm{ur}} \xrightarrow{\sim} J^{\cdot},$$

where

$$J^{\cdot} = \{\alpha \in \mathrm{Aut}^o_K(\mathbb{Y}) \mid \theta_{\mathbb{Y}}(\alpha(x), \alpha(y)) = \mu(\alpha)\theta_{\mathbb{Y}}(x, y),\ \text{for some}\ \mu(\alpha) \in F^{\times},\ x, y \in P_{\mathbb{Y}} \otimes \mathbb{Q}\}$$

□

The group $J^{\cdot}$ acts via the rigidification ρ on the functor $\tilde{\mathcal{N}}$.

Theorem 5.3.3 ([17])**.** *The morphisms of functors (5.3.4) extend to a functor isomorphism*

$$\tilde{\mathcal{M}}_{\mathrm{Dr}} \xrightarrow{\sim} \tilde{\mathcal{N}}$$

which commutes with the Weil descent data, the actions of $J^{*\mathrm{ur}} = J^{\cdot}$, and the translations on both sides.

Proof. We already checked that (5.3.4) extends to a functor morphism which respects translations and Weil descent data on both sides. Therefore it suffices to see that (5.3.4) is an isomorphism for $i = 0$,

$$\mathcal{M}_{\mathrm{Dr}} \xrightarrow{\sim} \mathcal{N}. \tag{5.3.6}$$

We begin with the case where $R = k$ is an algebraically closed field. Let $Y \in \mathcal{N}(k)$. Let M be the O_F-Dieudonné module of Y and let $\mathbb{M}$ be the O_F-Dieudonné module of $\mathbb{Y}$. The quasi-isogeny ρ induces an isomorphism $M \otimes \mathbb{Q} \cong \mathbb{M} \otimes \mathbb{Q}$. The polarization $\psi_{\mathbb{M}}$ induces a polarization ψ_1 on $M \otimes \mathbb{Q}$. Since ρ is of height zero and $\mathrm{ord}_{\pi} \det \psi_{\mathbb{M}} = 0$ we conclude that $\mathrm{ord}_{\pi} \det \psi_1 = 0$. On the other hand, we have by Proposition 5.1.1 a perfect pairing ψ on M which differs from ψ_1 by an element $f \in F'$. This shows that ψ_1 is perfect on M. Then we define the action $\iota(\Pi_1)$ by the equation

$$\theta(x, y) = \psi(\iota(\Pi_1)x, y), \quad x, y \in M. \tag{5.3.7}$$

Therefore the morphism (5.3.6) evaluated at k is bijective.

Since both functors of (5.3.6) are representable by formal schemes locally of finite type, it suffices now to check the following statement. Let $S \longrightarrow R$ be a surjective $O_{\breve{F}}$-algebra homomorphism such that S and R are artinian local rings with algebraically closed residue class field. Assume that (5.3.6) is bijective when evaluated at R. Then it is bijective when evaluated at S. We may assume that the kernel of $S \longrightarrow R$ is endowed with divided powers.

We consider a point $\widetilde{Y} \in \mathcal{N}(S)$ and we denote by $Y \in \mathcal{N}(R)$ its reduction. By our assumption, Y carries the structure of an O_D-module compatible with ρ. Therefore $\psi_{\mathbb{Y}}$ induces a perfect polarization on the O_F-display $\mathcal{P}$ of Y. The O_F-display $\widetilde{\mathcal{P}}$ of $\widetilde{Y}$ is a lifting of $\mathcal{P}$. By the crystalline property of displays [1], cf. end of subsection 3.1, we obtain a perfect pairing

$$\widetilde{\psi} : \widetilde{P} \times \widetilde{P} \longrightarrow W_{O_F}(S).$$

The involution induced by $\widetilde{\psi}$ on $O_{F'}$ is trivial. It follows that the decomposition

$$\widetilde{P} = \widetilde{P}_0 \oplus \widetilde{P}_1$$

according to the two O_F-algebra embeddings $O_{F'} \longrightarrow O_{\breve{F}}$ is orthogonal with respect to $\widetilde{\psi}$. The Hodge filtration

$$\tilde{Q}_i/I_{O_F}\tilde{P}_i \subset \tilde{P}_i/I_{O_F}\tilde{P}_i, \quad i = 0, 1$$

is isotropic with respect to $\widetilde{\psi}$ because these direct summands are of rank 1. Therefore $\widetilde{\psi}$ is a polarization of the O_F-display $\widetilde{\mathcal{P}}$. Using the given polarization $\widetilde{\theta}$ on $\widetilde{\mathcal{P}}$, we can define the endomorphism $\iota(\Pi_1) = \iota(\delta\Pi)$ of $\widetilde{P}$ by

$$\widetilde{\theta}(x, y) = \widetilde{\psi}(\iota(\Pi_1)x, y).$$

This gives the desired O_D-module structure on $\widetilde{\mathcal{P}}$ and therefore on $\widetilde{Y}$. □

The analogue of Corollary 5.2.6 follows as before from the properties of the Drinfeld moduli space.

Corollary 5.3.4. *The formal scheme $\mathcal{N}$ is π-adic and has semi-stable reduction. The special fiber $\mathcal{N} \otimes_{O_{\breve{F}}} \overline{\kappa}_F$ of $\mathcal{N}$ is a reduced scheme.* □

We next prove the uniqueness of the framing object, cf. (i) of subsection 2.5. We start with the following statement.

Proposition 5.3.5. *Let k be an algebraically closed field which contains κ_F. Let M be a $\mathcal{W}_{O_F}(k)$-Dieudonné module of height 4 and dimension 2. Let ι be a homomorphisms of O_F-algebras*

$$\iota : O_{F'} \longrightarrow \operatorname{End} M.$$

Assume that M/VM is a free $\kappa_{F'} \otimes_{\kappa_F} k$-module of rank 1. Let θ be a relative polarization on M which satisfies

$$\theta(\iota(a)x_1, x_2) = \theta(x_1, \iota(\bar{a})x_2), \quad a \in O_{F'},$$
$$\operatorname{ord}_\pi \det \theta = 2.$$

Then the action ι extends to an action $\iota : O_D \longrightarrow \operatorname{End} M$ such that θ satisfies (5.3.1). In particular M is isoclinic of slope $1/2$. Furthermore, $\operatorname{inv}(M, \iota, \theta) = -1$ (see Definition 8.3.1 for this invariant).

If (M', ι', θ') is a second triple with the same properties, then there exists a quasi-isogeny $(M, \iota) \longrightarrow (M', \iota')$ of height 0 which respects the polarizations θ and θ' up to a factor in $O_F^\times$.

Proof. Let ψ be the principal relative polarization on M which exists by Proposition 5.1.1. We define an endomorphism $\rho : M \longrightarrow M$ by the equation

$$\theta(x, y) = \psi(x, \rho(y)), \quad x, y \in M.$$

One checks that ρ is an endomorphism of the Dieudonné module M such that

$$\rho(\iota(a)x) = \iota(\bar{a})\rho(x), \quad a \in O_{F'}. \tag{5.3.8}$$

As in the proof of Proposition 5.1.1, we choose an embedding $\lambda : O_{F'} \longrightarrow W_{O_F}(k)$ and obtain a decomposition $M = M_0 \oplus M_1$. We note that

$$M_1 = \{x \in M \mid \iota(a)x = \lambda(\bar{a})x\}.$$

It follows from (5.3.8) that $\rho(M_0) \subset M_1$ and $\rho(M_1) \subset M_0$. We obtain a commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{\rho} & M_1 \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V} \\ M_1 & \xrightarrow{\rho} & M_0. \end{array}$$

By our assumption on M/VM, the cokernels of both vertical maps have $W_{O_F}(k)$-length 1. Therefore the cokernels of the horizontal maps have also the same length. This length must be 1 because $\operatorname{ord}_\pi \det(\rho|M) = \operatorname{ord}_\pi \det \theta = 2$.

We have

$$\theta(\rho(x), y) = \theta(x, \rho(y)),$$

because both sides are equal to $\psi(\rho(x), \rho(y))$. We consider the form

$$\psi_1(x, y) := \psi(\rho(x), \rho(y)).$$

This relative polarization satisfies the assumptions of the last part of Proposition 5.1.1. Therefore there exists $c \in O_{F'}$ such that $\psi_1(x, y) = \psi(\iota(c)x, y)$. We find

$$\psi(\iota(c)x, y) = \theta(\rho(x), y) = -\theta(y, \rho(x)) = -\psi(y, \rho^2(x)) = \psi(\rho^2(x), y).$$

This shows that

$$\rho^2 = \iota(c).$$

Since ρ commutes with the left hand side it commutes with $\iota(c)$. Comparing this with (5.3.8), we obtain $c \in O_F$. Since ρ^2 has height 4 we obtain $\mathrm{ord}_\pi c = 1$.

For $\alpha \in O_{F'}^\times$ we consider the endomorphism $\rho_\alpha(x) = \iota(\alpha)\rho(x)$ of M. We obtain $\rho_\alpha^2 = \iota(\alpha\bar{\alpha})\rho^2$. Since each unit of F is a norm in the unramified extension F'/F we can arrange that $\rho_\alpha^2 = -\pi$. We set $\Pi = \rho_\alpha$. Then we obtain an action of $O_D = O_{F'}[\Pi]$ on the Dieudonné module M. Since $M_0/\Pi M_1$ and $M_1/\Pi M_0$ have length 1, we have obtained a special formal O_D-module. The equations (5.3.1) are satisfied for θ. Therefore θ is up to a factor in $O_F^\times$ uniquely determined by the O_D-action. This implies the first and the last assertion of the Proposition.

Because the invariant depends only on the isogeny class, it is enough to compute it for a special formal O_D-module with two critical indices and the canonical form θ from (5.3.1).

We use the isomorphism

$$O_{F'} \otimes_{O_F} W_{O_F}(k) \longrightarrow W_{O_F}(k) \times W_{O_F}(k), \tag{5.3.9}$$

which maps $a \otimes \xi$ to $(\lambda(a)\xi, \lambda(\bar{a})\xi)$. Let $\sigma = F$ be the Frobenius automorphism of $W_{O_F}(k)$. It acts on the left hand side of (5.3.9) via the second factor. This induces on the right hand side the action $\sigma : (\xi_1, \xi_2) \mapsto (\sigma(\xi_2), \sigma(\xi_1))$.

We set $N_i = M_i \otimes \mathbb{Q}$. This is a $W_{O_F}(k)_\mathbb{Q}$-vector space of dimension 2. In the decomposition $N = M \otimes \mathbb{Q} = N_0 \otimes N_1$, the summands are isotropic with respect to θ. We consider the invariants

$$U_i = N_i^{V^{-1}\iota(\Pi)}.$$

(In the notation of the proof of Proposition 5.1.5, this is $U_i \otimes \mathbb{Q}$.) The U_i are F-vector spaces of dimension 2. Let

$$\wedge^2\theta : \wedge^2 N_0 \times \wedge^2 N_1 \longrightarrow W_{O_F}(k)_\mathbb{Q}$$

be the bilinear form defined by

$$\theta(n_0 \wedge n_0', n_1 \wedge n_1') = \det \begin{pmatrix} \theta(n_0, n_1) & \theta(n_0, n_1') \\ \theta(n_0', n_1) & \theta(n_0', n_1') \end{pmatrix},$$

for $n_0, n_0' \in N_0$, $n_1, n_1' \in N_1$. In the same way we can define $\wedge^2\theta : \wedge^2 N_1 \times \wedge^2 N_0 \longrightarrow W_{O_F}(k)_\mathbb{Q}$. Then we obtain $\wedge^2\theta(x_0, x_1) = \wedge^2\theta(x_1, x_0)$ for $x_0 \in \wedge^2 N_0, x_1 \in \wedge^2 N_1$. From θ we pass to $\varkappa$, cf. (8.1.2) (there our F' is called K),

$$\varkappa : N \times N \longrightarrow F' \otimes_F W_{O_F}(k)_\mathbb{Q} \cong W_{O_F}(k)_\mathbb{Q} \times W_{O_F}(k)_\mathbb{Q}.$$

Explicitly we have

$$\varkappa(n_0 + n_1, n_0' + n_1') = (\theta(n_0, n_1'), \theta(n_1, n_0')) \in W_{O_F}(k)_\mathbb{Q} \times W_{O_F}(k)_\mathbb{Q}.$$

We take $\wedge^2\varkappa$ on the $F' \otimes_F W_{O_F}(k)_\mathbb{Q}$-module

$$\bigwedge^2_{F' \otimes_F W_{O_F}(k)_\mathbb{Q}} N \cong \wedge^2 N_0 \oplus \wedge^2 N_1.$$

From the expression for $\varkappa$ we obtain

$$\wedge^2 \varkappa(x_0 + x_1, x_0' + x_1') = (\wedge^2\theta(x_0, x_1'), \wedge^2\theta(x_1, x_0')) \in W_{O_F}(k)_\mathbb{Q} \times W_{O_F}(k)_\mathbb{Q} \tag{5.3.10}$$

The restriction of θ to $U_0 \times U_1$ induces a nondegenerate F-bilinear form

$$\theta : U_0 \times U_1 \longrightarrow \delta F \subset F' \subset W_{O_F}(k)_\mathbb{Q}, \tag{5.3.11}$$

where δ was defined after (5.3.1). Indeed, for $u_0 \in U_0$ and $u_1 \in U_1$ we have by definition

$$\theta(Vu_0, Vu_1) = \theta(\iota(\Pi)u_0, \iota(\Pi)u_1) = \theta(\iota(\Pi)^2 u_0, u_1) = -\pi\theta(u_0, u_1).$$

Because θ is a polarization, we have on the other hand

$$\theta(Vu_0, Vu_1) = \pi\sigma^{-1}\big(\theta(u_0, u_1)\big).$$

Therefore $\theta(u_0, u_1)$ is anti-invariant by σ and (5.3.11) is proved.

We choose a nonzero element $u_0 \in U_0$. Then we find $u_1 \in U_1$ such that $\theta(u_0, u_1) = \delta$. We remark that $\theta(\iota(\Pi)n, n) = 0$, for an arbitrary $n \in N$. This is clear from the third equation of (5.3.1) because θ is alternating. Since $\theta(u_0, \iota(\Pi)u_0) = 0$, the vectors $u_1, \iota(\Pi)u_0 \in U_1$ are linearly independent. We set

$$x = x_0 + x_1 := u_0 \wedge \iota(\Pi)u_1 + u_1 \wedge \iota(\Pi)u_0 \in \wedge^2_{F' \otimes_F W_{O_F}(k)_{\mathbb{Q}}} N.$$

It satisfies $\wedge^2 Vx = \pi x$. Indeed,

$$\wedge^2 V(u_0 \wedge \iota(\Pi)u_1) = (Vu_0 \wedge \iota(\Pi)Vu_1) = (\iota(\Pi)u_0 \wedge \iota(\Pi)^2 u_1) = (\iota(\Pi)u_0 \wedge (-\pi)u_1) = \pi(u_1 \wedge \iota(\Pi)u_0).$$

The similiar equation holds for the second summand in the definition of x. By Definition 8.3.1 we obtain

$$\mathrm{inv}(M, \iota, \theta) = (-1)^{\mathrm{ord}_\pi \wedge^2 \varkappa(x,x)}.$$

By (5.3.10) we have $\mathrm{ord}_\pi \wedge^2 \varkappa(x, x) = \mathrm{ord}_\pi \wedge^2 \theta(x_0, x_1)$. We compute

$$\wedge^2\theta(x_0, x_1) = \det\begin{pmatrix} \theta(u_0, u_1) & \theta(u_0, \iota(\Pi)u_0) \\ \theta(\iota(\Pi)u_1, u_1) & \theta(\iota(\Pi)u_1, \iota(\Pi)u_0) \end{pmatrix} = \det\begin{pmatrix} \delta & 0 \\ 0 & \delta\pi \end{pmatrix} = \pi\delta^2.$$

This shows $\mathrm{ord}_\pi \wedge^2\theta(x_0, x_1) = 1$ and therefore $\mathrm{inv}(M, \iota, \theta) = -1$. □

We can now prove the uniqueness of the framing object.

Proposition 5.3.6. *Let r be special and K/F unramified. Let k be an algebraically closed field in* Nilp_{O_E}. *Let $(\mathcal{P}, \iota, \beta) \in \mathfrak{P}^{\mathrm{pol}}_{r,k}$ be such that $\mathrm{ord}_p \det_{W(k)} \beta = 2f$, cf. Definition 4.1.2. Then $\mathcal{P}$ is isoclinic of slope $1/2$ and $\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = -1$.*

If $(\mathcal{P}_1, \iota_1, \beta_1)$ is a second triple with the same properties, then there exists a quasi-isogeny of height zero,

$$\rho : (\mathcal{P}, \iota, \beta) \longrightarrow (\mathcal{P}_1, \iota_1, \beta_1),$$

such that there is an $f \in O_F^\times$ with

$$\beta_1(\rho(x), \rho(y)) = \beta(fx, y), \quad x, y \in P.$$

Proof. We apply the functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$ to $(\mathcal{P}, \iota, \beta)$ and obtain $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$, cf. (4.4.14). By Theorem 4.4.11 we find $\mathrm{ord}_\pi \det_{W_{O_F}(k)} \beta_{\mathrm{c}} = 2$. Therefore we can apply Proposition 5.3.5 to $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$. We obtain that $\mathcal{P}_{\mathrm{c}}$ is isoclinic of slope $1/2$ and $\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}) = -1$. By Corollary 4.3.3 and Proposition 3.3.17, we find that $\mathcal{P}$ is isoclinic of slope $1/2$, and by Proposition 8.3.2 we obtain $\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = -1$.

By Proposition 5.3.5, there is a quasi-isogeny of height 0 between $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$ and $(\mathcal{P}_{1,c}, \iota_{1,c}, \beta_{1,c})$. It induces by Theorem 4.4.11 a quasi-isogeny of height zero as claimed in the Proposition. □

We end this section by justifying the footnote in Definition 2.6.1. Let $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ and let $(Y, \iota, \theta, \rho) \in \mathcal{N}(S)$. Since there is an O_D-module structure on Y such that θ is of the form (5.3.7), the kernel of $\theta : Y \longrightarrow Y^\wedge$, considered as morphism to the dual relative to O_F, is annihilated by π. More generally we prove:

Proposition 5.3.7. *Let K/F be an unramified quadratic field extension. Let R be an O_K-algebra. Let $\mathcal{P}$ and $\mathcal{P}'$ be $\mathcal{W}_{O_F}(R)$-displays of height 4 with an action $\iota : O_K \longrightarrow \mathrm{End}\,\mathcal{P}$, resp. $\iota' : O_K \longrightarrow \mathrm{End}\,\mathcal{P}'$. Assume that* $\mathrm{Lie}\,\mathcal{P}$, *resp.* $\mathrm{Lie}\,\mathcal{P}'$, *is locally on* $\mathrm{Spec}\,R$ *a free $O_K \otimes_{O_F} R$-module of rank 2. Let $\alpha : \mathcal{P} \longrightarrow \mathcal{P}'$ be an isogeny of O_F-height 2. Then there exists locally on* $\mathrm{Spec}\,R$ *an isogeny $\beta : \mathcal{P}' \longrightarrow \mathcal{P}$ such that*

$$\beta \circ \alpha = \pi\mathrm{id}_{\mathcal{P}}, \quad \alpha \circ \beta = \pi\mathrm{id}_{\mathcal{P}'}.$$

Let $\mathcal{P}$ and $\mathcal{P}'$ be the displays of formal p-divisible groups X and X' with an O_K-action. Then the kernel of any isogeny $\alpha : X \longrightarrow X'$ of height 2 is annihilated by π.

Proof. The proof is a variant of the proof of Proposition 1.6.4 in [35]. We will use notation from that proof. The O_K-algebra structure on R induces a natural homomorphism $O_K \longrightarrow W_{O_F}(R)$ which is equivariant with respect to the Frobenius $\tau \in \mathrm{Gal}(K/F)$ and the Frobenius on $W_{O_F}(R)$. The composition with τ gives a second homomorphism $O_K \longrightarrow W_{O_F}(R)$. We denote by Ψ the set of these two homomorphism. We write $\bar{\psi} = \psi \circ \tau$ for $\psi \in \Psi$.

The O_K-action gives the usual decompositions,

$$P = \oplus_{\psi \in \Psi} P_\psi, \quad P' = \oplus_{\psi \in \Psi} P'_\psi.$$

We have the same kind of decompositions for $Q \subset P$ and $Q' \subset P'$. We choose normal decompositions

$$P_\psi = T_\psi \oplus L_\psi, \qquad P'_\psi = T'_\psi \oplus L'_\psi.$$

The T and L on the right hand sides are by assumption locally free of rank 1. Using these decompositions, we write $\alpha_\psi : P_{\bar{\psi}} \longrightarrow P'_\psi$ in matrix form

$$M_\psi = \begin{pmatrix} X_\psi & {}^V Y_\psi \\ U_\psi & Z_\psi \end{pmatrix}. \tag{5.3.12}$$

The maps $\dot{F}_\psi : I_{O_F}(R) T_\psi \oplus L_\psi \longrightarrow T_{\psi\tau} \oplus L_{\psi\tau}$, resp. $\dot{F}'_\psi : I_{O_F}(R) T'_\psi \oplus L'_\psi \longrightarrow T'_{\psi\tau} \oplus L'_{\psi\tau}$, are given by invertible matrices

$$\Phi_\psi = \begin{pmatrix} A_\psi & B_\psi \\ C_\psi & D_\psi \end{pmatrix}, \qquad \Phi'_\psi = \begin{pmatrix} A'_\psi & B'_\psi \\ C'_\psi & D'_\psi \end{pmatrix}.$$

The matrices (5.3.12) define a morphism α of displays iff

$$M_{\psi\tau} \Phi_\psi = \Phi'_\psi \; {}^s M_\psi, \quad \text{for } \psi \in \Psi. \tag{5.3.13}$$

We will argue as in [35]. The meaning of the upper left indes ^{s} is the same as there. Taking the determinants we obtain

$$\det M_{\psi\tau} \; \det \Phi_\psi = \det \Phi'_\psi \; \det \; {}^s M_\psi. \tag{5.3.14}$$

In particular $\det M_\psi$ and ${}^{F^2} \det M_\psi$ differ by a unit in $W_{O_F}(R)$. As in [35] we obtain that

$$\det M_\psi = \pi^{\mathtt{h}} \epsilon_\psi,$$

for some units $\epsilon_\psi \in W_{O_F}(R)$. By (5.3.14), $\mathtt{h}$ is independent of ψ. Since α is an isogeny of height 2 we conclude that $\mathtt{h} = \mathtt{1}$. Now we pass to the adjucate matrices

$${}^{\mathrm{ad}}\Phi_\psi \; {}^{\mathrm{ad}}M_{\psi\tau} = {}^{\mathrm{ad}}({}^s M_\psi) \; {}^{\mathrm{ad}}\Phi'_\psi.$$

Since the matrices Φ are invertible, we conclude

$$(\det \Phi_\psi) \; {}^{\mathrm{ad}}M_{\psi\tau} \; \Phi'_\psi = (\det \Phi'_\psi) \Phi_\psi \; {}^{\mathrm{ad}}({}^s M_\psi). \tag{5.3.15}$$

We consider first the case where R is reduced. Then π is not a zero divisor in $W_{O_F}(R)$. In this case, we conclude from (5.3.14)

$$\det \Phi_\psi \; \epsilon_{\psi\tau} = \det \Phi'_\psi \; {}^F \epsilon_\psi.$$

Thus we may rewrite equation (5.3.15) as

$$\epsilon_{\psi\tau}^{-1} \; {}^{\mathrm{ad}}M_{\psi\tau} \; \Phi'_\psi = \Phi_\psi \; {}^F \epsilon_\psi^{-1} \; {}^{\mathrm{ad}}({}^s M_\psi).$$

This shows that the matrices $\epsilon_\psi^{-1} \; {}^{\mathrm{ad}}M_\psi$ define the desired morphism $\beta : \mathcal{P}' \longrightarrow \mathcal{P}$. In the case where R is not reduced we may argue as in the proof of Proposition 1.6.4 in [35]. $\square$

6. Moduli spaces of formal local CM-triples

In this section we prove Theorems 2.6.2 and 2.6.3. Recall that $d = [F : \mathbb{Q}_p]$, and that we write $d = ef$.

6.1. **The case r special and K/F ramified.** Let $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ be a special formal O_D-module over $\bar{\kappa}_F$ with a Drinfeld polarization $\lambda_{\mathbb{Y}}$, cf. Definition 5.1.4 and section 5.2. We denote by $\breve{F}$ resp. $\breve{E}$, the completions of the maximal unramified extension of F, resp. of the reflex field E. Their residue class fields are identified with $\bar{\kappa}_F$ resp. $\bar{\kappa}_E$. We note that in the ramified case $E = E'$. We extend the embedding $\varphi_0 : O_F \longrightarrow O_E$ to an embedding $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$. The base change $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})_{\bar{\kappa}_E}$ is the base change via $\breve{\varphi}_0$. It is an object of $\mathfrak{d}\mathfrak{R}^{\mathrm{nilp,pol}}_{\bar{\kappa}_E}$. We refer to Theorem 4.4.11 for the latter notation. It is therefore isomorphic to the image of an object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \in \mathfrak{d}\mathfrak{P}^{\mathrm{ss,pol}}_{r,R}$. The polarization $\lambda_{\mathbb{X}}$ is again principal. We consider the functor $\mathcal{M}_r = \mathcal{M}_{K/F,r}$ of Definition 2.6.1 in the ramified case (where $\mathtt{h} = 0$). By Proposition 4.2.9, this Definition may be reformulated as follows.

Proposition 6.1.1. *Let S be a scheme over $\operatorname{Spf} O_{\breve{E}}$. A point of $\mathcal{M}_r(S)$ consists of*

(1) a local CM-pair (X, ι) of CM-type r over S which satisfies the Eisenstein conditions (EC_r) *relative to a fixed uniformizer Π of K and such that*

$$\mathrm{Tr}(\iota(\Pi) \mid \mathbf{E}_{A_{\psi_0}} \mathrm{Lie}_{\psi_0} X) = 0. \qquad (6.1.1)$$

(2) an isomorphism of p-divisible O_K-modules

$$\lambda : X \xrightarrow{\sim} X^{\wedge},$$

which is a polarization of X.

(3) a quasi-isogeny of height zero of p-divisible O_K-modules

$$\rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \bar{S},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $\lambda_{|X \times_S \overline{S}}$ by a scalar in $O_F^{\times}$, locally on $\overline{S}$. Here $\overline{S} = S \times_{\operatorname{Spec} O_{\breve{E}}} \operatorname{Spec} \bar{\kappa}_E$.

In Definition 2.6.1 we required that the scalar is in $F^{\times}$ but because the polarizations λ and $\lambda_{\mathbb{X}}$ are principal and ρ has height 0 the scalar is automatically in $O_F^{\times}$. We remark that the condition (6.1.1) depends only on the restriction of the structure morphism $S \longrightarrow \operatorname{Spf} O_E$.

We define for $i \in \mathbb{Z}$ the functor $\mathcal{M}_r(i)$ on the category of schemes $S \longrightarrow \operatorname{Spf} O_{\breve{E}}$ by replacing (3) in Proposition 6.1.1 by

(3′) *a quasi-isogeny of p-divisible O_K-modules*

$$\rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \bar{S},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $p^i \lambda_{|X \times_S \bar{S}}$ by a scalar in $O_F^{\times}$, locally on $\bar{S}$.

It follows from the last condition that

$$2 \,\mathrm{height}\, \rho = \mathrm{height}(p^i \mid X) = 4di. \qquad (6.1.2)$$

We have an isomorphism of functors

$$\mathcal{M}_r \longrightarrow \mathcal{M}_r(i),$$

which associates to a point $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(S)$ the point $(X, \iota, \lambda, \Pi^{ei}\rho) \in \mathcal{M}_r(i)(S)$. We set

$$\tilde{\mathcal{M}}_r = \coprod_{i \in \mathbb{Z}} \mathcal{M}_r(i).$$

We define a Weil descent datum on $\tilde{\mathcal{M}}_r$ relative to $O_{\breve{E}}/O_E$. Let τ_E be the Frobenius of $\breve{E}/E$. It is enough to consider the functor $\tilde{\mathcal{M}}_r$ for affine schemes $S = \operatorname{Spec} R$. We write $\varepsilon : O_{\breve{E}} \longrightarrow R$ for the given algebra structure. We write $R_{[\tau_E]}$ for the ring R with the new algebra structure $\varepsilon \circ \tau_E$. By base change to $\bar{\kappa}_E$, we obtain

$$\bar{\varepsilon} : \bar{\kappa}_E \longrightarrow \bar{R} := R \otimes_{O_{\breve{E}}} \bar{\kappa}_E.$$

We consider a point $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(i)(R)$, where ρ is a quasi-isogeny

$$\rho : X_{\bar{R}} \longrightarrow \bar{\varepsilon}_* \mathbb{X}.$$

Since the notion of a CM-triple depends only on the induced O_E-algebra structure on R, we may regard (X, ι, λ) as a CM-triple on $R_{[\tau_E]}$. We set

$$\tilde{\rho} : X_{\bar{R}} \xrightarrow{\rho} \bar{\varepsilon}_* \mathbb{X} \xrightarrow{F_{\mathbb{X},\tau_E}} \bar{\varepsilon}_*(\tau_E)_* \mathbb{X}.$$

The assignment $(X, \iota, \lambda, \rho) \mapsto (X, \iota, \lambda, \tilde{\rho})$ defines a morphism

$$\omega_{\mathcal{M}_r} : \mathcal{M}_r(i)(R) \longrightarrow \mathcal{M}_r(i + f_E)(R_{[\tau_E]}) \tag{6.1.3}$$

where $f_E = [\kappa_E : \mathbb{F}_p]$. Here we note that the inverse image of the polarization $(\tau_E)_*\lambda_{\mathbb{X}}$ on $(\tau_E)_*\mathbb{X}$ by $F_{\mathbb{X},\tau_E} : \mathbb{X} \longrightarrow (\tau_E)_*\mathbb{X}$ is $p^{f_E}\lambda_{\mathbb{X}}$. From (6.1.3) we obtain the Weil descent datum

$$\omega_{\mathcal{M}_r} : \tilde{\mathcal{M}}_r \longrightarrow \tilde{\mathcal{M}}_r^{(\tau_E)}, \tag{6.1.4}$$

where the upper index (τ_E) denotes the base change via $\operatorname{Spec} \tau_E : \operatorname{Spf} O_{\breve{E}} \longrightarrow \operatorname{Spf} O_{\breve{E}}$.

Let $\mathcal{N}(i)$ be the functor of Definition 5.2.1. Note that we took $\mathbb{Y}$ for the framing object appearing in the definition of $\mathcal{N}(i)$. We consider a point $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(i)(R)$, where $R \in \mathrm{Nilp}_{O_{\breve{E}}}$. Applying the contracting functor $\mathfrak{C}^{\mathrm{pol}}_{r,R}$, we obtain a quadruple $(X_{\mathrm{c}}, \iota_{\mathrm{c}}, \lambda_{X_{\mathrm{c}}}, \rho_{\mathrm{c}})$. It follows from the isomorphism (4.3.11) (which also holds in the ramified case, cf. a few lines below (4.3.11)) that the condition (6.1.1) implies

$$\operatorname{Tr}(\iota_{\mathrm{c}}(\Pi) \mid \operatorname{Lie} X_{\mathrm{c}}) = 0.$$

By functoriality, the polarizations $\rho_{\mathrm{c}}^*\lambda_{\mathbb{Y}}$ and $p^i\lambda_{(X_{\mathrm{c}})_{\bar{R}}}$ differ by a unit in O_F. Hence $(X_{\mathrm{c}}, \iota_{\mathrm{c}}, \lambda_{X_{\mathrm{c}}}, \rho_{\mathrm{c}})$ defines a point of $\mathcal{N}(i)$. Therefore we obtain from Theorem 4.4.11 an isomorphism of functors,

$$\mathcal{M}_r(i) \xrightarrow{\sim} \mathcal{N}(ei) \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spf} O_{\breve{E}}. \tag{6.1.5}$$

The base change on the right hand side is via $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$

We set

$$\tilde{\mathcal{N}}[e] = \coprod_{i \in \mathbb{Z}} \mathcal{N}(ei). \tag{6.1.6}$$

We endow $\tilde{\mathcal{N}}[e]$ with a Weil descent datum relative to $O_{\breve{E}}/O_E$. Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$. We consider the map

$$\Pi^{(d-1)f_E/f} \omega_{\mathcal{N}}^{f_E/f} : \mathcal{N}(ei)(R) \longrightarrow \mathcal{N}(e(i + f_E)(R_{[\tau_E]}).$$

Here on the left hand side appears the iterate of the Weil descent datum $\omega_{\mathcal{N}} : \mathcal{N}(i) \longrightarrow \mathcal{N}(i{+}1)^{(\tau)}$ of $\tilde{\mathcal{N}}$ relative to $O_{\breve{F}}/O_F$ from (5.2.5) and the translation functor $\Pi : \mathcal{N}(i) \longrightarrow \mathcal{N}(i+1)$, cf. (5.2.4). This defines a Weil descent datum relative to $O_{\breve{E}}/O_E$,

$$\Pi^{(d-1)f_E/f} \omega_{\mathcal{N}}^{f_E/f} : \tilde{\mathcal{N}}[e] \longrightarrow \tilde{\mathcal{N}}[e]^{(\tau_E)}. \tag{6.1.7}$$

We define

$$\begin{aligned} J' &= \{\alpha \in \operatorname{Aut}^o_K \mathbb{X} \mid \alpha^*\lambda_{\mathbb{X}} = u\lambda_{\mathbb{X}}, \text{ for } u \in p^{\mathbb{Z}} O_F^\times\} \\ &= \{\alpha \in \operatorname{Aut}^o_K \mathbb{Y} \mid \alpha^*\lambda_{\mathbb{Y}} = u\lambda_{\mathbb{Y}}, \text{ for } u \in p^{\mathbb{Z}} O_F^\times\}. \end{aligned} \tag{6.1.8}$$

The last equation holds because of the contraction functor. This group acts via the rigidifications ρ on the functors $\tilde{\mathcal{M}}_r$ and $\tilde{\mathcal{N}}[e]$. By the last equation of (6.1.8), we may regard J' as a subgroup of $J^{*\mathrm{r}}$ of Lemma 5.2.2.

Proposition 6.1.2. *There is an isomorphism of formal schemes over* $\operatorname{Spf} O_{\breve{E}}$

$$\tilde{\mathcal{M}}_r \longrightarrow \tilde{\mathcal{N}}[e] \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spf} O_{\breve{E}},$$

where the right hand side denotes the base change via $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$. *This isomorphism is compatible with the action of* J' *on both sides.*

Under the isomorphism the Weil descent datum (6.1.4) *on the left hand side corresponds to the Weil descent datum*

$$\Pi^{(d-1)f_E/f} \omega_{\mathcal{N}}^{f_E/f} : \tilde{\mathcal{N}}[e] \longrightarrow \tilde{\mathcal{N}}[e]^{(\tau_E)}$$

on the right hand side. More explicitly, for any $i \in \mathbb{Z}$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_r(i) & \xrightarrow{\sim} & \mathcal{N}(ei)_{O_{\breve{E}}} \\ \downarrow{\scriptstyle \omega_{\mathcal{M}_r}} & & \downarrow{\scriptstyle \Pi^{(d-1)f_E/f}\omega_{\mathcal{N}}^{f_E/f}} \\ \mathcal{M}_r(i+f_E)^{(\tau_E)} & \xrightarrow{\sim} & \mathcal{N}(e(i+f_E))_{O_{\breve{E}}}^{(\tau_E)}. \end{array} \tag{6.1.9}$$

Proof. The isomorphism of formal schemes over $\operatorname{Spf} O_{\breve{E}}$ comes from (6.1.5). It remains to show that the diagram is commutative. Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$ with structure morphism $\varepsilon : O_{\breve{E}} \longrightarrow R$. Let $\bar{\varepsilon} : \bar{\kappa}_E \longrightarrow \bar{R} = R \otimes_{O_{\breve{E}}} \bar{\kappa}_E$ be the induced morphism. We start with a point $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(i)(R)$. If we apply $\omega_{\mathcal{M}_r}$, we change ρ to

$$\tilde{\rho} : X_{\bar{R}} \longrightarrow \bar{\varepsilon}_* \mathbb{X} \xrightarrow{F_{\mathbb{X},\tau_E}} \bar{\varepsilon}_*(\tau_E)_* \mathbb{X}.$$

The lower horizontal arrow in (6.1.9) applies to $\tilde{\rho}$ the contracting functor $\mathfrak{C}_{r,R}$, cf. Definition 4.4.2. We have $\mathbb{X}_{\mathrm{c}} = \mathbb{Y}$. Let $P_{\mathbb{X}}$ be the $\mathcal{W}(\kappa_{\bar{E}})$-Dieudonné module of $\mathbb{X}$. Then the $\mathcal{W}_{O_F}(\kappa_{\bar{E}})$-Dieudonné module of $\mathbb{Y}$ is by definition the degree-zero component of $P'_{\mathbb{X}}$ defined by (4.3.21), comp. Remark 4.4.4. From this definition we obtain

$$(V')^f = \Pi^{1-d} V^f.$$

In terms of Dieudonné modules, $F_{\mathbb{X},\tau_E}$ is given by

$$P_{\mathbb{X}} \longrightarrow W(\kappa_{\bar{E}}) \otimes_{W(\tau_E), W(\kappa_{\bar{E}})} P_{\mathbb{X}}, \quad x \longmapsto 1 \otimes V^{f_E} x.$$

In terms of the relative Dieudonné module, $F_{\mathbb{Y},\tau_E}$ is given by $(V')^{f_E}$. Therefore the contracting functor applied to $F_{\mathbb{X},\tau_E}$ gives

$$\iota_{\mathbb{Y}}(\Pi^{(d-1)f_E/f}) F_{\mathbb{Y},\tau_E} : \mathbb{Y} \longrightarrow (\tau_E)_* \mathbb{Y},$$

since $(V')^{f_E} = \Pi^{(1-d)f_E/f} V^{f_E}$. On the other hand $\omega_{\mathcal{N}}$ just multiplies ρ_{c} by $F_{\mathbb{Y},\tau_E}$. Therefore we obtain the commutativity of the diagram. □

Corollary 6.1.3. *Let ω_{τ_E} denote the action of τ_E on the formal scheme scheme $\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{E}}$ via the second factor. There exists an isomorphism of formal schemes over* $\operatorname{Spf} O_{\breve{E}}$

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} (\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{E}}) \times \mathbb{Z},$$

such that the Weil descent datum $\omega_{\mathcal{M}_r}$ induces on the right hand side the Weil descent datum

$$(\xi, i) \longmapsto (\omega_{\tau_E}(\xi), i + f_E).$$

In particular the formal scheme $\mathcal{M}_{K/F,r}$ over $\operatorname{Spf} O_{\breve{E}}$ is a p-adic formal scheme which has semi-stable reduction; hence it is also flat, with reduced special fiber.

Proof. We consider the isomorphism $\tilde{\mathcal{M}}_{\mathrm{Dr}} \longrightarrow \tilde{\mathcal{N}}$ of formal schemes over $\operatorname{Spf} O_{\breve{F}}$ from Theorem 5.2.3. It is compatible with translations and the Weil descent data on both sides. Combining this with the Drinfeld isomorphism (5.1.22), we obtain an isomorphism

$$(\widehat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} O_{\breve{F}}) \times \mathbb{Z}e \xrightarrow{\sim} \tilde{\mathcal{N}}[e]. \tag{6.1.10}$$

We consider on the right hand side the Weil descent datum $\Pi^{(d-1)f_E/f}\omega_{\mathcal{N}}^{f_E/f}$ which is a composite of an iterate of the translation functor and an iterate of the Weil descent datum $\omega_{\mathcal{N}}$. By (5.1.23) we see that $\Pi^{(d-1)f_E/f}\omega_{\mathcal{N}}^{f_E/f}$ induces on the left hand side of (6.1.10) the Weil descent datum

$$(\xi, ei) \longmapsto (\omega_{\tau_E}, e(i + f_E)).$$

The assertion about descent data follows by forgetting e. The last assertion follows from Corollary 5.2.6. □

6.2. **The case r special and K/F unramified.** Let $\varphi_0, \bar{\varphi}_0 \in \Phi$ be the special embeddings. Their restrictions to F are the same. We extend the resulting embedding $O_F \longrightarrow O_E$ to an embedding $O_{\breve{F}} \longrightarrow O_{\breve{E}}$. The two embeddings $\varphi_0, \bar{\varphi}_0$ then factor over the two O_F-algebra homomorphisms $\breve{\varphi}_0, \bar{\breve{\varphi}}_0 : O_K \longrightarrow O_{\breve{F}}$.

Let $(\mathbb{Y}, \iota_{\mathbb{Y}})$ be a special formal O_D-module over $\bar{\kappa}_F$. We endow the $\mathcal{W}_{O_F}(\bar{\kappa}_F)$-Dieudonné module $P_{\mathbb{Y}}$ of $\mathbb{Y}$ with the polarization θ, cf. (5.3.1). (One should note that we call now K what was F' in that section.) Then θ defines an isogeny to the Faltings dual,

$$\lambda_{\mathbb{Y}} : \mathbb{Y} \longrightarrow \mathbb{Y}^{\nabla}.$$

If we compose the action $\iota_{\mathbb{Y}}^{\nabla}$ of $(O_D)^{\mathrm{opp}}$ on $\mathbb{Y}^{\nabla}$ with the anti-involution $\dagger$ of (5.3.3), the isogeny $\lambda_{\mathbb{Y}}$ becomes an isogeny of special formal O_D-modules. We indicate this by rewriting the isogeny as

$$\lambda_{\mathbb{Y}} : \mathbb{Y} \longrightarrow \mathbb{Y}^{\Delta}. \tag{6.2.1}$$

In section 5.3 we used $(\mathcal{P}_{\mathbb{Y}}, \theta)$ to define the functor $\mathcal{N}(i)$. Together with the restriction to O_K of the action of O_D on $\mathcal{P}_{\mathbb{Y}}$, we obtain an object of $\mathfrak{dR}^{\mathrm{pol}}_{\bar{\kappa}_F}$, cf. Definition 4.4.10. By Theorem 4.4.11, this object is the image of an object in $\mathfrak{dP}^{\mathrm{ss,pol}}_{r,\bar{\kappa}_E}$. The latter is the display of an object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \in \mathfrak{P}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$, cf. Definition 4.1.2. The height of the polarization $\lambda_{\mathbb{X}}$ is $2f$, and the associated Rosati involution induces on O_K the conjugation over O_F.

We consider the functor $\mathcal{M}_r = \mathcal{M}_{K/F,r}$ of Definition 2.6.1 with the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as defined above. We can give an alternative description of that functor.

Proposition 6.2.1. *Let S be a scheme over $\mathrm{Spf}\, O_{\breve{E}}$. A point of $\mathcal{M}_r(S)$ consists of*

(1) *a local CM-pair (X, ι) of CM-type r over S which satisfies the Eisenstein conditions* (EC_r) *relative to the fixed uniformizer π of F.*

(2) *an isogeny of height $2f$ of p-divisible O_K-modules*

$$\lambda : X \longrightarrow X^{\wedge},$$

which is a polarization of X.

(3) *a quasi-isogeny of p-divisible O_K-modules*

$$\rho : X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \bar{R} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}_E} \mathrm{Spec}\, \bar{R},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $\lambda_{|X \times_S \mathrm{Spec}\, \bar{S}}$ by a scalar in $O_F^{\times}$, locally on $\bar{S}$, where $\bar{S} = S \otimes_{\mathrm{Spf}\, O_{\breve{E}}} \mathrm{Spec}\, \bar{\kappa}_E$.

Proof. We may assume that $S = \mathrm{Spec}\, R$. Let $(X_0, X_1, \lambda, \rho_X) \in \mathcal{M}_r(R)$ be a point as in Definition 2.6.1. We obtain a point (X, λ, ρ_X) as in the Proposition if we set $X := X_0$, keep ρ_X and redefine λ to be the composite

$$X_0 \longrightarrow X_1 \xrightarrow{\lambda} X_0^{\wedge}.$$

We note that ρ is automatically of height zero because by the last condition $\rho^*(\lambda_{\mathbb{X}})$ and $\lambda_{\mathbb{X}}$ have the same height $2f$.

Conversely, assume (X, λ, ρ_X) is as in the Proposition. Then we set $X_0 = X$ and $X_1 = X^{\wedge}$. By Corollary 4.2.8, X_1 satisfies the condition (EC_r) and, by Proposition 4.2.6, X_0 and X_1 satisfy (KC_r) . The polarization λ defines an isogeny $a : X_0 \longrightarrow X_1$ of p-divisible O_K-modules which has height $2f$. To the morphism induced by a on the displays we apply the contracting functor of Definition 4.4.2. We obtain an isogeny of $\mathcal{W}_{O_F}(R)$-displays $\alpha : \mathcal{P}_0 \longrightarrow \mathcal{P}_1$ of height 2. To this isogeny we may apply Proposition 5.3.7. We find an isogeny $\beta : \mathcal{P}_1 \longrightarrow \mathcal{P}_0$ such that $\beta \circ \alpha = \pi \mathrm{id}_{\mathcal{P}_0}$.

The existence of ρ guarantees that X_0, X_1 and a are defined over an $O_{\breve{E}}$-subalgebra $R' \subset R$ which is of finite type over $O_{\breve{E}}$. Therefore we may apply Theorem 4.4.3. It gives us a morphism $b : X_1 \longrightarrow X_0$ such that $b \circ a = \pi \mathrm{id}_{X_0}$. We see that X_0, X_1, ρ together with the defining isomorphism $X_1 \xrightarrow{\sim} X_0$ defines a point of the functor $\mathcal{M}_r$ of Definition 2.6.1. □

Definition 6.2.2. We define the functor $\mathcal{M}_r(i)$ in the same way as in Definition 6.2.1, but we replace the data (4) by the following

(4′) A quasi-isogeny of p-divisible O_K-modules

$$\rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \bar{S},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $p^i\lambda_{|X\times_S\bar{S}}$ by a scalar in $O_F^\times$, locally on $\bar{S}$.

As in the ramified case (6.1.2) we conclude that height $\rho = 2di$. We set

$$\tilde{\mathcal{M}}_r = \coprod_{i\in\mathbb{Z}} \mathcal{M}_r(i).$$

Let $R \in \mathrm{Nilp}_{O_{\breve{E}}}$. Exactly as in the ramified case we obtain a morphism

$$\omega_{\mathcal{M}_r} : \mathcal{M}_r(i)(R) \longrightarrow \mathcal{M}_r(i+f_E)(R_{[\tau_E]}), \tag{6.2.2}$$

where, as in the ramified case, $f_E = [\kappa_E : \mathbb{F}_p]$. With the notation used in the ramified case, it changes the datum ρ in point (4′) of Definition 6.2.2 to

$$\tilde{\rho} : (X_0)_{\bar{R}} \xrightarrow{\rho} \bar{\varepsilon}_*\mathbb{X} \xrightarrow{F_{\mathbb{X},\tau_E}} \bar{\varepsilon}_*(\tau_E)_*\mathbb{X}.$$

From (6.2.2) we obtain the Weil descent datum,

$$\omega_{\mathcal{M}_r} : \tilde{\mathcal{M}}_r \longrightarrow \tilde{\mathcal{M}}_r^{(\tau_E)}. \tag{6.2.3}$$

We define an isomorphism of functors on $\mathrm{Nilp}_{O_{\breve{E}}}$,

$$\mathcal{M}_r(i) \xrightarrow{\sim} \mathcal{N}(ei). \tag{6.2.4}$$

Let $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(i)(R)$. Applying the contracting functor $\mathfrak{C}^{\mathrm{pol}}_{r,R}$, we obtain a quadruple $(X_{\mathrm{c}}, \iota_{\mathrm{c}}, \lambda_{X_{\mathrm{c}}}, \rho_{\mathrm{c}})$, where $\rho_{\mathrm{c}} : X_{\mathrm{c}}\otimes_R, \bar{R} \longrightarrow \mathbb{Y} \otimes_{\bar{\kappa}_E} \bar{R}$. This gives a point of $\mathcal{N}(i)(R)$. The functor $\mathfrak{C}^{\mathrm{pol}}_{r,R}$ is an equivalence of categories if the ideal of nilpotent elements of R is nilpotent, cf. Theorem 4.4.11.Therefore we may reverse the construction of (6.2.4). Therefore $\mathcal{M}_r(i)(R) \longrightarrow \mathcal{N}(ei)(R)$ is bijective for those R. For a general R we obtain the bijectivity as in the proof of Proposition 6.2.1. With the notation (6.1.6) we have a bijection $\tilde{\mathcal{M}}_r(R) \xrightarrow{\sim} \tilde{\mathcal{N}}[e](R)$.

We define

$$\begin{aligned} J' = &\quad \{\alpha \in \mathrm{Aut}^o_K \mathbb{X} \mid \alpha^*\lambda_{\mathbb{X}} = u\lambda_{\mathbb{X}},\ \text{for } u \in p^{\mathbb{Z}}O_F^\times\} \\ = &\quad \{\alpha \in \mathrm{Aut}^o_K \mathbb{Y} \mid \alpha^*\lambda_{\mathbb{Y}} = u\lambda_{\mathbb{Y}},\ \text{for } u \in p^{\mathbb{Z}}O_F^\times\}. \end{aligned} \tag{6.2.5}$$

The last equation holds because of the contraction functor. This group acts via the rigidifications ρ on the functors $\tilde{\mathcal{M}}_r$ and $\tilde{\mathcal{N}}[e]$. By the last equation of (6.2.5) we may regard J' as a subgroup of $J^{*\mathrm{ur}}$ of Lemma 5.3.2.

Proposition 6.2.3. *There exists an isomorphism of formal schemes* $\operatorname{Spf} O_{\breve{E}}$

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} \tilde{\mathcal{N}}[e] \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spf} O_{\breve{E}},$$

where the right hand side denotes the base change via $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$*. This isomorphism is compatible with the action of* J' *on both sides, and the Weil descent datum* (6.1.4) *on the left hand side corresponds to the Weil descent datum*

$$\pi^g\omega_{\mathcal{N}}^{f_E/f} : \tilde{\mathcal{N}}[e] \longrightarrow \tilde{\mathcal{N}}[e]^{(\tau_E)}.$$

on the right hand side. Here $g = f_E(d-1)/2f$*. More explicitly, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_r(i) & \xrightarrow{\quad\sim\quad} & \mathcal{N}_{O_{\breve{E}}}(ei) \\ {\scriptstyle \omega_{\mathcal{M}_r}}\downarrow & & \downarrow{\scriptstyle \pi^g\omega_{\mathcal{N}}^{f_E/f}} \\ \mathcal{M}_r(i+f_E)^{(\tau_E)} & \xrightarrow{\quad\sim\quad} & \mathcal{N}_{O_{\breve{E}}}(e(i+f_E))^{(\tau_E)}. \end{array} \tag{6.2.6}$$

The multiplication by π *is the morphism* $\mathcal{N}(j) \to \mathcal{N}(j+2)$ *which is obtained by multiplying* ρ *in Definition 5.3.1 by* π*. Equivalently one can apply two times the translation functor (5.3.5).*

Proof. We have already proved the isomorphism of formal schemes over $\operatorname{Spf} O_{\breve{E}}$. The compatibility with the Weil descent datum follows from the following lemma. ☐

Lemma 6.2.4. *Let $\mathbb{X}$ be the framing object over $\bar{\kappa}_E$, with corresponding Frobenius morphism $F_{\mathbb{X},\kappa_E} : \mathbb{X} \longrightarrow (\tau_E)_* \mathbb{X}$. Let $\mathbb{Y}$ be its image under the contracting functor $\mathfrak{C}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$. Then the image of $F_{\mathbb{X},\kappa_E}$ under $\mathfrak{C}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$ is given as*

$$\pi^g F_{\mathbb{Y},\kappa_E} : \mathbb{Y} \longrightarrow (\tau_E)_* \mathbb{Y}.$$

Proof. Let $M = P_{\mathbb{X}}$ be the $\mathcal{W}(\bar{\kappa}_E)$-Dieudonné module of $\mathbb{X}$. Then the $\mathcal{W}_{O_F}(\kappa_{\breve{E}})$-Dieudonné module M' of $\mathbb{Y}$ is by definition the ψ_0-component of $P'_{\mathbb{X}}$ defined by (4.3.18). In terms of Dieudonné modules, $F_{\mathbb{X},\tau_E}$ is given by

$$M \longrightarrow W(\kappa_{\breve{E}}) \otimes_{\tau_E, W(\kappa_{\breve{E}})} M, \quad x \longmapsto 1 \otimes V^{f_E} x.$$

This induces

$$V^{f_E} : M' \longrightarrow W(\bar{\kappa}_E) \otimes_{\tau_E, W(\bar{\kappa}_E)} M'. \tag{6.2.7}$$

We consider the decomposition

$$M = \oplus M_\psi.$$

We denote by σ the Frobenius on the Witt ring $W(\bar{\kappa}_E)$ and also the Frobenius of $K^t/\mathbb{Q}_p$. We note that

$$W(\bar{\kappa}_E) \otimes_{\tau_E, W(\bar{\kappa}_E)} M_\psi = (W(\bar{\kappa}_E) \otimes_{\tau_E, W(\bar{\kappa}_E)} M)_{\tau_E \psi} = (W(\bar{\kappa}_E) \otimes_{\tau_E, W(\bar{\kappa}_E)} M)_{\psi \sigma^{f_E}}.$$

In terms of the relative Dieudonné module, $F_{\mathbb{Y},\tau_E}$ is given by $(V')^{f_E}$. Our problem is to express (6.2.7) in terms of V'. By definition we have

$$V' = \pi^{-a_\psi} V : M_{\psi\sigma} \longrightarrow M_\psi.$$

We consider

$$V^{f_E} : M_\psi \xrightarrow{V} M_{\psi\sigma^{-1}} \xrightarrow{V} \ldots \xrightarrow{V} M_{\psi\sigma^{-f_E}}.$$

Therefore we obtain for this map

$$V^{f_E} = \pi^{a_{\psi\sigma^{-1}}} \cdot \ldots \cdot \pi^{a_{\psi\sigma^{-f_E}}} (V')^{f_E}.$$

By definition of the reflex field E, we have $a_\psi = a_{\psi\sigma^{-f_E}}$. This implies that the number

$$g = a_{\psi\sigma^{-1}} + \ldots + a_{\psi\sigma^{-f_E}}$$

is independent of ψ. We add to the last equation

$$g = a_{\bar{\psi}\sigma^{-1}} + \ldots + a_{\bar{\psi}\sigma^{-f_E}}.$$

Since $a_\psi + a_{\bar{\psi}}$ is e or $e-1$ we obtain

$$2g = \frac{f_E}{f} \cdot (d-1).$$

This proves the Lemma. □

Corollary 6.2.5. *Let ω_{τ_E} denote the action of τ_E on the formal scheme scheme $\widehat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{E}}$ via the second factor, i.e., in the notation of (5.1.23), $\omega_{\tau_E} = \omega_\tau^{f_E/f}$. There exists an isomorphism of formal schemes over* $\mathrm{Spf}\, O_{\breve{E}}$

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} (\widehat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{E}}) \times \mathbb{Z},$$

such that the Weil descent datum $\omega_{\mathcal{M}_r}$ induces on the right hand side the Weil descent datum

$$(\xi, i) \longmapsto (\omega_{\tau_E}(\xi), i + f_E).$$

In particular the formal scheme $\mathcal{M}_{K/F,r}$ over $\mathrm{Spf}\, O_{\breve{E}}$ is a p-adic formal scheme which has semi-stable reduction; hence it is also flat, with reduced special fiber.

Proof. We consider the isomorphisms of functors

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} \tilde{\mathcal{N}}_{O_{\breve{E}}}[e] \xleftarrow{\sim} (\tilde{\mathcal{M}}_{\mathrm{Dr}}[e])_{O_{\breve{E}}} \xrightarrow{\sim} (\widehat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{E}}) \times \mathbb{Z}e. \tag{6.2.8}$$

The last arrow is the isomorphism (5.1.22) and the left arrow in the middle is the isomorphism of Theorem 5.3.3. We must see what the Weil descent data $\pi^g \omega_{\mathcal{N}}^{f_E/f}$ on $\tilde{\mathcal{N}}_{O_{\breve{E}}}[e]$ does on the last functor. By Theorem 5.3.3, it induces on $(\tilde{\mathcal{M}}_{\mathrm{Dr}}[e])_{O_{\breve{E}}}$ the Weil descent datum $\omega_{\mathcal{M}_{\mathrm{Dr}}}^{f_E/f}$ multiplied

$2g$-times with the translation (cf. last sentence of Proposition 6.2.3). By (5.1.23), the induced Weil descent datum on the last functor of (6.2.8) is

$$(\xi, ie) \longmapsto (\omega_{\tau_E}, ie + (f_E/f) + 2g).$$

But we have

$$ie + (f_E/f) + 2g = ie + (f_E/f) + (d-1)(f_E/f) = ie + f_E e.$$

This proves the Corollary. □

6.3. **The case r banal and K/F ramified.** Let $\varepsilon \in \{\pm 1\}$. There is up to isomorphism a unique anti-hermitian K-vector space (V, ψ) of dimension 2 such that $\mathrm{inv}(V, \psi) = \varepsilon$, cf. Definition 8.1.1. Let $\Lambda \subset V$ be an O_K-lattice such that ψ induces a perfect pairing on Λ, cf. Lemma 8.1.3. By Theorem 4.5.11, (Λ, ψ) corresponds to a display $(\mathcal{P}, \iota, \beta) \in \mathfrak{dP}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$ over the residue class field of $O_{\breve{E}}$ which is unique up to isomorphism. Let $(\mathbb{X}, \iota_{\mathbb{X}}, \beta_{\mathbb{X}})$ be the corresponding polarized p-divisible O_K-module of type r. It is uniquely determined by the conditions that β is principal and that $\mathrm{inv}^r(\mathbb{X}, \iota_{\mathbb{X}}, \beta_{\mathbb{X}}) = \varepsilon$. Then $(C_{\mathbb{X}} \otimes \mathbb{Q}, \phi) \simeq (V, \psi)$.

Definition 6.3.1. We define a functor $\mathcal{M}_{r,\varepsilon}(i)$ on the category $\mathrm{Nilp}_{O_{\breve{E}}}$. For $R \in \mathrm{Nilp}_{O_{\breve{E}}}$, a point of $\mathcal{M}_{r,\varepsilon}(i)(R)$ is given by the following data:

(1) a local CM-pair (X, ι) of CM-type r over R which satisfies the Eisenstein conditions (EC_r) relative to a fixed uniformizer $\Pi \in K$.

(2) an isomorphism of p-divisible O_K-modules
$$\lambda : X \longrightarrow X^{\wedge},$$
which is a polarization of X.

(3) a quasi-isogeny of p-divisible O_K-modules
$$\rho : X_{\bar{R}} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}_E} \mathrm{Spec}\, \bar{R},$$
such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $p^i \lambda_{|X \times_S \bar{S}}$ by a scalar in $O_F^{\times}$, locally on $\mathrm{Spec}\, \bar{R}$. Here $\bar{R} = R \otimes_{O_{\breve{E}}} \bar{\kappa}_E$.

Two data $(X, \iota, \lambda, \rho)$ and $(X_1, \iota_1, \lambda_1, \rho_1)$ define the same point iff there is an isomorphism of O_K-modules $\alpha : X \to X_1$ such that $\rho_1 \circ \alpha_{\bar{R}} = \rho$. (This implies that α respects the polarizations up to a factor in $O_F^{\times}$).

By Proposition 4.2.2, $\mathcal{M}_{r,\varepsilon}(0)$ is the functor $\mathcal{M}_{K/F,r,\varepsilon}$ used in Theorem 2.6.3. We will also consider the functor
$$\tilde{\mathcal{M}}_{r,\varepsilon} = \coprod_{i \in \mathbb{Z}} \mathcal{M}_{r,\varepsilon}(i).$$

Let $i \in \mathbb{Z}$. We consider the following functor $\mathcal{G}_{\varepsilon}(i)$ on the category $\mathrm{Nilp}_{O_{\breve{E}}}$. A point of $\mathcal{G}_{\varepsilon}(i)(R)$ is given by the following data:

(1) a locally constant p-adic étale sheaf C on $\mathrm{Spec}\, R$ which is $\mathbb{Z}_p$-flat with $\mathrm{rank}_{\mathbb{Z}_p} C = 4d$ and with an action
$$\iota : O_K \longrightarrow \mathrm{End}_{\mathbb{Z}_p} C.$$

(2) a perfect alternating O_F-bilinear pairing
$$\phi : C \times C \longrightarrow O_F,$$
such that $\phi(\iota(a)c_1, c_2) = \phi(c_1, \iota(\bar{a})c_2)$ for $c_1, c_2 \in C$ and $a \in O_K$.

(3) a quasi-isogeny of O_K-module sheaves on $\mathrm{Spec}\, R$
$$\rho : (C, \iota) \longrightarrow (\underline{C}_{\mathbb{X}}, \iota) \tag{6.3.1}$$
such that locally on $\mathrm{Spec}\, R$ there is an $f \in O_F^{\times}$ with
$$p^i f \phi(c_1, c_2) = \phi_{\mathbb{X}}(\rho(c_1), \rho(c_2)).$$

Another set of data $(C', \iota', \lambda', \rho')$ defines the same point iff there is an isomorphism $\alpha : C \xrightarrow{\sim} C'$ such that $\rho' \circ \alpha = \rho$. Then α respects ϕ and ϕ' up to a factor in $O_F^\times$.

We remark that in (6.3.1) we regard $\underline{C}_{\mathbb{X}}$ as the constant sheaf on $\operatorname{Spec} R$. The existence of the quasi-isogeny implies that C is locally constant for the Zariski topology. Therefore locally on $\operatorname{Spec} R$ the sheaf C is the constant sheaf associated to an O_K-submodule $C \subset C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and ρ is given by the last inclusion.

The polarized contraction functor $\mathfrak{C}_r^{\mathrm{pol}}$ defines a morphism of functors

$$\mathcal{M}_{r,\varepsilon}(i) \longrightarrow \mathcal{G}_\varepsilon(i). \tag{6.3.2}$$

To describe the functor $\mathcal{G}_\varepsilon(i)$, we may restrict to the case where the sheaf C is given by an O_K-submodule of $C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then C defines a point of $\mathcal{G}_\varepsilon(i)(R)$ iff $(1/p^i)\phi_{\mathbb{X}}$ is a perfect alternating pairing on C. We define an algebraic group over $\mathbb{Z}_p$, and its $\mathbb{Z}_p$-rational points,

$$J'(\mathbb{Z}_p) = \{g \in \mathrm{GL}_{O_K}(C_{\mathbb{X}}) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = f \cdot \phi_{\mathbb{X}}(c_1, c_2) \text{ for some } f \in O_F^\times\}.$$

By Lemma 8.1.3, there is an isomorphism $g : (C_{\mathbb{X}}, \phi_{\mathbb{X}}) \longrightarrow (C, \frac{1}{p^i}\phi_{\mathbb{X}})$. This means that $gC_{\mathbb{X}} = C$ and

$$\phi_{\mathbb{X}}(gc_1, gc_2) = p^i \phi_{\mathbb{X}}(c_1, c_2), \quad c_1, c_2 \in C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{6.3.3}$$

We define

$$J'(i) = \{g \in \mathrm{GL}_K(C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = p^i f \phi_{\mathbb{X}}(c_1, c_2), \text{ for some } f \in O_F^\times\}.$$

This construction gives us a functor isomorphism

$$\mathcal{G}_\varepsilon(i) \xrightarrow{\sim} J'(i)/J'(\mathbb{Z}_p),$$

where the right hand side is considered as the restriction of the constant sheaf to $\mathrm{Nilp}_{O_{\breve{E}}}$. Let $J' \subset \mathrm{GL}(C_{\mathbb{X}})$ be the union of the $J'(i)$. Using the contraction functor we may write

$$J' = \{\alpha \in \mathrm{Aut}_K^o \mathbb{X} \mid \alpha^* \lambda_{\mathbb{X}} = \mu(\alpha)\lambda_{\mathbb{X}} \text{ for some } \mu(\alpha) \in p^{\mathbb{Z}} O_F^\times\}. \tag{6.3.4}$$

Therefore J' acts via ρ on the functor $\tilde{\mathcal{M}}_{r,\varepsilon}$.

Proposition 6.3.2. *The morphism of functors on* $\mathrm{Nilp}_{O_{\breve{E}}}$ *obtained from (6.3.2) is a* J'*-equivariant isomorphism,*

$$\tilde{\mathcal{M}}_{r,\varepsilon} \xrightarrow{\sim} J'/J'(\mathbb{Z}_p). \tag{6.3.5}$$

The left hand side is endowed with the Weil descent datum $\omega_{\mathcal{M}_r}$ *relative to* $O_{\breve{E}}/O_E$ *which is defined exactly in the same way as (6.1.4). This Weil descent datum corresponds on the right hand side to the Weil descent datum given by multiplication with* Π^{ef_E}. *Here we regard* Π^{ef_E} *as an automorphism of the* K*-vector space* $C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. We show that (6.3.2) is an isomorphism of functors. By the Grothendieck-Messing criterion both are formally étale, hence we may restrict to the category of $\bar{\kappa}_E$-algebras. Both functors commute with inductive limits of rings, i.e., they are locally of finite presentation. To see this, we consider the special fiber $\mathcal{M}_{r,\varepsilon,\bar{\kappa}_E}(0)$. Let $m \in \mathbb{N}$. We consider the subfunctor U_m which consists of points such that $p^m\rho^{-1}$ is an isogeny. Then X is the quotient of $\mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \operatorname{Spec} R$ by a finite locally free subgroup scheme of $\mathbb{X}(4dm) \times_{\operatorname{Spec} \bar{\kappa}_E} \operatorname{Spec} R$. (We have denoted by $\mathbb{X}(4dm)$ the kernel of the multiplication by p^{4dm}.) This shows that U_m is a scheme of finite presentation over $\operatorname{Spec} \bar{\kappa}_E$. Therefore U_m is locally of finite representation as a functor, cf. EGA IV, Thm. (8.8.2). One easily deduces from this that $\mathcal{M}_{r,\varepsilon,\bar{\kappa}_E}(0)$ is locally of finite presentation as a functor. In the same way we see that $\mathcal{G}_\varepsilon(i)$ is locally of finite presentation. To show that (6.3.2) is an isomorphism of functors we can therefore restrict to $\bar{\kappa}_E$-algebras R which are of finite type over $\bar{\kappa}_E$. For such R, (6.3.2) induces a bijection by Theorem 4.5.11.

It remains to compare the Weil descent data on both sides of (6.3.5). It is enough to make the comparison for the restriction of (6.3.5) to the category of $\bar{\kappa}_E$-algebras. Let $\varepsilon : \bar{\kappa}_E \longrightarrow R$ be a $\bar{\kappa}_E$-algebra. The restriction of the functor $\mathcal{M}_{r,\varepsilon}$ to $\bar{\kappa}_E$-algebras has a Weil descent datum $\omega_{\mathcal{M}_{r,\varepsilon},\mathbb{F}_p} : \mathcal{M}_{r,\varepsilon}(R) \longrightarrow \mathcal{M}_{r,\varepsilon}(R_{[\sigma]})$ over $\mathbb{F}_p$, given by

$$\omega_{\mathcal{M}_{r,\varepsilon},\mathbb{F}_p}((X, \iota, \lambda, \rho)) = (X, \iota, \lambda, \rho_{[\sigma]}), \tag{6.3.6}$$

where $\rho_{[\sigma]}$ is the composite

$$X \xrightarrow{\rho} \varepsilon_* \mathbb{X} \xrightarrow{\varepsilon_* F_{\mathbb{X}}} \varepsilon_* \sigma_* \mathbb{X}.$$

Here σ denotes the Frobenius automorphism of $\bar{\kappa}_E$ over $\mathbb{F}_p$. To see that this makes sense, we have to check that all p-divisible O_K-modules above satisfy the rank condition (RC_r) and the Eisenstein conditions (EC_r). The condition (EC_r) says that $\Pi^e \operatorname{Lie} X = 0$. This remains true if we regard X as a p-divisible O_K-module over $R_{[\sigma]}$. For the condition (RC_r), the claim is obvious.

Therefore it suffices to show that $\omega_{\mathcal{M}_r,\mathbb{F}_p}$ induces on the right hand side of (6.3.5) the Weil descent datum $g \mapsto \Pi^e g$. This follows from the following Lemma. □

Lemma 6.3.3. *There is an identification $C_{\mathbb{X}} = C_{\sigma_*\mathbb{X}}$. The functor $\mathfrak{C}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$ applied to the Frobenius morphism $F_{\mathbb{X}} : \mathbb{X} \longrightarrow \sigma_*\mathbb{X}$ yields $\Pi^e : C_{\mathbb{X}} \longrightarrow C_{\mathbb{X}}$.*

Proof. The first assertion follows because the functor $\mathfrak{C}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$ commutes with base change. Consider the Dieudonné module $P_{\mathbb{X}}$ of $\mathbb{X}$. We have

$$C_{\mathbb{X}} = \{c \in P_{\mathbb{X}} \mid Vc = \Pi^e c\},$$

cf. Remark 4.5.13. The map

$$P_{\mathbb{X}} \longrightarrow W(\bar{\kappa}_E) \otimes_{\sigma, W(\bar{\kappa}_E)} P_{\mathbb{X}}, \quad c \longmapsto 1 \otimes c$$

defines the identification $C_{\mathbb{X}} = C_{\sigma_*\mathbb{X}}$. The Frobenius $F_{\mathbb{X}}$ induces on the Dieudonné modules

$$P_{\mathbb{X}} \xrightarrow{V^\sharp} W(\bar{\kappa}_E) \otimes_{\sigma, W(\bar{\kappa}_E)} P_{\mathbb{X}}, \quad x \longmapsto 1 \otimes Vx.$$

For $c \in C_{\mathbb{X}}$ we obtain $V^\sharp c = 1 \otimes Vc = 1 \otimes \Pi^e c$. □

We can reformulate a part of Proposition 6.3.2 as follows. We consider the algebraic group over $\mathbb{Z}_p$ such that

$$J(\mathbb{Z}_p) = \{g \in \mathrm{GL}_{O_K}(C_{\mathbb{X}}) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = u \cdot \phi_{\mathbb{X}}(c_1, c_2) \text{ for some } u \in \mathbb{Z}_p^\times\}.$$

We define

$$J(i) = \{g \in \mathrm{GL}_K(C_{\mathbb{X}} \otimes \mathbb{Q}) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = up^i \cdot \phi_{\mathbb{X}}(c_1, c_2) \text{ for some } u \in \mathbb{Z}_p^\times\}.$$

The union of the $J(i)$ is the group $J(\mathbb{Q}_p)$ of unitary similitudes with similitude factor in $\mathbb{Q}_p^\times$.

Corollary 6.3.4. *Let $J(\mathbb{Q}_p)$ be the group of unitary similitudes of $C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with similitude factor in $\mathbb{Q}_p^\times$, and let $J(\mathbb{Z}_p)$ be its subgroup stabilizing the lattice $C_{\mathbb{X}}$. There are isomorphisms of functors on $\mathrm{Nilp}_{O_{\breve{E}}}$,*

$$\tilde{\mathcal{M}}_{r,\varepsilon} \xrightarrow{\sim} J(\mathbb{Q}_p)/J(\mathbb{Z}_p), \qquad \mathcal{M}_{K/F,r,\varepsilon} \xrightarrow{\sim} J(\mathbb{Q}_p)^o/J(\mathbb{Z}_p).$$

Here $J(\mathbb{Q}_p)^o$ denotes the group of unitary similitudes with similitude factor in $\mathbb{Z}_p^\times$.

Proof. It is enough to show that $\mathcal{G}_\varepsilon(i)(\bar{\kappa}_E)$ is in bijection with $J(i)/J'(\mathbb{Z}_p)$. For this is enough to show that for each $C \in \mathcal{G}_\varepsilon(i)(\bar{\kappa}_E)$ there exists $g \in J(i)$ such that $gC_{\mathbb{X}} = C$. This we have already shown before (6.3.3). □

In this reformulation it is less obvious what the Weil descent datum is.

6.4. **The case r banal and K/F unramified.** Let $\varepsilon \in \{\pm 1\}$. We consider a CM-triple $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ over $\bar{\kappa}_E$ such that $\lambda_{\mathbb{X}}$ is principal if $\varepsilon = 1$ and is almost principal if $\varepsilon = -1$. By Proposition 4.5.14 such a CM-triple exists and $\mathrm{inv}^r(\mathbb{X}, \iota, \lambda_{\mathbb{X}}) = \varepsilon$. In fact, by Lemma 8.1.2 and Theorem 4.5.11, $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ is unique up to isomorphism.

We recall the functor $\mathcal{M}_{K/F,r,\varepsilon}$ from section 2.6.

Definition 6.4.1. Let $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ be a framing object with an almost principal polarization. We define a functor $\mathcal{M}_{r^-}(i)$ on the category $\mathrm{Nilp}_{O_{\breve{E}}}$. For $R \in \mathrm{Nilp}_{O_{\breve{E}}}$, a point of $\mathcal{M}_{r^-}(i)(R)$ is given by the following data:

(1) a local CM-triple (X, ι, λ) of type r over R which satisfies the Eisenstein conditions (EC_r) relative to the fixed uniformizer $\pi \in K$.

(2) the polarization λ is almost principal.

(3) a quasi-isogeny of p-divisible O_K-modules
$$\rho : X_{\bar{R}} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{\kappa}_E} \operatorname{Spec} \bar{R},$$
such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $p^i \lambda_{|X_{\bar{R}}}$ by a scalar in $O_F^\times$, locally on $\operatorname{Spec} \bar{R}$. Here $\bar{R} = R \otimes_{O_{\breve{E}}} \bar{\kappa}_E$.

Now let $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ be a framing object with a principal polarization. Then we have $\mathtt{h}(\lambda_{\mathbb{X}}) = \mathtt{0}$ and $\operatorname{inv}^r(\mathbb{X}, \iota, \lambda_{\mathbb{X}}) = 1$. We define the functor $\mathcal{M}_{r^+}(i)$ by exactly the same data but we replace the condition (2) above by the condition

(2′) the polarization λ is principal.

In the almost principal case there exists an isogeny $X^\wedge \longrightarrow X$ such that the composite
$$X \xrightarrow{\lambda} X^\wedge \longrightarrow X$$
is $\pi \mathrm{id}_X$. This follows from the following analogue of Proposition 5.3.7.

Proposition 6.4.2. *Let $\alpha : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an isogeny of CM-pairs of type r over $R \in \operatorname{Nilp}_{O\breve{E}}$ which both satisfy the Eisenstein condition* (EC_r). *Let $\alpha_C : C_1 \longrightarrow C_2$ be the morphism in $\operatorname{Et}(O_K)_R$ associated by the contracting functor $\mathfrak{C}_{r,R}$. Then*
$$\operatorname{height} \alpha = 2f \cdot \operatorname{length}_{O_K} \operatorname{Coker} \alpha_C.$$
If $\operatorname{height} \alpha = 2f$ *then there exists a unique morphism $\beta : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$ such that*
$$\alpha \circ \beta = \pi \mathrm{id}_{\mathcal{P}_2}, \quad \beta \circ \alpha = \pi \mathrm{id}_{\mathcal{P}_1},$$

Proof. To prove the first assertion we can assume that $R = k$ is an algebraically closed field. Then we can use that $\operatorname{Coker} \alpha = \operatorname{Coker} \alpha_C \otimes_{\mathbb{Z}_p} W(k)$. If $\operatorname{length}_{O_K} \operatorname{Coker} \alpha_C = 1$, then β_C clearly exists. □

In the case where $i = 0$, the quasi-isogeny ρ is of height zero because the polarizations $\lambda_{\mathbb{X}}$ and λ have the same degree. We set $X_0 = X$ and $X_1 = X^\wedge$. Since X and $X^\wedge$ satisfy the Eisenstein condition (EC_r) and, by Proposition 4.2.2, also the Kottwitz condition (KC_r), we obtain a point of the functor $\mathcal{M}_{K/F,r,-1}$ of Definition 2.6.1. We conclude that $\mathcal{M}_{r^-}(0) = \mathcal{M}_{K/F,r,1}$. The index r^- on the left hand side indicates that we are in the case where the adjusted invariant of the framing object is -1. Similarly, $\mathcal{M}_{r^+}(0) = \mathcal{M}_{K/F,r,1}$. The index r^+ on the left hand side indicates that we are in the case where the adjusted invariant of the framing object is 1.

We will describe the formal scheme which represents the functor $\mathcal{M}_{K/F,r,\varepsilon}$. More precisely, consider the functors on $\operatorname{Nilp}_{O_{\breve{E}}}$,
$$\tilde{\mathcal{M}}_{r^\pm} = \coprod_{i \in \mathbb{Z}} \mathcal{M}_{r^\pm}(i).$$
These functors are endowed with a Weil descent datum $\omega_{\mathcal{M}_{r^\pm}} : \mathcal{M}_{r^\pm}(i) \longrightarrow \mathcal{M}_{r^\pm}(i + f_E)^{\tau_E}$ relative to $\breve{E}/E$ using exactly the same definition as in (6.1.4). Recall $(C_{\mathbb{X}}, \phi_{\mathbb{X}}) = (C_{\mathbb{X}}, \iota_{\mathbb{X}}, \phi_{\mathbb{X}})$. We define
$$J'(\mathbb{Z}_p) = \{g \in \operatorname{Aut}_{O_K} C_{\mathbb{X}} \mid \phi_{\mathbb{X}}(gc_1, gc_2) = f \cdot \phi_{\mathbb{X}}(c_1, c_2) \text{ for some } f \in O_F^\times\},$$
$$J'(i) = \{g \in \operatorname{Aut} C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \mid \phi_{\mathbb{X}}(gc_1, gc_2) = p^i f \phi_{\mathbb{X}}(c_1, c_2), \text{ for some } f \in O_F^\times\}.$$

As in the ramified case, we see that a point of $\mathcal{M}_{r^\pm}(i)(R)$ is locally on $\operatorname{Spec} R$ given by a lattice $C \subset C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that the restriction of $(1/p^i)\phi_{\mathbb{X}}$ to C induces a O_F-bilinear form
$$\frac{1}{p^i} \phi_{\mathbb{X}} : C \times C \longrightarrow O_F$$
which is perfect in the case of r^+ and such that $\operatorname{ord}_\pi \frac{1}{p^i} \phi_{\mathbb{X}} = 1$ in the case of r^-. For the case of r^-, we are using here Proposition 6.4.2.

We deduce that there is an isometry $g : (C_{\mathbb{X}}, \phi_{\mathbb{X}}) \longrightarrow (C, (1/p^i)\phi_{\mathbb{X}})$, cf. Lemma 8.1.2. Consequently we have $g \in J'(i)$. Conversely, if $g \in J'(i)$, the sublattice $C = gC_{\mathbb{X}}$ with the bilinear form $(1/p^i)\phi_{\mathbb{X}}$ gives rise to a point of $\mathcal{M}_{r^\pm}(i)(R)$.

We will denote by $J' \subset \operatorname{Aut} C_{\mathbb{X}}$ the union of the $J'(i)$. We can identify J' with a subgroup of $\operatorname{Aut}^o_K \mathbb{X}$ exactly as in the ramified case, cf. (6.3.4). It acts via ρ on the functor $\tilde{\mathcal{M}}_{r^\pm}$.

Proposition 6.4.3. *There is a J'-equivariant isomorphism of functors on* $\mathrm{Nilp}_{O_{\breve{E}}}$,

$$\tilde{\mathcal{M}}_{r^\pm} \xrightarrow{\sim} J'/J'(\mathbb{Z}_p). \tag{6.4.1}$$

Here the right hand side is the constant sheaf on $\mathrm{Nilp}_{O_{\breve{E}}}$. *The Weil descent datum* $\omega_{\mathcal{M}_{r,\pm}}$ *on the left hand side corresponds on the right hand side to the Weil descent datum given by multiplication with* $\pi^{ef_E/2}$. *Here we view* $\pi^{ef_E/2}$ *as an automorphism of the* K*-vector space* $C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *by multiplication.*

Proof. That (6.4.1) is an isomorphism of functors follows from Theorem 4.5.11 in the same way as in the proof of Proposition 6.3.2.

Let us recall the definition of the Weil descent relative to $O_{\breve{E}}/O_E$ on the functor $\tilde{\mathcal{M}}_{r^\pm}$. We write

$$F_{\mathbb{X},\tau_E} : \mathbb{X} \longrightarrow (\tau_E)_*\mathbb{X} \tag{6.4.2}$$

for the Frobenius relative to κ_E. Let $\varepsilon : O_{\breve{E}} \longrightarrow R$ be an object of $\mathrm{Nilp}_{O_{\breve{E}}}$. We write $\bar{\varepsilon} = \varepsilon \otimes \bar{\kappa}_E : \bar{\kappa}_E \longrightarrow \bar{R}$. Let $(X, \iota, \lambda, \rho) \in \mathcal{M}_{r^\pm}(i)(R)$ be a point. We view (X, ι, λ) as a CM-triple on $R_{[\tau_E]}$ and we endow it with the framing

$$\tilde{\rho} : X_{\bar{R}} \xrightarrow{\rho} \bar{\varepsilon}_*\mathbb{X} \xrightarrow{\bar{\varepsilon}_* F_{\mathbb{X},\tau_E}} \bar{\varepsilon}_*(\tau_E)_*\mathbb{X}.$$

Then $(X, \iota, \lambda, \tilde{\rho})$ defines a point of $\mathcal{M}_{r^\pm}(i + f_E)(R_{[\tau_E]})$. Varying $i \in \mathbb{Z}$, we obtain the Weil descent datum

$$\omega_{\mathcal{M}_{r^\pm}} : \tilde{\mathcal{M}}_{r^\pm} \longrightarrow \tilde{\mathcal{M}}_{r^\pm}^{(\tau_E)}.$$

We note that the inverse image of $(\tau_E)_*\lambda$ by (6.4.2) is $p^{f_E}\lambda$. The compatibility of the Weil descent data follows as in the proof of Proposition 6.3.2 from the following Lemma. □

Lemma 6.4.4. *The contracting functor applied to the Frobenius morphism* $F_{\mathbb{X},\tau_E} : \mathbb{X} \longrightarrow (\tau_E)_*\mathbb{X}$ *yields the multiplication by* $\pi^{ef_E/2} : C_{\mathbb{X}} \longrightarrow C_{\mathbb{X}}$.

Proof. We use the Dieudonné module P of $\mathbb{X}$ over $\bar{\kappa}_E$. The map $F_{\mathbb{X},\tau_E}$ induces on the Dieudonné modules the map

$$V^{f_E,\sharp} : P \longrightarrow W(\bar{\kappa}_E) \otimes_{F^{f_E},\bar{\kappa}_E} P, \quad x \longmapsto 1 \otimes V^{f_E}x.$$

By definition we have

$$C := C_{\mathbb{X}} = \{c \in P \mid Vc = \pi_r c\},$$

where we recall π_r from (4.5.12), cf. Remark 4.5.13. With the identification $C_{\mathbb{X}} = C_{(\tau_E)_*\mathbb{X}}$, the restriction of $V^{f_E,\sharp}$ to C gives

$$^{F^{-f_E+1}}\pi_r \cdot \ldots \cdot {}^{F^{-1}}\pi_r \cdot \pi_r : C \longrightarrow C. \tag{6.4.3}$$

The right hand side is a module over

$$O_K \otimes_{\mathbb{Z}_p} W(\bar{\kappa}_E) = \prod_{\psi \in \Psi} O_K \otimes_{O_{K^t},\tilde{\psi}} W(\bar{\kappa}_E).$$

On the right hand side, F^{-1} is given by the map

$$O_K \otimes_{O_{K^t},\tilde{\psi\sigma}} W(\bar{\kappa}_E) \longrightarrow O_K \otimes_{O_{K^t},\tilde{\psi}} W(\bar{\kappa}_E), \quad a \otimes \xi \longmapsto a \otimes {}^{F^{-1}}\xi.$$

Therefore the components of the element on the left hand side of (6.4.3) are

$$\pi^{a_{\psi\sigma^{(f_E-1)}}} \cdot \ldots \cdot \pi^{a_{\psi\sigma}} \cdot \pi^{a_\psi}.$$

Since σ^{f_E} fixes κ_E we have $a_{\psi\sigma^{f_E}} = a_\psi$. It follows that the numbers

$$g_\psi := a_{\psi\sigma^{(f_E-1)}} + \ldots a_{\psi\sigma} + a_\psi$$

are independent of ψ. We call this number g. We find:

$$2g = g_\psi + g_{\bar{\psi}} = ef_E$$

because $a_\psi + a_{\bar{\psi}} = e$. We conclude that (6.4.3) is the multiplication by $\pi^{ef_E/2}$. □

Corollary 6.4.5. *Let $J(\mathbb{Q}_p)$ be the group of unitary similitudes of $C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with similitude factor in $\mathbb{Q}_p^{\times}$, and let $J(\mathbb{Z}_p)$ be its subgroup stabilizing the lattice $C_{\mathbb{X}}$. There are isomorphisms of functors on* $\mathrm{Nilp}_{O_{\breve{E}}}$,

$$\tilde{\mathcal{M}}_{r,\varepsilon} \xrightarrow{\sim} J(\mathbb{Q}_p)/J(\mathbb{Z}_p), \qquad \mathcal{M}_{K/F,r,\varepsilon} \xrightarrow{\sim} J(\mathbb{Q}_p)^o/J(\mathbb{Z}_p).$$

Here $J(\mathbb{Q}_p)^o$ denotes the group of unitary similitudes with similitude factor in $\mathbb{Z}_p^{\times}$. □

6.5. **The banal split case.** Let $R = k$ be an algebraically closed field. There is up to isomorphism a unique hermitian O_K-module (C, ϕ) of rank 2 with ϕ perfect. Hence there is a unique CM-triple

$$(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \in \mathfrak{P}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$$

with principal $\lambda_{\mathbb{X}}$. We take this as framing object.

We define functors $\mathcal{M}_r(i)$ on the category $\mathrm{Nilp}_{O_{\breve{E}}}$. For $R \in \mathrm{Nilp}_{O_{\breve{E}}}$, a point of $\mathcal{M}_r(i)(R)$ is given by the following data:

(1) a local CM-triple (X, ι, λ) of type r over R which satisfies the Eisenstein conditions (EC_r) relative to the fixed uniformizer $\pi \in F$.

(2) the polarization $\lambda \colon X \longrightarrow X^\wedge$ is principal.

(3) a quasi-isogeny of p-divisible O_K-modules

$$\rho : X_{\bar{R}} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}_E} \mathrm{Spec}\, \bar{R},$$

such that the pullback quasi-isogeny $\rho^*(\lambda_{\mathbb{X}})$ differs from $p^i \lambda_{|X_{\bar{R}}}$ by a scalar in $O_F^{\times}$, locally on $\mathrm{Spec}\, \bar{R}$. Here $\bar{R} = R \otimes_{O_{\breve{E}}} \bar{\kappa}_E$.

Note that $\mathcal{M}_r(0) = \mathcal{M}_{r,1}$ of before Theorem 2.6.3. Consider a point $(X, \iota, \lambda, \rho)$ as above. Let (C, ϕ) be the p-adic étale sheaf associated to $(X, \iota, \lambda)_{\bar{R}}$ on $(\mathrm{Spec}\, \bar{R})_{\text{ét}} = (\mathrm{Spec}\, R)_{\text{ét}}$. Let $\underline{C}_{\mathbb{X}}$ be the constant sheaf on $(\mathrm{Spec}\, R)_{\text{ét}}$ of the O_K-module $C_{\mathbb{X}}$. The existence of ρ implies that C is locally constant for the Zariski topology. Therefore, locally on $\mathrm{Spec}\, R$, we may regard C as a submodule of $C_{\mathbb{X}} \otimes \mathbb{Q}$. By the definition of $\mathcal{M}_r(i)$, we have

$$f p^i \phi(x, y) = \phi_{\mathbb{X}}(x, y), \quad x, y \in C_{\mathbb{X}} \otimes \mathbb{Q},$$

for some $f \in O_F^{\times}$. We see by Theorem 4.5.11 that a point of $\mathcal{M}_r(i)(R)$ is the same as a O_K-sublattice $C \subset (C_{\mathbb{X}})_R \otimes \mathbb{Q}$ such that the restriction of $(1/p^i)\phi_{\mathbb{X}}$ to C induces a perfect pairing

$$C \times C \longrightarrow O_F.$$

This is directly clear if the ideal of nilpotent elements of R is nilpotent, and follows from the argument in the proof of Proposition 6.3.2 in the general case.

Again we set

$$J'(\mathbb{Z}_p) = \{g \in \mathrm{GL}_{O_K}(C_{\mathbb{X}}) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = f \cdot \phi_{\mathbb{X}}(c_1, c_2), \text{ for some } f \in O_F^{\times}\},$$
$$J'(i) = \{g \in \mathrm{GL}(C_{\mathbb{X}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \mid \phi_{\mathbb{X}}(gc_1, gc_2) = p^i f \phi_{\mathbb{X}}(c_1, c_2), \text{ for some } f \in O_F^{\times}\}.$$

There is an isometry up to a constant in $O_F^{\times}$,

$$g : (C_{\mathbb{X}}, \phi_{\mathbb{X}}) \longrightarrow (C, \frac{1}{p^i}\phi_{\mathbb{X}}).$$

Then $g \in J'(i)$ and $gC_{\mathbb{X}} = C$. Any other isometry of this type is of the form gh, with $h \in J'(\mathbb{Z}_p)$. Therefore we have associated to the point $(X, \iota, \lambda, \rho)$ a section of the constant sheaf $J'(i)/J'(\mathbb{Z}_p)$.

We set

$$\tilde{\mathcal{M}}_r = \coprod_{i \in \mathbb{Z}} \mathcal{M}_r(i), \quad J' = \cup_{i \in \mathbb{Z}} J'(i).$$

As in the ramified case, the group J' acts via ρ on the functor $\tilde{\mathcal{M}}_r$, cf. (6.3.4). Moreover, this functor is endowed with the Weil descent datum $\omega_{\mathcal{M}_r} : \mathcal{M}_r(i) \longrightarrow \mathcal{M}_r(i + f_E)^{(\tau_E)}$ relative to $O_{\breve{E}}/O_E$.

Let $\sigma \in \mathrm{Gal}(F^t/F)$ be the Frobenius. We use the notation introduced below (4.3.22). We fix $\theta \in \Theta$, with $\theta_1, \theta_2 \in \Psi$. Set

$$a_{1,E} = a_{\theta_1 \sigma^{f_E - 1}} + \ldots + a_{\theta_1 \sigma} + a_{\theta_1} \tag{6.5.1}$$

This number is independent of the choice of θ because, by the definition of the reflex field E,

$$a_{\theta_1\sigma^{f_E}} = a_{\theta_1}.$$

In the same way we define $a_{2,E}$ by using θ_2. If we add both definitions we find

$$a_{1,E} + a_{2,E} = ef_E.$$

The endomorphism

$$\pi^{a_E} := \pi^{a_{1,E}} \oplus \pi^{a_{2,E}} : C_{\mathbb{X},1} \oplus C_{\mathbb{X},2} \longrightarrow C_{\mathbb{X},1} \oplus C_{\mathbb{X},2} \tag{6.5.2}$$

is an element of $J'(f_E)$.

Proposition 6.5.1. *The polarized contraction functor defines an isomorphism of functors on* $\mathrm{Nilp}_{O_{\breve{E}}}$,

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} J'/J'(\mathbb{Z}_p). \tag{6.5.3}$$

The Weil descent datum $\omega_{\mathcal{M}_r}$ relative to $O_{\breve{E}}/O_E$ corresponds on the right hand side to the Weil descent datum given by multiplication with $\pi^{a_E} \in J'(f_E)$.

We note that $J'/J'(\mathbb{Z}_p) = J'/J'(\mathbb{Z}_p)^{(\tau_E)}$ because this is true for any constant sheaf. Proposition 6.5.1 is the consequence of the definition of $\omega_{\mathcal{M}_r}$ and the following Lemma.

Lemma 6.5.2. *The Frobenius $F_{\mathbb{X},\tau_E} : \mathbb{X} \longrightarrow (\tau_E)_*\mathbb{X}$ induces on $C_{\mathbb{X}}$ the multiplication by π^{a_E}.*

Proof. The statement needs an explanation. Because the functor $\mathfrak{C}^{\mathrm{pol}}_{r,\bar{\kappa}_E}$ commutes with base change, we have a canonical isomorphism $C_{\mathbb{X}} = C_{(\tau_E)_*\mathbb{X}}$. Indeed, the inverse image of the constant sheaf $C_{\mathbb{X}}$ by $\operatorname{Spec}\tau_E$ is the constant sheaf $C_{\mathbb{X}}$.

Let $M = P_{\mathbb{X}}$ be the Dieudonné module of $\mathbb{X}$. The Frobenius $F_{\mathbb{X},\tau_E}$ is induced by the Verschiebung

$$V^{f_E} : M \longrightarrow M.$$

We write in this proof $C := C_{\mathbb{X}}$. By definition we have

$$C = M^{\pi_r V^{-1}} = C_1 \oplus C_2,$$

cf. Remark 4.5.13. Therefore the action of V^{f_E} on C coincides with the action of

$${}^{F^{-f_E+1}}\pi_r \cdot \ldots \cdot {}^{F^{-1}}\pi_r \cdot \pi_r : C \longrightarrow C. \tag{6.5.4}$$

We look at the components of the element on the left hand side in (4.3.22). Let us consider the components of the first set of factors of (4.3.22) which act on C_1. The component of (6.5.4) in the factor $O_F \otimes_{O_{F^t},\tilde{\theta}} W(O_{E'})$ is

$$\pi^{a_{\theta_1\sigma^{f_E-1}}} \cdot \ldots \cdot \pi^{a_{\theta_1\sigma}} \cdot \pi^{a_{\theta_1}} \otimes 1 = \pi^{a_{1,E}} \otimes 1.$$

Therefore V^{f_E} induces on C_1 the multiplication by $\pi^{a_{1,E}}$. By the same argument it induces on C_2 the multiplication by $\pi^{a_{2,E}}$. □

Corollary 6.5.3. *Let $J(\mathbb{Q}_p) = \mathrm{GL}_K(C_{\mathbb{X},1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ and $J(\mathbb{Z}_p) = \mathrm{GL}_{O_K}(C_{\mathbb{X},1})$. There are isomorphisms of functors on* $\mathrm{Nilp}_{O_{\breve{E}}}$,

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} J(\mathbb{Q}_p)/J(\mathbb{Z}_p), \quad \mathcal{M}_{K/F,r,1} \xrightarrow{\sim} J(\mathbb{Q}_p)^o/J(\mathbb{Z}_p).$$

Here $J(\mathbb{Q}_p)^o$ denotes the subgroup of elements with determinant in $O_K^\times$. □

7. Application to p-adic uniformization

In this section, we reap the global fruits from our local work in the preceding sections. This section is modelled on the case of *p-adic uniformization of the first kind* of the previous paper [18, section 4].

7.1. **The Shimura variety and its p-integral model.** In this section, K and F will be number fields. Let K/F be a CM-field. We fix an archimedean place w_0 of F. We denote by $a \mapsto \bar{a}$ the conjugation acting on K.

Let V be a K-vector space of dimension 2. Let

$$\varsigma(\ ,\ ) : V \times V \longrightarrow \mathbb{Q}$$

be a non-degenerate alternating $\mathbb{Q}$-bilinear form such that

$$\varsigma(av_1, v_2) = \varsigma(v_1, \bar{a}v_2), \quad a \in K,\ v_1, v_2 \in V.$$

We define three algebraic groups over $\mathbb{Q}$. For a $\mathbb{Q}$-algebra R, the R-valued points are:

$$\begin{aligned} U(V,\varsigma)(R) &= \{g \in \mathrm{GL}_{K\otimes R}(V \otimes R) \mid \varsigma_R(gx_1, gx_2) = \varsigma_R(x_1, x_2)\} \\ G(V,\varsigma)(R) &= \{g \in \mathrm{GL}_{K\otimes R}(V \otimes R) \mid \varsigma_R(gv_1, gv_2) = \mu(g)\varsigma_R(v_1, v_2),\ \mu(g) \in R^\times\}, \\ \dot{G}(V,\varsigma)(R) &= \{g \in \mathrm{GL}_{K\otimes R}(V \otimes R) \mid \varsigma_R(gv_1, gv_2) = \varsigma_R(\mu(g)v_1, v_2),\ \mu(g) \in (F \otimes R)^\times\}. \end{aligned} \tag{7.1.1}$$

If (V, ς) is fixed, we write $U, G, \dot{G}$.

Equivalently, we can replace the form ς by the anti-hermitian form

$$\varkappa : V \times V \longrightarrow K$$

on the K-vector space V which is defined by the equation

$$\mathrm{Tr}_{K/\mathbb{Q}}\, a\varkappa(v_1, v_2) = \varsigma(av_1, v_2), \quad a \in K. \tag{7.1.2}$$

Then $\varkappa$ is linear in the first argument and anti-linear in the second.

For each place w of F we obtain an anti-hermitian pairing

$$\varkappa_w : V \otimes_F F_w \times V \otimes_F F_w \longrightarrow K \otimes_F F_w. \tag{7.1.3}$$

Let $w : F \longrightarrow \mathbb{R}$ be an archimedean place. We choose an extension of w to $\varphi : K \longrightarrow \mathbb{C}$. This defines an isomorphism $K \otimes_F F_w \cong \mathbb{C}$ and $\varkappa_w$ becomes an anti-hermitian pairing

$$\varkappa_\varphi : V \otimes_{K,\varphi} \mathbb{C} \times V \otimes_{K,\varphi} \mathbb{C} \longrightarrow \mathbb{C}.$$

Note that the space V is determined up to isomorphism by the signature at the archimedean places of F and the local invariants $\mathrm{inv}_v(V)$ at the non-archimedean places v of F. We will impose the following signature condition on V. Let $\Phi = \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(K, \mathbb{C})$ and let r be a special CM-type of rank 2 wrt. w_0, i.e., a function

$$r : \Phi \longrightarrow \mathbb{Z}_{\geqslant 0}, \qquad \varphi \longmapsto r_\varphi, \tag{7.1.4}$$

such that $r_\varphi + r_{\bar{\varphi}} = 2$ for all $\varphi \in \Phi$ and such that there is exactly one pair $\{\varphi_0, \bar{\varphi}_0\}$ such that $r_{\varphi_0} = r_{\bar{\varphi}_0} = 1$. The archimedean place determined by φ_0 and $\bar{\varphi}_0$ is supposed to be w_0.

We require that $\varkappa_\varphi$ is isomorphic to the anti-hermitian form on $\mathbb{C}^2$ given by the matrix

$$\begin{pmatrix} \mathbf{i}E_{r_\varphi} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i}E_{r_{\bar{\varphi}}} \end{pmatrix}, \tag{7.1.5}$$

for every φ. Here E_{r_φ} is denotes the unit matrix of size r_φ, and $\mathbf{i}$ the imaginary unit. We note that the last requirement is independent of the choice of φ above w. We endow

$$V \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{w : F \longrightarrow \mathbb{R}} V \otimes_{F,w} \mathbb{R},$$

with the complex structure $\mathcal{J}$ such that $\varkappa_w(v_1, \mathcal{J}v_2)$ is hermitian and positive-definite for all w. This defines a Shimura datum (G, h), resp., $(\dot{G}, h)$, cf. [8, 4.9, 4.13] and an associated Shimura variety $\mathrm{Sh}(G, h)$ with canonical model over the reflex field E of r.

Let p be a prime number. We choose a place v of F of residue characteristic p which does not split in K and such that $\mathrm{inv}_v(V) = -1$. We choose an embedding $E \longrightarrow \bar{\mathbb{Q}}_p$ and denote by ν the induced place of E, with corresponding prime ideal $\mathfrak{p}_\nu$. We assume that the place v_ν of F induced by

$$F \xrightarrow{\varphi_0} E \longrightarrow \bar{\mathbb{Q}}_p$$

is equal to v, i.e, that $\mathfrak{p}_\nu$ induces the prime ideal $\mathfrak{p}_v$ corresponding to v.

We have an isomorphism

$$V \otimes \mathbb{Q}_p \cong \oplus_{\mathfrak{p}|p} V \otimes_F F_{\mathfrak{p}}. \tag{7.1.6}$$

Here $\mathfrak{p}$ runs over all prime ideals of F which divide p. We will write

$$V_{\mathfrak{p}} := V \otimes_F F_{\mathfrak{p}}, \quad K_{\mathfrak{p}} = K \otimes_F F_{\mathfrak{p}}.$$

The decomposition (7.1.6) is orthogonal with respect to ς and, for each prime ideal $\mathfrak{p}$ of F over p, we obtain a bilinear form

$$\varsigma_{\mathfrak{p}} : V_{\mathfrak{p}} \times V_{\mathfrak{p}} \longrightarrow \mathbb{Q}_p.$$

It is related to $\varkappa_{\mathfrak{p}}$ by

$$\mathrm{Tr}_{K\otimes_F F_{\mathfrak{p}}/\mathbb{Q}_p} a\varkappa_{\mathfrak{p}}(x_1, x_2) = \varsigma_{\mathfrak{p}}(ax_1, x_2), \quad a \in K \otimes_F F_{\mathfrak{p}},\ x_1, x_2 \in V_{\mathfrak{p}}.$$

One defines algebraic groups over $\mathbb{Q}_p$ as in (7.1.1) above:

$$U_{\mathfrak{p}} = U(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}), \quad G_{\mathfrak{p}} = G(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}), \quad \dot{G}_{\mathfrak{p}} = \dot{G}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}).$$

Let $\mathfrak{p}|p$ be such that $\mathfrak{p}$ is unramified (and hence non-split) in K/F and such that $\mathrm{inv}(V_{\mathfrak{p}}, \varkappa_{\mathfrak{p}}) = -1$, cf. Definition 8.1.1. Then there is a $O_{K_{\mathfrak{p}}}$-lattice

$$\Lambda_{\mathfrak{p}} \subset V_{\mathfrak{p}}$$

such that $\varsigma_{\mathfrak{p}}$ induces a bilinear form

$$\varsigma_{\mathfrak{p}} : \Lambda_{\mathfrak{p}} \times \Lambda_{\mathfrak{p}} \longrightarrow \mathbb{Z}_p \tag{7.1.7}$$

such that $\Lambda_{\mathfrak{p}}$ is almost self-dual, i.e., $\mathtt{h}(\Lambda_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = 1$ (compare (8.1.4)). Any other lattice with these properties has the form $g\Lambda_{\mathfrak{p}}$ where $g \in \mathrm{U}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}})(\mathbb{Q}_p)$. This follows from Lemma 8.1.2.

In all other cases, we apply the following lemma.

Lemma 7.1.1. *Assume that if $\mathfrak{p}$ is unramified in K/F, then $\mathrm{inv}(V_{\mathfrak{p}}, \varkappa_{\mathfrak{p}}) = 1$. Then there is an $O_{K_{\mathfrak{p}}}$-lattice $\Lambda_{\mathfrak{p}} \subset V \otimes_F F_{\mathfrak{p}}$ such that $\varsigma_{\mathfrak{p}}$ induces a perfect pairing*

$$\varsigma_{\mathfrak{p}} : \Lambda_{\mathfrak{p}} \times \Lambda_{\mathfrak{p}} \longrightarrow \mathbb{Z}_p.$$

Any other such lattice is of the form $g\Lambda_{\mathfrak{p}}$ where $g \in U_{\mathfrak{p}}(\mathbb{Q}_p)$.

Proof. Indeed, because of Lemmas 8.1.2 and 8.1.3, we need only a justification in the case where $\mathfrak{p}$ is split. In this case we have $K_{\mathfrak{p}} = F_{\mathfrak{p}} \times F_{\mathfrak{p}}$ and, accordingly, a decomposition $V \otimes_F F_{\mathfrak{p}} = U_1 \oplus U_2$. The vector spaces U_1 and U_2 are isotropic with respect to $\varsigma_{\mathfrak{p}}$ and therefore $\varsigma_{\mathfrak{p}}$ induces an isomorphism $U_2 = \mathrm{Hom}_{\mathbb{Q}_p}(U_1, \mathbb{Q}_p)$. The form $\varsigma_{\mathfrak{p}}$ becomes

$$\varsigma_{\mathfrak{p}}(x + x^*, y + y^*) = x^*(y) - y^*(x), \quad x, y \in U_1,\ x^*, y^* \in U_2.$$

The existence and uniqueness of $\Lambda_{\mathfrak{p}}$ follows easily. □

To pass to a p-integral model over $O_{E,(\mathfrak{p}_\nu)}$ of $\mathrm{Sh}(G, h)$, we restrict the choice of the level structure. To do this, we choose for each $\mathfrak{p}|p$ a $O_{K\otimes_F F_{\mathfrak{p}}}$-lattice $\Lambda_{\mathfrak{p}} \subset V_{\mathfrak{p}}$ as above. We define

$$\begin{aligned} \mathbf{K}_{\mathfrak{p}} &= \{g \in G_{\mathfrak{p}} \mid g\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}\} \\ \mathbf{K}_p &= \{g \in G(\mathbb{Q}_p) \mid g\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \text{ for all } \mathfrak{p}|p\}. \end{aligned} \tag{7.1.8}$$

We choose an open compact subgroup $\mathbf{K}^p \in G(\mathbb{A}_f^p)$ and set $\mathbf{K} = \mathbf{K}_p \cdot \mathbf{K}^p \subset G(\mathbb{A}_f)$.

We extend the embedding $\nu\colon E \longrightarrow \bar{\mathbb{Q}}_p$ to an embedding $\bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$. We obtain a decomposition

$$\Phi = \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(K, \bar{\mathbb{Q}}) = \coprod_{\mathfrak{p}} \mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(K_{\mathfrak{p}}, \bar{\mathbb{Q}}_p) = \coprod_{\mathfrak{p}} \Phi_{\mathfrak{p}}. \tag{7.1.9}$$

The restriction of the function r to $\Phi_{\mathfrak{p}}$ will be denoted by $r_{\mathfrak{p}}$. The group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_\nu)$ acts on Φ via the restriction

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_\nu) \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/E).$$

Therefore $r_{\tau\varphi} = r_\varphi$ for $\varphi \in \Phi$ and $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/E_\nu)$. This implies that the local reflex fields $E(K_{\mathfrak{p}}/F_{\mathfrak{p}}, r_{\mathfrak{p}})$ are all contained in E_ν.

Let R be an O_{E_ν}-algebra. Let $\mathcal{L}$ be an R-module with an O_K-action. We have decompositions

$$O_K \otimes R = \prod_{\mathfrak{p}} O_{K_{\mathfrak{p}}} \otimes_{\mathbb{Z}_p} R, \quad \mathcal{L} = \oplus_{\mathfrak{p}} \mathcal{L}_{\mathfrak{p}}.$$

We will say that $\mathcal{L}$ satisfies the Eisenstein condition (EC_r) if each $\mathcal{L}_{\mathfrak{p}}$ satisfies the Eisenstein condition ($\mathrm{EC}_{r_{\mathfrak{p}}}$), cf. (2.2.12). We use a similar terminology for the Kottwitz condition (KC_r).

Definition 7.1.2. We define the groupoid $\mathcal{A}_{\mathbf{K}}$ on the category of $O_{E,(\mathfrak{p}_\nu)}$-algebras. A point of $\mathcal{A}_{\mathbf{K}}(R)$ consists of the following data:

(1) An abelian scheme A over R, up to isogeny of degree prime to p, with an algebra homomorphism

$$\iota : O_K \longrightarrow \operatorname{End} A \otimes \mathbb{Z}_{(p)}.$$

such that $\operatorname{Lie} A \otimes_{O_{E,(\mathfrak{p}_\nu)}} O_{E_\nu}$ satisfies the conditions (KC_r) and (EC_r).

(2) A $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}$ of A such that the Rosati involution induces the conjugation of K/F.

(3) A class of O_K-linear isomorphisms

$$\bar{\eta}^p : V \otimes \mathbb{A}_f^p \longrightarrow V^p(A) \bmod \mathbf{K}^p$$

which respect the forms on both sides up to a constant in $\mathbb{A}_f^p(1)^\times$.

We impose the following two conditions.

(i) There exists a polarization $\lambda \in \bar{\lambda}$ such that the induced map to the dual variety $\lambda : A \to A^\wedge$ has the following property. Let $(\ker \lambda)_p$ be the p-primary part of the kernel of λ. It has the decomposition $(\ker\lambda)_p = \prod_{\mathfrak{p}|p} (\ker\lambda)_{\mathfrak{p}}$. We require that $(\ker\lambda)_{\mathfrak{p}}$ is trivial, unless $\mathfrak{p}$ is unramified in K and $\operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = -1$. In the latter case the height of $(\ker\lambda)_{\mathfrak{p}}$ is $2f_{\mathfrak{p}}$.

(ii) There is an identity of invariants,

$$\operatorname{inv}^r_{\mathfrak{p}}(A, \iota, \lambda) = \operatorname{inv}_{\mathfrak{p}}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}), \qquad \text{for all } \mathfrak{p}|p.$$

An isomorphism of such data $(A, \iota, \bar{\lambda}, \bar{\eta}^p) \longrightarrow (A', \iota', \bar{\lambda}', \bar{\eta}'^p)$ is given by a O_K-linear quasi-isogeny $\phi\colon A \longrightarrow A'$ of degree prime to p compatible with the $\mathbb{Q}$-homogeneous polarizations and the level structures.

We will denote a point of $\mathcal{A}_{\mathbf{K}}(R)$ simply by $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$.

Remark 7.1.3. It is equivalent to consider in (2) a $\mathbb{Z}_{(p)}$-homogeneous polarization $\bar{\lambda}$ of A such that the elements $\lambda \in \bar{\lambda}$ satisfy the condition (i) on the p-primary part of the kernel of λ.

Remark 7.1.4. Let $(A, \iota, \bar{\lambda}, \bar{\eta}^p) \in \mathcal{A}_{\mathbf{K}}(R)$. Let $\lambda \in \bar{\lambda}$ be as in condition (i) of Definition 7.1.2. For each geometric point ω of characteristic 0 of $\operatorname{Spec} R$ the pairing induced by λ on the $\mathfrak{p}$-adic Tate module,

$$\mathcal{E}_{\mathfrak{p}} : T_{\mathfrak{p}}(A_\omega) \times T_{\mathfrak{p}}(A_\omega) \longrightarrow \mathbb{Z}_p(1) \tag{7.1.10}$$

has the following properties. If $\mathfrak{p}$ is ramified in K/F, the pairing is perfect and $\operatorname{inv}(V_{\mathfrak{p}}(A_\omega), \mathcal{E}_{\mathfrak{p}}) = \operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}})$. If $\mathfrak{p}$ is unramified, then (7.1.10) is perfect if $\operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = 1$ and is almost perfect if $\operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = -1$. If $\mathfrak{p}$ is split in K/F, then (7.1.10) is perfect.

For each geometric point ω of R of characteristic p, the polarization λ induces a pairing on the Dieudonné module M of A_ω and therefore for each prime $\mathfrak{p}|p$ of F a pairing

$$\mathcal{E}_{\mathfrak{p}} : M_{\mathfrak{p}}(A_\omega) \times M_{\mathfrak{p}}(A_\omega) \longrightarrow W(\kappa(\omega)), \tag{7.1.11}$$

with the following properties. If $\mathfrak{p}$ is ramified in K/F, the pairing is perfect and $\operatorname{inv}^r(M_{\mathfrak{p}}(A_\omega), \mathcal{E}_{\mathfrak{p}}) = \operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}})$. If $\mathfrak{p}$ is unramified, then (7.1.11) is perfect if $\operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = 1$ and is almost perfect if $\operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) = -1$. If $\mathfrak{p}$ is split in K/F, then (7.1.11) is perfect.

Proposition 7.1.5. *Assume that $\mathbf{K}^p$ is small enough. Then the functor $\mathcal{A}_{\mathbf{K}}$ is representable by a projective scheme over $\operatorname{Spec} O_{E,(\mathfrak{p}_\nu)}$ whose generic fiber is the Shimura variety $\operatorname{Sh}_{\mathbf{K}}$ associated to the Shimura datum (G, h). For general $\mathbf{K}^p$, $\mathcal{A}_{\mathbf{K}}$ is a DM-stack proper over $\operatorname{Spec} O_{E,(\mathfrak{p}_\nu)}$ whose generic fiber is the Shimura variety $\operatorname{Sh}_{\mathbf{K}}$ considered as an orbifold.*

Proof. Let $(A, \iota, \bar{\lambda}, \bar{\eta}^p) \in \mathcal{A}_{\mathbf{K}}(R)$, and fix $\eta^p \in \bar{\eta}^p$. Let $\Lambda \subset V$ be a O_K-lattice on which ς is integral. We find an abelian variety A_1 in the class A up to isogeny prime to p such that for each $\ell \neq p$

$$\eta^p(T_\ell(A_1)) = \Lambda \otimes \mathbb{Z}_\ell.$$

In this way we obtain also a polarization on A_1 whose degree is bounded in terms of ς and Λ. If $\mathbf{K}^p$ is small enough we obtain a level structure on the m-division points for some $m \geq 3$.

The fact that the moduli problem of abelian varieties with a polarization of given degree and a m-level structure for $m \geq 3$ is a quasi-projective scheme implies that the functor of $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ as in (1)–(3) of Definition 7.1.2, is representable by a quasi-projective scheme. Now the conditions (i) and (ii) define open and closed subschemes (this is easy for condition (i), and follows from Proposition 8.2.1 for condition (ii)). The representability by a Deligne-Mumford stack for general $\mathbf{K}^p$ follows.

To compare the generic fiber of $\mathcal{A}_{\mathbf{K}}$ with $\mathrm{Sh}_{\mathbf{K}}$, recall from [15, §8] that $\mathrm{Sh}_{\mathbf{K}}$ represents the following functor $\mathcal{A}_{\mathbf{K},E}$ on the category of E-algebras, comp. section 1.2. A point of $\mathcal{A}_{\mathbf{K},E}(R)$ consists of the following data.

(1) An abelian scheme A over R, up to isogeny, with an algebra homomorphism

$$\iota : O_K \longrightarrow \mathrm{End}\, A \otimes \mathbb{Q}$$

such that the Kottwitz condition (KC_r) is satisfied.

(2) A $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}$ of A such that the Rosati involution induces the conjugation of K/F.

(3) A class of K-linear isomorphisms

$$\bar{\eta} : V \otimes \mathbb{A}_f \longrightarrow \hat{V}(A) \bmod \mathbf{K}$$

which respect the forms on both sides up to a constant in $\mathbb{A}_f(1)^\times$.

Here we are implicitly using the fact that G satisfies the Hasse principle, cf. [15, §7]. We define a map $\mathcal{A}_{\mathbf{K},E}(R) \longrightarrow \mathcal{A}_{\mathbf{K}}(R)$. We fix a O_K-lattice Λ in V whose localizations at $\mathfrak{p}|p$ are the given lattices $\Lambda_{\mathfrak{p}}$ above. Let $(A, \iota, \bar{\lambda}, \bar{\eta}) \in \mathcal{A}_{\mathbf{K},E}(R)$, and fix $\eta \in \bar{\eta}$. We find an abelian variety A_1 in the isogeny class A such that for each ℓ

$$\eta(T_\ell(A_1)) = \Lambda \otimes \mathbb{Z}_\ell.$$

Then we obtain $\iota_1 \colon O_K \longrightarrow \mathrm{End}(A_1) \otimes \mathbb{Z}_{(p)}$. The Eisenstein condition (EC_r) is automatically satisfied, cf. Proposition 2.2.1. We also find a polarization $\lambda_1 \in \bar{\lambda}$ which satisfies the condition (i) in Definition 7.1.2. The existence of η implies Condition (ii). By forgetting the p-component of $\bar{\eta}$, we have associated to $(A, \iota, \bar{\lambda}, \bar{\eta})$ a well-defined object $(A_1, \iota_1, \bar{\lambda}_1, \bar{\eta}^p)$ of $\mathcal{A}_{\mathbf{K}}(R)$. By the uniqueness property of the lattices $\Lambda_{\mathfrak{p}}$ mentioned above, this map is bijective.

The properness of $\mathcal{A}_{\mathbf{K}} \longrightarrow \mathrm{Spec}\, O_{E,(\mathfrak{p}_\nu)}$ is a consequence of Proposition 7.1.7 below. □

Remark 7.1.6. If $\mathfrak{p}_v$ is the only prime ideal of F over p, then it follows from the product formula that condition (ii) of Definition 7.1.2 is automatically satisfied. Indeed, condition (ii) defines an open and closed subscheme which in this case has the same generic fiber.

Proposition 7.1.7. *The morphism $\mathcal{A}_{\mathbf{K}} \longrightarrow \mathrm{Spec}\, O_{E,(\mathfrak{p}_\nu)}$ is proper.*

For the proof of Proposition 7.1.7, we need two lemmas.

Lemma 7.1.8. *Let K/F be a CM-field and let r be a generalized CM-type of rank 2. Let $E \subset \bar{\mathbb{Q}}$ be the reflex field. Let R be a complete discrete valuation ring with an O_E-algebra structure. Let L be the field of fractions of R. We assume that the residue characteristic of R is $p > 0$. Let w be a finite place of F of residue characteristic ℓ, such that K_w/F_w is a field extension. We assume that L is of characteristic zero or that $\ell \neq p$.*

Let (A, ι, λ) be a CM-triple of type r over L. The polarization induces on the rational Tate module $V_w(A)$ an alternating pairing

$$\psi_w : V_w(A) \times V_w(A) \longrightarrow \mathbb{Q}_\ell(1). \tag{7.1.12}$$

If $\mathrm{inv}(V_w(A), \psi_w) = -1$, then the abelian variety A has potentially good reduction.

Proof. We consider only the case $\ell = p$. We may assume that A has semistable reduction. We choose an isomorphism $\mathbb{Q}_p \cong \mathbb{Q}_p(1)$ over $\bar{L}$. We obtain from ψ_w the anti-hermitian form

$$\varkappa_w : V_w(A) \times V_w(A) \longrightarrow K_w,$$

cf. (8.1.2). Let T be the toric part of the special fiber of the Néron model of A. Then O_K acts on the character group $X_*(T)$. If T is non-trivial, we obtain that

$$\dim T = [K : \mathbb{Q}] = \dim A.$$

This implies that the toric part $V_w^t(A) \subset V_w(A)$ is a K_w-vector subspace of dimension 1. By the orthogonality theorem [SGA7, Exp IX, Thm. 5.2], the anti-hermitian form $\varkappa_w$ is zero on this subspace. Let u_1, u_2 be a basis of $V_w(A)$ such that u_1 is a basis of $V_w^t(A)$. Then we obtain, in the notation of (8.1.1),

$$\mathfrak{d}_{K/F}(V_w(A), \varkappa_w) = -\varkappa_w(u_1, u_2)\varkappa_w(u_2, u_1) = \varkappa_w(u_1, u_2)\overline{\varkappa_w(u_1, u_2)} \equiv 1$$

modulo $\mathrm{Nm}_{K_w/F_w} K_w^\times$. This contradicts the assumption $\mathrm{inv}(V_w(A), \varkappa_w) = -1$. □

With the notation of the last lemma, we consider the case where $\ell = p$ and where the characteristic of L is also p. The O_E-algebra structure on R factors

$$O_E \longrightarrow \kappa_\nu \longrightarrow R,$$

where κ_ν is the residue field of E_ν. We fix a commutative diagram

$$\begin{array}{ccc} E_\nu & \longrightarrow & \bar{\mathbb{Q}}_p \\ \uparrow & & \uparrow \\ E & \longrightarrow & \bar{\mathbb{Q}}. \end{array}$$

Let w be a p-adic place of F. By the last diagram we can restrict r to a local CM-type r_w of K_w/F_w. Then E_ν is the composite of the subfields $E(K_w/F_w, r_w)$, for w running over all places of F over p.

Let (A, ι, λ) be a CM-triple of type r over L. The action of $O_F \otimes \mathbb{Z}_p = \prod_w O_{F_w}$, where w runs over all p-adic places of F, induces a decomposition of the p-divisible group of A:

$$X = \prod_w X_w.$$

Then $(X_w, \iota_w, \lambda_w)$ is a local CM-triple with respect to $K_w/F_w, r_w$ over the field L.

Lemma 7.1.9. *Let R be a discrete valuation ring of equal characteristic $p > 0$, and let L be the field of fractions. Let R be an O_E-algebra. Let ν be the p-adic place of E induced by this algebra structure.*

Let (A, ι, λ) be a CM-triple of type r over L. We assume that there is a p-adic place w of F such that one of the following conditions is satisfied.

(1) K_w/F_w is a ramified field extension. The local CM-triple $(X_w, \iota_w, \lambda_w)$ satisfies the Eisenstein condition (EC_{r_w}), and $\mathrm{inv}^r(X_w, \iota_w, \lambda_w) = -1$.

(2) K_w/F_w is an unramified field extension. The local CM-triple $(X_w, \iota_w, \lambda_w)$ satisfies the Eisenstein condition (EC_{r_w}), and λ_w is almost principal.

Then the abelian variety A has potentially good reduction over R.

Proof. We may assume that A has semistable reduction over R. Let $\tilde{A}$ be the Néron model over R, and let B be the identity component of the special fibre of $\tilde{A}$. Let us assume that the torus part $T \subset B$ is nontrivial. Since O_K acts on T, we obtain that $X_*(T)_{\mathbb{Q}}$ is a K-vector space of dimension one. Let Y be the p-divisible group of T. We obtain a decomposition

$$Y = \prod_u Y_u$$

where u runs over all places of K over p and Y_u is an O_{K_u}-module which is of height $[K_u : \mathbb{Q}_p]$ and of multiplicative type. We pass from R to the completion $\hat{R}$. Let $\hat{X} = X_{\hat{K}}$ be the p-divisible group of $A_{\hat{K}}$. By [SGA7, Exp IX, §5], the multiplicative group Y_w lifts to a multiplicative group $\tilde{Y}_w \subset \hat{X}_w^f$ of the finite part of $\hat{X}_w$ over $\hat{R}$. If we pass to the general fibre of the last inclusion we obtain a nontrivial multiplicative subgroup $(\tilde{Y}_w)_{\hat{L}} \subset \hat{X}_w$. But our assumption implies, by Lemma 5.2.8 in the ramified case, and by Proposition 5.3.5 in the unramified case, that $\hat{X}_w$ is isoclinic of slope $1/2$. This contradicts the existence of a nontrivial multiplicative part and therefore the assumption that the torus part of B is nontrivial. □

Proof. (of Proposition 7.1.7) We check the valuative criterion. Let R a discrete valuation ring with a O_{E_ν}-algebra structure. Let L be the field of fractions of R. Let $\alpha : \mathrm{Spec}\, L \longrightarrow \mathcal{A}_{\mathbf{K}}$ be a O_{E_ν}-morphism. We have to show that α extends to $\mathrm{Spec}\, R \longrightarrow \mathcal{A}_{\mathbf{K}}$. It is enough to show that for a discrete valuation ring R' which dominates R, the morphism $\mathrm{Spec}\, L' \longrightarrow \mathcal{A}_{\mathbf{K}}$ induced by α

extends $\operatorname{Spec} R' \longrightarrow \mathcal{A}_{\mathbf{K}}$. The map α gives a point $(A, \iota, \bar{\lambda}, \bar{\eta}^p) \in \mathcal{A}_{\mathbf{K}}(L)$. Since we may replace L by L' we may assume that A has semistable reduction. Let ω be a geometric point concentrated in the generic point of $\operatorname{Spec} R$. We are assuming that $\operatorname{inv}^r(T_{\mathfrak{p}_{\mathfrak{v}}}(A_\omega), \mathcal{E}_{\mathfrak{p}_{\mathfrak{v}}}) = \operatorname{inv}(V_{\mathfrak{p}_{\mathfrak{v}}}, \varsigma_{\mathfrak{p}_v}) = -1$ when $\operatorname{char} L = 0$, resp. $\operatorname{inv}^r(M_{\mathfrak{p}_{\mathfrak{v}}}(A_\omega), \mathcal{E}_{\mathfrak{p}_{\mathfrak{v}}}) = \operatorname{inv}(V_{\mathfrak{p}_{\mathfrak{v}}}, \varsigma_{\mathfrak{p}_v}) = -1$, when $\operatorname{char} L = p$. Hence we conclude by the last two lemmas that A has good reduction. Let $\tilde{A}/R$ denote the abelian scheme which extends A. Then ι extends to an action $\tilde{\iota}$ of O_K on $\tilde{A}$. The Kottwitz condition and the Eisenstein condition are closed conditions and hold therefore for $\tilde{A}$. The polarization λ extends to $\tilde{\lambda} : \tilde{A} \longrightarrow \tilde{A}^\wedge$. The condition (i) from Definition 7.1.2 extends from A to $\tilde{A}$. For a geometric point ω_0 concentrated in the closed point of $\operatorname{Spec} R$ we find

$$\operatorname{inv}^r_{\mathfrak{p}}(\tilde{A}_{\omega_0}, \tilde{\iota}_{\omega_0}, \lambda_{\omega_0}) = \operatorname{inv}^r(M_{\mathfrak{p}}(\tilde{A}_{\omega_0}), \mathcal{E}_{\mathfrak{p}}) = \operatorname{inv}(V_{\mathfrak{p}}, \varsigma_{\mathfrak{p}_v})$$

because the left hand is by Proposition 8.2.1 equal to $\operatorname{inv}^r_{\mathfrak{p}}(\tilde{A}_L, \iota_L, \lambda_L)$. Hence condition (ii) from Definition 7.1.2 also extends from A to $\tilde{A}$. From this we obtain an extension of $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ to a point of $\mathcal{A}_{\mathbf{K}}(R)$. □

Remark 7.1.10. The scheme $\mathcal{A}_{\mathbf{K}}$ turns out to be flat over $\operatorname{Spec} O_{E,(\mathfrak{p}_v)}$, cf. Theorem 7.3.3, (i). Hence its generic fiber is dense. It follows that it is enough to check the valuative criterion on discrete valuation rings R with fraction field L of characteristic zero. Hence Lemma 7.1.9 is not needed.

The following proposition shows that there is only one isogeny class in the special fiber of $\mathcal{A}_{\mathbf{K}}$. This is the underlying reason why there is p-adic uniformization.

Proposition 7.1.11. *Let κ_ν be the residue class field of E_ν. Let $(A_1, \iota_1, \bar{\lambda}_1, \bar{\eta}_1^p)$ and $(A_2, \iota_2, \bar{\lambda}_2, \bar{\eta}_2^p)$ be two points of $\mathcal{A}_{\mathbf{K}}(\bar{\kappa}_\nu)$. Then there exists a quasi-isogeny*

$$(A_1, \iota_1, \bar{\lambda}_1) \longrightarrow (A_2, \iota_2, \bar{\lambda}_2),$$

i.e., a quasi-isogeny which respects the actions ι_i and the $\mathbb{Q}$-homogeneous polarizations $\bar{\lambda}_i$. In fact, there exists such a quasi-isogeny of degree prime to p.

Proof. Let X_i be the p-divisible group of A_i, with its decomposition $X_i = \prod_{\mathfrak{p}|p} X_{i,\mathfrak{p}}$. It follows from Proposition 5.2.7 (jointly with Lemma 5.2.8) and Proposition 5.3.6 that $X_{i,\mathfrak{p}_v}$ is isoclinic. In the banal cases $\mathfrak{p} \neq \mathfrak{p}_v$, the same follows from Lemma 4.3.3 for $X_{i,\mathfrak{p}}$. By [27, Cor. 6.29] we find a quasi-isogeny

$$a : (A_1, \iota_1) \longrightarrow (A_2, \iota_2).$$

We choose $\lambda_i \in \bar{\lambda}_i$. We set $\lambda = a^*(\lambda_2)$. We find an endomorphism $u \in \operatorname{End}^o(A_1)$ such that

$$\lambda = \lambda_1 u.$$

Since λ_1 and λ induce the conjugation on K, we conclude that $u \in \operatorname{End}^o_K A_1$. Moreover u is fixed by the Rosati involution $*$ induced by λ_1 on $D := \operatorname{End}^o_K A_1$. It is enough to find an element $d \in D^\times$ such that

$$u = f d^* d \tag{7.1.13}$$

for some element $f \in \mathbb{Q}^\times$. The solutions of these equations form a torsor under the algebraic group J over $\mathbb{Q}$ such that

$$J(\mathbb{Q}) = \{e \in D^\times \mid e^* e \in \mathbb{Q}^\times\}. \tag{7.1.14}$$

By [15, §7], this group satisfies the Hasse principle. Therefore it is enough to find a solution of the equation (7.1.13) in $D \otimes \mathbb{Q}_w^\times$ for all places w of $\mathbb{Q}$. If w is a finite place $w \neq p$ we have, by [27, Cor. 6.29], that

$$D \otimes \mathbb{Q}_w = \operatorname{End}_{K \otimes \mathbb{Q}_w} V_w(A_1)$$

such that the Riemann form $\mathcal{E}_w^{\lambda_1}$ induces the involution $*$. A solution of (7.1.13) exists iff the symplectic $K \otimes \mathbb{Q}_w$-modules

$$(V_w(A_1), \mathcal{E}_w^{\lambda_1}), \qquad (V_w(A_2), \mathcal{E}_w^{\lambda_2})$$

are similar up to a factor in $\mathbb{Q}_w^\times$. But this follows from the existence of $\bar{\eta}_1^p$ and $\bar{\eta}_2^p$.

In the case $w = p$ we can use Dieudonné modules. In this case we know, by condition (ii) in Definition 7.1.2, that the rational Dieudonné modules of A_1 and A_2 together with their polarizations are isomorphic.

If w is the infinite place, one can deduce the assertion from the fact that u in (7.1.13) is totally positive.

Now let us prove the second assertion. We consider the Dieudonné modules M_1, resp. M_2, of A_1, resp. A_2. We choose the polarizations $\lambda_1 \in \bar{\lambda}_1$, resp. $\lambda_2 \in \bar{\lambda}_2$, as in condition (i) of Definition 7.1.2. Using the contracting functor, it is clear that there is a quasi-isogeny of height zero $\alpha : (M_1, \lambda_1) \longrightarrow (M_2, \lambda_2)$. Let ρ: $(M_1, \bar{\lambda}_1) \longrightarrow (M_2, \bar{\lambda}_2)$ be an arbitrary quasi-isogeny. Consider the morphism

$$\alpha \circ \rho^{-1} : (M_2, \bar{\lambda}_2) \longrightarrow (M_2, \bar{\lambda}_2).$$

We consider the group J for $(A_2, \iota_2, \bar{\lambda}_2)$ (compare (7.1.14)). Then $\alpha \circ \rho^{-1}$ is an element of $J(\mathbb{Q}_p)$ by Tate's theorem ([27, Cor. 6.29]). We approximate it by an element $\alpha_1 \in J(\mathbb{Q})$. Then

$$\rho \circ \alpha_1 : (A_1, \iota_1, \bar{\lambda}_1) \longrightarrow (A_2, \iota_2, \bar{\lambda}_2)$$

is the desired quasi-isogeny of order prime to p. □

7.2. **The RZ-space $\tilde{\mathcal{M}}_r$.** We fix a point $(A_0, \iota_0, \bar{\lambda}_0, \bar{\eta}_0^p)$ of $\mathcal{A}_{\mathbf{K}}(\bar{\kappa}_\nu)$. We also fix a polarization $\lambda_0 \in \bar{\lambda}_0$ which satisfies the condition (i) of Definition 7.1.2. We denote by $\mathbb{X}$ the p-divisible group of A_0. The action ι_0 induces an action $\iota_{\mathbb{X}}$ on $\mathbb{X}$ and λ_0 induces a polarization $\lambda_{\mathbb{X}}$ on $\mathbb{X}$. We denote by q_ν the number of elements in $\kappa_\nu = \kappa_{E_\nu}$.

Let $R \in \mathrm{Nilp}_{O_{E_\nu}}$ and let (X, ι) be a p-divisible group over $\operatorname{Spec} R$ with an action

$$\iota : O_K \otimes \mathbb{Z}_p \longrightarrow \operatorname{End} X.$$

The notion of a *semi-local CM-triple* (X, ι, λ) *relative to* $K \otimes \mathbb{Q}_p / F \otimes \mathbb{Q}_p$ *and* r should be obvious but we explain it more precisely: the decomposition

$$O_F \otimes \mathbb{Z}_p = \prod_{\mathfrak{p}} O_{F_\mathfrak{p}}$$

induces the decomposition

$$X = \prod X_\mathfrak{p}.$$

Let λ be a polarization of X which induces the conjugation on K/F. Then the decomposition extends to

$$(X, \iota, \lambda) = \prod (X_\mathfrak{p}, \iota_\mathfrak{p}, \lambda_\mathfrak{p}). \tag{7.2.1}$$

We call (X, ι, λ) a semi-local CM-triple of type r if each $(X_\mathfrak{p}, \iota_\mathfrak{p}, \lambda_\mathfrak{p})$ is a local CM-triple of type $r_\mathfrak{p}$ with respect to $K_\mathfrak{p}/F_\mathfrak{p}$. This makes sense because $E(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p}) \subset E_\nu$.

Definition 7.2.1. A semi-local CM-triple (X, ι, λ) of type $(K \otimes \mathbb{Q}_p / F \otimes \mathbb{Q}_p, r)$ over an algebraically closed field with a κ_{E_ν}-algebra structure is said to be *compatible with* (V, ς) if, for each $\mathfrak{p}|p$,

$$\operatorname{inv}^r(X_\mathfrak{p}, \iota_\mathfrak{p}, \lambda_\mathfrak{p}) = \operatorname{inv}(V_\mathfrak{p}, \varsigma_\mathfrak{p}),$$

and if $\lambda_\mathfrak{p}$ is principal, except in the case where $K_\mathfrak{p}/F_\mathfrak{p}$ is unramified and $\operatorname{inv}(V_\mathfrak{p}, \varsigma_\mathfrak{p}) = -1$. In the latter case $\lambda_\mathfrak{p}$ is almost principal.

We note that the CM-triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\bar{\kappa}_\nu$ is compatible with (V, ς) and satisfies the conditions (KC_r) and (EC_r), in the sense explained before Definition 7.1.2.

Definition 7.2.2. Let $i \in \mathbb{Z}$. Let $\mathcal{M}_r(i)$ be the following functor on the category $\mathrm{Nilp}_{O_{\breve{E}_\nu}}$. For an object $R \in \mathrm{Nilp}_{O_{\breve{E}_\nu}}$, write $\bar{R} = R \otimes_{O_{\breve{E}_\nu}} \bar{\kappa}_{E_\nu}$. A point of $\mathcal{M}_r(i)(R)$ is given by the following data:

(1) A CM-triple (X, ι, λ) of type $(K \otimes \mathbb{Q}_p / F \otimes \mathbb{Q}_p, r)$ over $\operatorname{Spec} R$ which satisfies the conditions (KC_r) and (EC_r) and is compatible with (V, ς).

(2) A $O_K \otimes \mathbb{Z}_p$-linear quasi-isogeny

$$\rho : \bar{X} := X \times_{\operatorname{Spec} R} \operatorname{Spec} \bar{R} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \kappa_{\breve{E}_\nu}} \operatorname{Spec} \bar{R}$$

such that ρ respects the polarization $p^i \lambda$ on X and $\lambda_{\mathbb{X}}$ up to a factor in $(O_F \otimes \mathbb{Z}_p)^\times$.

We denote these data by $(X, \iota, \lambda, \rho)$. Two data $(X, \iota, \lambda, \rho)$ and $(X', \iota', \lambda', \rho')$ define the same point of $\mathcal{M}_r(i)$ iff there is an isomorphism $\alpha : (X, \iota) \longrightarrow (X', \iota')$, such that $\rho' \circ \alpha_{\bar{R}} = \rho$. In particular, α respects the polarizations λ and λ' up to a factor in $(O_F \otimes \mathbb{Z}_p)^\times$.

Remark 7.2.3. In (2) we could replace the last condition on ρ by

($2'$) *The quasi-isogeny ρ respects the polarizations as follows,*

$$p^i\lambda = \rho^*\lambda_{\mathbb{X}}.$$

Then we obtain a functor which is naturally isomorphic to $\mathcal{M}_r(i)$. This follows because for $a \in (O_F \otimes \mathbb{Z}_p)^\times$ the points $(X, \iota, a\lambda, \rho)$ and $(X, \iota, \lambda, \rho)$ of $\mathcal{M}_r(R)$ are isomorphic. We could also require $up^i\lambda = \rho^*\lambda_{\mathbb{X}}$ for some $u \in \mathbb{Z}_p^\times$ without changing the functor. We use different descriptions of the functor $\mathcal{M}_r(i)$ in order to describe better different group actions.

Let $\tau_{E_\nu} \in \mathrm{Gal}(\breve{E}_\nu/E_\nu)$ be the Frobenius automorphism and let f_{E_ν} be the inertia index of $E_\nu/\mathbb{Q}_p$, i.e., $q_\nu = p^{f_{E_\nu}}$. As earlier, the Frobenius $F_{\mathbb{X},\tau_{E_\nu}}$ defines a Weil descent datum on these functors,

$$\omega_{\mathcal{M}_r} : \mathcal{M}_r(i)(R) \longrightarrow \mathcal{M}_r(i + f_{E_\nu})(R_{[\tau_{E_\nu}]}), \tag{7.2.2}$$

cf. (6.1.3), (6.2.2). Since the degrees of the polarizations λ and $\lambda_{\mathbb{X}}$ are the same, it follows that

$$2\,\mathrm{height}\,\rho = \mathrm{height}(p^i \mid X) = 4[F : \mathbb{Q}]i.$$

More precisely, $\rho = \prod_{\mathfrak{p}} \rho_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over the prime ideals of F over p. For each $\mathfrak{p}$ we have

$$2\,\mathrm{height}\,\rho_{\mathfrak{p}} = \mathrm{height}(p^i \mid X_{\mathfrak{p}}) = 4[F_{\mathfrak{p}} : \mathbb{Q}_p]i.$$

We define

$$\tilde{\mathcal{M}}_r = \coprod_{i\in\mathbb{Z}} \mathcal{M}_r(i). \tag{7.2.3}$$

We describe the functor $\tilde{\mathcal{M}}_r$ with its Weil descent datum. Let

$$J(\mathbb{Q}_p) = \{\alpha \in \mathrm{End}^o_{K\otimes\mathbb{Q}_p} \mathbb{X} \mid \alpha^*(\lambda_{\mathbb{X}}) = c\lambda_{\mathbb{X}},\ \text{for some } c \in \mathbb{Q}_p^\times\}. \tag{7.2.4}$$

This group acts naturally on $\tilde{\mathcal{M}}_r$ via the rigidification ρ. We consider the decomposition (7.2.1) for $\mathbb{X}$. We set

$$J_{\mathfrak{p}} = \{\alpha \in \mathrm{End}^o_{K_{\mathfrak{p}}} \mathbb{X}_{\mathfrak{p}} \mid \alpha^*(\lambda_{\mathfrak{p}}) = c\lambda_{\mathfrak{p}},\ \text{for some } c \in \mathbb{Q}_p^\times\} \tag{7.2.5}$$

For all $\mathfrak{p}$ the groups $J_{\mathfrak{p}}$ are subgroups of J' as introduced in section 6 in the local cases and they agree with J introduced in the banal cases.

We will give an explicit description of these groups. For this, it is convenient to replace the bilinear form $\varsigma_{\mathfrak{p}}$ by the $F_{\mathfrak{p}}$-bilinear form

$$\tilde{\varsigma}_{\mathfrak{p}} : V_{\mathfrak{p}} \times V_{\mathfrak{p}} \to F_{\mathfrak{p}},$$

which is defined by

$$\mathbf{t}(a\tilde{\varsigma}_{\mathfrak{p}}(x_1, x_2)) = \varsigma_{\mathfrak{p}}(ax_1, x_2), \quad a \in F_{\mathfrak{p}},$$

for $\mathbf{t}(a) = \mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \vartheta^{-1}a$, where as usual $\vartheta \in O_F$ is the different of $F/\mathbb{Q}_p$. The restriction to the lattices $\Lambda_{\mathfrak{p}}$ gives

$$\tilde{\varsigma}_{\mathfrak{p}} : \Lambda_{\mathfrak{p}} \times \Lambda_{\mathfrak{p}} \to O_{F_{\mathfrak{p}}}.$$

Let us consider the prime $\mathfrak{p} = \mathfrak{p}_v$. We denote by D_v the quaternion division algebra over F_v. We choose a two-dimensional K_v-vector space with an anti-hermitian form

$$\bar{\varsigma}_{\mathfrak{p}_v} : \bar{V}_{\mathfrak{p}_v} \times \bar{V}_{\mathfrak{p}_v} \to F_{\mathfrak{p}_v}$$

of invariant $+1$. The contraction functor associates to $(\mathbb{X}_{\mathfrak{p}_v}, \iota_{\mathbb{X}_{\mathfrak{p}_v}}, \lambda_{\mathbb{X}_{\mathfrak{p}_v}})$ a special formal O_{D_v}-module $\mathbb{Y}$ with the relative polarization $\lambda_v = \psi_v$, resp. $\lambda_v = \theta_v$, as in section 5.2 resp. 5.3. Since the endomorphism ring is not changed by the contraction functor, it follows from Lemmas 5.2.2 and 5.3.2 that there is an isomorphism

$$J_{\mathfrak{p}_v} = G(\bar{V}_{\mathfrak{p}_v}, \bar{\varsigma}_{\mathfrak{p}_v}). \tag{7.2.6}$$

For a banal prime $\mathfrak{p}|p$ of F, we consider the image $(C_{\mathbb{X}_{\mathfrak{p}}}, \phi_{\mathbb{X}_{\mathfrak{p}}})$ by the polarized contraction functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$ of Theorem 4.5.11. By Proposition 8.3.6, it follows from Condition (ii) in Definition 7.1.2 that there is an isomorphism

$$(C_{\mathbb{X}_{\mathfrak{p}}}, \phi_{\mathbb{X}_{\mathfrak{p}}}) \cong (\Lambda_{\mathfrak{p}}, \tilde{\varsigma}_{\mathfrak{p}}). \tag{7.2.7}$$

More precisely, Condition (ii) implies that the corresponding vector spaces are isomorphic; the integral isomorphism follows from Lemmas 8.1.2 and 8.1.3. Therefore we obtain

$$J_{\mathfrak{p}} = G(V_{\mathfrak{p}}, \tilde{\varsigma}_{\mathfrak{p}}) = G_{\mathfrak{p}}, \quad \text{for } \mathfrak{p} \neq \mathfrak{p}_v. \tag{7.2.8}$$

Since we want a uniform notation, we set $(\bar{V}_{\mathfrak{p}}, \bar{\varsigma}_{\mathfrak{p}}) = (V_{\mathfrak{p}}, \tilde{\varsigma}_{\mathfrak{p}})$ for $\mathfrak{p} \neq \mathfrak{p}_v$. We set

$$\bar{G}_{\mathfrak{p}} = G(\bar{V}_{\mathfrak{p}}, \bar{\varsigma}_{\mathfrak{p}}).$$

We now have fixed an isomorphism $J_{\mathfrak{p}} \cong \bar{G}_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. For $\mathfrak{p}$ banal, we have $G_{\mathfrak{p}} = \bar{G}_{\mathfrak{p}}$.

Let

$$\bar{V}_p = \oplus_{\mathfrak{p}|p} \bar{V}_{\mathfrak{p}}.$$

This is an $K \otimes \mathbb{Q}_p$-module. Let

$$\bar{\varsigma}_p : \bar{V}_p \times \bar{V}_p \to F \otimes \mathbb{Q}_p$$

be the orthogonal sum of the forms $\bar{\varsigma}_{\mathfrak{p}}$. We define

$$\bar{G}(\mathbb{Q}_p) := G(\bar{V}_p, \bar{\varsigma}_p) := \{g \in \mathrm{Aut}_{K\otimes\mathbb{Q}_p}(\bar{V}_p) \mid \bar{\varsigma}_p(gx, gy) = c\bar{\varsigma}_p(x, y), \text{ for some } c \in \mathbb{Q}_p^\times\}. \tag{7.2.9}$$

We have shown that $\bar{G}(\mathbb{Q}_p) = J(\mathbb{Q}_p)$. In the description of the descent data, the following slightly larger group will be needed. We define the group $\bar{G}'_p \supset \bar{G}(\mathbb{Q}_p)$ via

$$\bar{G}'_p = \{g \in \mathrm{Aut}_{K\otimes\mathbb{Q}_p}(\bar{V}_p) \mid \bar{\varsigma}_p(gx, gy) = \mu_p(g)\bar{\varsigma}_p(x, y), \text{ for } \mu_p(g) \in p^{\mathbb{Z}}(O_F \otimes \mathbb{Z}_p)^\times\},$$

and

$$\bar{G}'_{\mathfrak{p}} = \{g \in \mathrm{Aut}_{K_{\mathfrak{p}}}(\bar{V}_{\mathfrak{p}}) \mid \bar{\varsigma}_{\mathfrak{p}}(gx, gy) = \mu_{\mathfrak{p}}(g)\bar{\varsigma}_{\mathfrak{p}}(x, y), \text{ for } \mu_{\mathfrak{p}}(g) \in p^{\mathbb{Z}} O_{F_{\mathfrak{p}}}^\times\}.$$

The groups $\bar{G}'_{\mathfrak{p}}$ are isomorphic to the groups $J'_{\mathfrak{p}} = J'$ introduced in section 6 in the local cases. We fix these isomorphisms which are associated to the framing objects. Therefore the groups $\bar{G}'_{\mathfrak{p}}$ act on the local moduli spaces $\mathcal{M}$ of section 6 and the subgroup $\bar{G}'_p \subset \prod_{\mathfrak{p}|p} \bar{G}'_{\mathfrak{p}}$ acts on $\tilde{\mathcal{M}}_r$, cf. (7.2.3).

We define the group $\hat{G}'(\mathbb{Q}_p)$ as the union of the following sets for $i \in \mathbb{Z}$,

$$\hat{G}'(i) = \{(c, g_{\mathfrak{p}}) \in p^i O_{F_{\mathfrak{p}_v}}^\times \times \prod_{\mathfrak{p}\text{ banal}} \bar{G}'_{\mathfrak{p}} \mid \mu_{\mathfrak{p}}(g_{\mathfrak{p}}) \in p^i O_{F_{\mathfrak{p}}}^\times, \text{ for all } \mathfrak{p}\}. \tag{7.2.10}$$

Let $\hat{G}'(\mathbb{Z}_p) \subset \hat{G}'(\mathbb{Q}_p)$ be the subgroup of elements $(c, g_{\mathfrak{p}})$ such that $c \in O_{F_{\mathfrak{p}_v}}^\times$ and $g_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}$. The multiplicator $\mu_{\mathfrak{p}_v} : \bar{G}'_{\mathfrak{p}_v} \to p^{\mathbb{Z}} O_{F_{\mathfrak{p}_v}}^\times$ induces homomorphisms

$$\bar{G}'_p \to \hat{G}'(\mathbb{Q}_p) \quad \text{and} \quad G(\mathbb{Q}_p) \to \hat{G}'(\mathbb{Q}_p). \tag{7.2.11}$$

For the second map we used the identification $\bar{G}_{\mathfrak{p}} = G_{\mathfrak{p}}$ for $\mathfrak{p}$ banal.

Definition 7.2.4. We consider the following element $w'_r = (c, w_{\mathfrak{p}}) \in \hat{G}'(\mathbb{Q}_p)$.

(1) $c = p^{f_{E_\nu}}$.

(2) If $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is ramified and hence $\lambda_{\mathbb{X}_{\mathfrak{p}}}$ is principal, $w_{\mathfrak{p}}$ is the multiplication

$$\Pi_{\mathfrak{p}}^{e_{\mathfrak{p}} f_{E_\nu}} : \bar{V}_{\mathfrak{p}} \longrightarrow \bar{V}_{\mathfrak{p}},$$

see Proposition 6.3.2.

(3) If $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is unramified, then both principal and almost principal $\lambda_{\mathbb{X}_{\mathfrak{p}}}$ are allowed. In both cases we define $w_{\mathfrak{p}}$ as the multiplication

$$\pi_{\mathfrak{p}}^{e_{\mathfrak{p}} f_{E_\nu}/2} : \bar{V}_{\mathfrak{p}} \longrightarrow \bar{V}_{\mathfrak{p}},$$

see Proposition 6.4.3.

(4) In the case where $K_{\mathfrak{p}} = F_{\mathfrak{p}} \times F_{\mathfrak{p}}$ is split and hence $\lambda_{\mathbb{X}_{\mathfrak{p}}}$ is principal, we have the decomposition $\bar{V}_{\mathfrak{p}} = \bar{V}_{\mathfrak{p},1} \oplus \bar{V}_{\mathfrak{p},2}$. Let $E(r_{\mathfrak{p}})$ be the reflex field of $(K_{\mathfrak{p}}/F_{\mathfrak{p}}, r_{\mathfrak{p}})$. Let $f_{r_{\mathfrak{p}}}$ be the inertia index of $E(r_{\mathfrak{p}})$. We consider the numbers $a_{1,E(r_{\mathfrak{p}})}$ and $a_{2,E(r_{\mathfrak{p}})}$ as defined by (6.5.1). We set

$$a_{1,E_\nu} = a_{1,E(r_{\mathfrak{p}})} \frac{f_{E_\nu}}{f_{r_{\mathfrak{p}}}}, \quad a_{2,E_\nu} = a_{2,E(r_{\mathfrak{p}})} \frac{f_{E_\nu}}{f_{r_{\mathfrak{p}}}}.$$

Then we have $a_{1,E_\nu} + a_{2,E_\nu} = e_{\mathfrak{p}} f_{E_\nu}$. We define $w_{\mathfrak{p}}$ to be the multiplication by $\pi_{\mathfrak{p}}^{a_{1,E_\nu}}$ on $\bar{V}_{\mathfrak{p},1}$ and the multiplication by $\pi_{\mathfrak{p}}^{a_{2,E_\nu}}$ on $\bar{V}_{\mathfrak{p},1}$.

The element w'_r is clearly an element of the center of $\hat{G}'(\mathbb{Q}_p)$.

Proposition 7.2.5. *There exists an isomorphism*

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} (\widehat{\Omega}_{F_v} \times_{\operatorname{Spf} O_{F_v}} \operatorname{Spf} O_{\breve{E}_\nu}) \times \hat{G}'(\mathbb{Q}_p)/\hat{G}'(\mathbb{Z}_p)$$

which is equivariant with respect to the action of $\bar{G}'_p$ on both sides. This extends the action of $J(\mathbb{Q}_p) = \bar{G}(\mathbb{Q}_p) \subset \bar{G}'_p$.

The Weil descent datum $\omega_{\mathcal{M}_r}$ relative to $O_{\breve{E}_\nu}/O_{E_\nu}$ on the left hand side (7.2.2) *corresponds on the right hand side to*

$$(\xi, g) \longmapsto (\omega_{\tau_{E_\nu}}(\xi), w'_r g), \quad g \in \hat{G}'(\mathbb{Q}_p).$$

Proof. We use the decomposition

$$\mathcal{M}_r(i) = \prod_{\mathfrak{p}|p} \mathcal{M}_{r_\mathfrak{p}}(i),$$

which follows immediately from (7.2.1). Then we conclude by the results of section 6, in particular Propositions 6.3.2, 6.4.3, 6.5.1. □

Remark 7.2.6. We may multiply each $w_\mathfrak{p}$ by a unit in $K_\mathfrak{p}$ in the Definition 7.2.4 of w'_r. This does not change the assertion of the last Proposition.

We introduce the group

$$\hat{G}(\mathbb{Q}_p) = \{(c, g_\mathfrak{p}) \in \mathbb{Q}_p^\times \times \prod_{\mathfrak{p}\text{ banal}} G_\mathfrak{p} \mid \mu_\mathfrak{p}(g_\mathfrak{p}) = c, \text{ for all } \mathfrak{p} \text{ banal}\}. \tag{7.2.12}$$

There are natural homomorphisms

$$G(\mathbb{Q}_p) \to \hat{G}(\mathbb{Q}_p) \quad \text{and} \quad \bar{G}(\mathbb{Q}_p) \to \hat{G}(\mathbb{Q}_p). \tag{7.2.13}$$

For the second map, we used that in the definition of $\hat{G}(\mathbb{Q}_p)$ we can replace $G_\mathfrak{p}$ by $\bar{G}_\mathfrak{p}$. In particular the groups $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$ act on $\hat{G}(\mathbb{Q}_p)$. We denote by $\hat{G}(\mathbb{Z}_p) \subset \hat{G}(\mathbb{Q}_p)$ the subgroup of all $(c, g_\mathfrak{p})$ such that $c \in \mathbb{Z}_p^\times$ and $g_\mathfrak{p}\Lambda_\mathfrak{p} = \Lambda_\mathfrak{p}$. By Corollaries 6.3.4, 6.4.5, 6.5.3, we obtain a bijection

$$\hat{G}(\mathbb{Q}_p)/\hat{G}(\mathbb{Z}_p) \xrightarrow{\sim} \hat{G}'(\mathbb{Q}_p)/\hat{G}'(\mathbb{Z}_p).$$

Corollary 7.2.7. *There exists an isomorphism*

$$\tilde{\mathcal{M}}_r \xrightarrow{\sim} (\widehat{\Omega}_{F_v} \times_{\operatorname{Spf} O_{F_v}} \operatorname{Spf} O_{\breve{E}_\nu}) \times \hat{G}(\mathbb{Q}_p)/\hat{G}(\mathbb{Z}_p)$$

which is equivariant with respect to the action of $J(\mathbb{Q}_p)$ on both sides. □

Note that in this version of Proposition 7.2.5 we loose control of the descent data.

7.3. **The p-adic uniformization.** We will now define a uniformization morphism in the sense of [27]. We fix a point $(A_0, \iota_0, \bar{\lambda}_0, \bar{\eta}_0^p)$ of $\mathcal{A}_\mathbf{K}(\bar{\kappa}_\nu)$. The uniformization morphism will depend on the choice of $\eta_0^p \in \bar{\eta}_0^p$. This choice defines a point of the proscheme $\operatorname{proj\,lim}_{\mathbf{K}^p} \mathcal{A}_\mathbf{K}$ for all congruence subgroups $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p$ as above. We also fix a polarization $\lambda_0 \in \bar{\lambda}_0$ which satisfies the condition (i) of Definition 7.1.2. Recall the p-divisible group with induced polarization $(\mathbb{X}, \iota_\mathbb{X}, \lambda_\mathbb{X})$ corresponding to A_0.

We denote by $\hat{\mathcal{A}}_\mathbf{K}$ the restriction of $\mathcal{A}_\mathbf{K}$ to the category $\operatorname{Nilp}_{O_{E_\nu}}$. The uniformization morphism

$$\Theta : \tilde{\mathcal{M}}_r \times G(\mathbb{A}_f^p)/\mathbf{K}^p \longrightarrow \hat{\mathcal{A}}_\mathbf{K} \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu} \tag{7.3.1}$$

is defined as follows. Let $(X, \iota, \lambda, \rho) \in \mathcal{M}_r(i)(R)$ and let $g \in G(\mathbb{A}_f^p)$. Recall the notation $\bar{R} = R \otimes_{O_{\breve{E}_\nu}} \bar{\kappa}_\nu$. The quasi-isogeny ρ extends uniquely to a quasi-isogeny of abelian varieties

$$\rho : \bar{A} \longrightarrow A_0 \times_{\operatorname{Spec} \bar{\kappa}_\nu} \operatorname{Spec} \bar{R}. \tag{7.3.2}$$

Because O_K acts on $\bar{X} = X \otimes_R \bar{R}$, we obtain a map $O_K \longrightarrow \operatorname{End}(\bar{A}) \otimes \mathbb{Z}_{(p)}$. Moreover the polarization $\lambda_0 : A_0 \longrightarrow A_0^\wedge$ induces on $\bar{A}$ a quasi-polarization $\lambda'_{\bar{A}} : \bar{A} \longrightarrow \bar{A}^\wedge$ and $\bar{\eta}_0^p$ induces

$$\bar{\eta}_{\bar{A}}^p = V^p(\rho^{-1}) \circ \eta_0^p : V \otimes \mathbb{A}_f^p \longrightarrow V^p(\bar{A}) \quad \operatorname{mod} \mathbf{K}^p.$$

On the p-divisible groups, $\lambda'_{\bar{A}}$ differs from $p^i\lambda$ by a factor from $(O_F \otimes \mathbb{Z}_p)^\times$ and therefore $\lambda_{\bar{A}} := p^{-i}\lambda'_{\bar{A}}$ satisfies the condition (i) in the Definition 7.1.2 of the functor $\mathcal{A}_{\mathbf{K}}$.

We associate to the pair (X, g) from the left hand side of (7.3.1) the point

$$(\bar{A}, \iota_{\bar{A}}, \lambda_{\bar{A}}, \bar{\eta}^p_{\bar{A}} g) \in \mathcal{A}_{\mathbf{K}}(\bar{R}). \tag{7.3.3}$$

The CM-triple (X, ι, λ) over R defines by the Serre-Tate theorem a lifting of (7.3.3) to a point of $\mathcal{A}_{\mathbf{K}}(R)$. This finishes the definition of the uniformization morphism Θ in (7.3.1).

Lemma 7.3.1. *The uniformization morphism is compatible with the Weil descent data $\omega_{\mathcal{M}_r}$ acting on the first factor on the left hand side and the natural Weil descent data on $\hat{\mathcal{A}}_{\mathbf{K}} \times_{\operatorname{Spf} O_{E_\nu}} \operatorname{Spf} O_{\breve{E}_\nu}$.*

Proof. This is essentially [27, Thm. 6.21] but we repeat the simple argument in our context. By definition of the Weil descent data repeated below, it is enough to consider both sides of (7.3.3) on the category of $\bar{\kappa}_\nu$-algebras R. We will denote by $\varepsilon : \bar{\kappa}_\nu \longrightarrow R$ the algebra structure. Consider a point $(X, \iota, \lambda, \rho) \in \tilde{\mathcal{M}}_r(R)$. The Weil descent datum $\omega_{\mathcal{M}_r}$ is obtained by changing ρ to ρ':

$$\rho' : X \xrightarrow{\rho} \varepsilon_* \mathbb{X} \overset{\varepsilon_* F_{\mathbb{X}, \tau_{E_\nu}}}{\longrightarrow} \varepsilon_* (\tau_{E_\nu})_* \mathbb{X}.$$

This gives a point $(X, \iota, \lambda, \rho') \in \tilde{\mathcal{M}}_r(R_{[\tau_{E_\nu}]})$. The point $(X, \iota, \lambda, \rho)$ defines a quasi-isogeny of abelian varieties

$$\rho : A \longrightarrow \varepsilon_* A_0,$$

as explained in the definition of Θ. The point $(X, \iota, \lambda, \rho')$ defines in the same way the quasi-isogeny of abelian varieties over $R_{[\tau_{E_\nu}]}$,

$$A \longrightarrow \varepsilon_* A_0 \overset{\varepsilon_* F_{A_0, \tau_{E_\nu}}}{\longrightarrow} (\varepsilon \tau_{E_\nu})_* A_0.$$

Here A with its additional structure is regarded as a point of $\hat{\mathcal{A}}_{\mathbf{K}^p}(R_{[\tau_{E_\nu}]})$. This makes sense because to be a point of $\hat{\mathcal{A}}_{\mathbf{K}^p}(R)$ depends only on the κ_ν-algebra structure on R. In other words

$$\hat{\mathcal{A}}_{\mathbf{K}^p}(R) = \hat{\mathcal{A}}_{\mathbf{K}^p}(R_{[\tau_{E_\nu}]}). \tag{7.3.4}$$

But this equation is the Weil descent datum on the right hand side of (7.3.1). □

We define the group

$$J(\mathbb{Q}) = \{\gamma \in \operatorname{End}^o_K A_0 \mid \gamma^* \lambda_0 = u \lambda_0, \ \text{ for some } u \in \mathbb{Q}^\times\}, \tag{7.3.5}$$

cf. (7.1.14). Regarded as an algebraic group over $\mathbb{Q}$, the group J is an inner form of G. In the proof of Proposition 7.1.11 we saw that the $\mathbb{Q}_p$-valued points of J coincide with the group $J(\mathbb{Q}_p)$ of (7.2.4). We proved in section 7.2 that $\bar{G}(\mathbb{Q}_p) = J(\mathbb{Q}_p)$. Let $\gamma \in J(\mathbb{Q})$. With the chosen η^p_0, we define $\omega(\gamma) \in G(\mathbb{A}^p_f)$ by the equation

$$V^p(\gamma) \circ \eta^p_0 = \eta^p_0 \omega(\gamma). \tag{7.3.6}$$

This defines a homomorphism

$$\omega : J(\mathbb{Q}) \longrightarrow G(\mathbb{A}^p_f),$$

and an isomorphism $J(\mathbb{A}^p_f) \cong G(\mathbb{A}^p_f)$. Therefore J and G are isomorphic over the finite places $w \neq p$ of $\mathbb{Q}$. At the infinite place J is anisotropic because the Rosati involution is positive.

The group $J(\mathbb{Q}_p)$ acts on $\tilde{\mathcal{M}}_r$,

$$(X, \iota, \lambda, \rho) \longmapsto (X, \iota, \lambda, \gamma\rho), \quad \gamma \in J(\mathbb{Q}_p).$$

Let $((X, \iota, \lambda, \rho), g)$, with $g \in G(\mathbb{A}^p_f)$ be a point from the left hand side of (7.3.1) and let $(A, \iota_A, \lambda_A, \eta^p_A g)$ be its image by Θ, cf. (7.3.3). If $\gamma \in J(\mathbb{Q})$, the quasi-isogeny $\gamma\rho$ extends to the quasi-isogeny of abelian schemes

$$\bar{A} \xrightarrow{\rho} (A_0)_{\bar{R}} \xrightarrow{\gamma} (A_0)_{\bar{R}}.$$

In follows from (7.3.6) that the image of $((X, \iota, \lambda, \gamma\rho), g)$ by the morphism Θ is

$$(A, \iota_A, \lambda_A, \eta^p_A \omega(\gamma^{-1}) g)$$

We define an action of $J(\mathbb{Q})$ on the left hand side of (7.3.1) by

$$((X,\iota,\lambda,\rho),g)\longmapsto ((X,\iota,\lambda,\gamma\rho),\omega(\gamma)g).$$

Proposition 7.3.2. *The uniformization morphism (7.3.1) factors through an isomorphism*

$$\Theta: J(\mathbb{Q})\backslash(\tilde{\mathcal{M}}_r\times G(\mathbb{A}_f^p)/\mathbf{K}^p)\xrightarrow{\sim}\hat{\mathcal{A}}_{\mathbf{K}}\times_{\operatorname{Spf}O_{E_\nu}}\operatorname{Spf}O_{\breve{E}_\nu}.$$

This isomorphism is compatible with the Weil descent data relative to $O_{\breve{E}_\nu}/O_{E_\nu}$. Here the Weil descent datum on the left is induced from $\omega_{\mathcal{M}_r}$, cf. Proposition 7.2.2.

Proof. We have just proved that the morphism is well-defined. The bijectivity follows from the Proposition 7.1.11 and [27, Thm. 6.30]. □

By inserting Proposition 7.2.5 in this result, we obtain our main theorem about uniformization.

Theorem 7.3.3. *(i) The O_{E_ν}-scheme $\mathcal{A}_{\mathbf{K}}$ is projective and flat, with semi-stable reduction.*

(ii) Let $\hat{\mathcal{A}}_{\mathbf{K}}$ be the completion of $\mathcal{A}_{\mathbf{K}}$ along its special fiber, which is a formal scheme over $\operatorname{Spf}O_{E_\nu}$. *There exists an isomorphism of formal schemes over* $\operatorname{Spf}O_{\breve{E}_\nu}$,

$$J(\mathbb{Q})\backslash\big[\big(\widehat{\Omega}_{F_v}\times_{\operatorname{Spf}O_{F_v}}\operatorname{Spf}O_{\breve{E}_\nu}\big)\times\hat{G}'(\mathbb{Q}_p)/\hat{G}'(\mathbb{Z}_p)\times G(\mathbb{A}_f^p)/\mathbf{K}^p\big]\xrightarrow{\sim}\hat{\mathcal{A}}_{\mathbf{K}}\times_{\operatorname{Spf}O_{E_\nu}}\operatorname{Spf}O_{\breve{E}_\nu}\,. \quad (7.3.7)$$

For varying $\mathbf{K}^p$, this isomorphism is compatible with the action of $G(\mathbb{A}_f^p)$ through Hecke correspondences on both sides.

Let w'_r the element in the center of $\hat{G}'(\mathbb{Q}_p)$ of Definition 7.2.4. We endow the left hand with the Weil descent datum

$$(\xi,h,g)\longmapsto(\omega_{\tau_{E_\nu}}(\xi),w'_rh,g),\quad h\in\hat{G}'(\mathbb{Q}_p),\ g\in G(\mathbb{A}_f^p).$$

Then the isomorphism (7.3.7) is compatible with the Weil descent data on both sides. □

There is a simpler version of this statement, as follows. When the inertia index f_{E_ν} is even, this simpler version can be used to describe the descent datum. We define $\hat{G}(\mathbb{A}_f)=\hat{G}(\mathbb{Q}_p)\times G(\mathbb{A}_f^p)$, where we recall $\hat{G}(\mathbb{Q}_p)$ from (7.2.12), and $\hat{\mathbf{K}}=\hat{G}(\mathbb{Z}_p)\times\mathbf{K}^p$.

Corollary 7.3.4. *There is a natural isomorphism of formal schemes*

$$J(\mathbb{Q})\backslash\big[\big(\widehat{\Omega}_{F_v}\times_{\operatorname{Spf}O_{F_v}}\operatorname{Spf}O_{\breve{E}_\nu}\big)\times\hat{G}(\mathbb{A}_f)/\hat{\mathbf{K}}\big]\xrightarrow{\sim}\hat{\mathcal{A}}_{\mathbf{K}}\times_{\operatorname{Spf}O_{E_\nu}}\operatorname{Spf}O_{\breve{E}_\nu}\,. \quad (7.3.8)$$

Assume that the inertia index f_{E_ν} is even. The multiplication by p on $V\otimes\mathbb{Q}_p$ defines an element of $G(\mathbb{Q}_p)$. Let $\hat{p}$ be the image in $\hat{G}(\mathbb{Q}_p)$. We also denote by $\hat{p}$ the element

$$(\hat{p},1)\in\hat{G}(\mathbb{Q}_p)\times G(\mathbb{A}_f^p)=\hat{G}(\mathbb{A}_f).$$

If we endow the left hand side of (7.3.8) with the Weil descent datum

$$(\xi,g)\longmapsto(\omega_{\tau_{E_\nu}}(\xi),\hat{p}^{f_{E_\nu}/2}g),\quad g\in\hat{G}(\mathbb{A}_f),$$

then the morphism (7.3.8) is compatible with the Weil descent data. □

Note that $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)\xrightarrow{\sim}\hat{G}(\mathbb{Q}_p)/\hat{G}(\mathbb{Z}_p)$, as follows from $\mathbf{K}_{\mathfrak{p}_v}=\ker(c\colon G_{\mathfrak{p}_v}(\mathbb{Q}_p)\longrightarrow\mathbb{Q}_p^\times/\mathbb{Z}_p^\times)$. Hence

$$G(\mathbb{A}_f)/\mathbf{K}\simeq\hat{G}(\mathbb{A}_f)/\hat{\mathbf{K}}. \quad (7.3.9)$$

Hence Corollary 7.3.4 implies Theorem 1.2.3 in the Introduction.

7.4. **The uniformization for deeper level structures at p.** We now pass to deeper level structures. For each prime ideal $\mathfrak{p}$ of O_F with $\mathfrak{p}|p$ we have the group

$$G_{\mathfrak{p}}=\{g\in\operatorname{GL}_{O_{K_\mathfrak{p}}}(V_\mathfrak{p})\mid\varsigma_\mathfrak{p}(gx_1,gx_2)=\mu(g)\varsigma_\mathfrak{p}(x_1,x_2),\ \text{for some }\mu_\mathfrak{p}(g)\in\mathbb{Q}_p^\times\},$$

and the open compact subgroup $\mathbf{K}_\mathfrak{p}\subset G_\mathfrak{p}$ cf. (7.1.8). We will assume that there exist prime ideals $\mathfrak{p}$ which are banal since our deeper level structures exist only in this case. For each banal $\mathfrak{p}$, we choose an open subgroup of $\mathbf{K}_\mathfrak{p}^*\subset\mathbf{K}_\mathfrak{p}$. For a natural number M we consider the subgroup $\mathbf{K}_\mathfrak{p}(p^M)\subset\mathbf{K}_\mathfrak{p}$ which consists of the elements that act trivial on $\Lambda_\mathfrak{p}/p^M\Lambda_\mathfrak{p}$. We will assume that for some M

$$\mathbf{K}_\mathfrak{p}(p^M)\subset\mathbf{K}_\mathfrak{p}^* \quad (7.4.1)$$

For the special prime $\mathfrak{p}_v$ we set $\mathbf{K}^\star_{\mathfrak{p}_v} = \mathbf{K}_{\mathfrak{p}_v}$. We set

$$\mathbf{K}^\star_p = \{g = (g_\mathfrak{p}) \in G(\mathbb{Q}_p) \mid g_\mathfrak{p} \in \mathbf{K}^\star_\mathfrak{p}\}.$$

This says that $\mu_\mathfrak{p}(g_\mathfrak{p})$ is independent of $\mathfrak{p}$. We also introduce

$$\mathbf{K}^{\star,\mathrm{ba}}_p = \{(g_\mathfrak{p}) \in \prod_{\mathfrak{p},\ \mathrm{banal}} \mathbf{K}^\star_\mathfrak{p} \mid \mu_\mathfrak{p}(g_\mathfrak{p}) = c \in \mathbb{Z}_p^\times,\ \text{independent of } \mathfrak{p}\}. \tag{7.4.2}$$

This is a subgroup of

$$G^{\mathrm{ba}}(\mathbb{Q}_p) = \{(g_\mathfrak{p}) \in \prod_{\mathfrak{p},\ \mathrm{banal}} G_\mathfrak{p} \mid \mu_\mathfrak{p}(g_\mathfrak{p}) = c \in \mathbb{Q}_p^\times,\ \text{independent of } \mathfrak{p}\}.$$

Also, let $O_K^{\mathrm{ba}} = \prod_{\mathfrak{p},\ \mathrm{banal}} O_{K_\mathfrak{p}}$.

We need with some generalities on p-divisible groups suited for our special case. Let X and Y be p-divisible groups on a scheme S. We consider the category of étale morphisms $U \to S$ with the étale topology.

Definition 7.4.1. Let $n \in \mathbb{N}$. We define a sub-presheaf

$$G_n^{\mathrm{p}} \subset \underline{\mathrm{Hom}}^{\mathrm{et}}(X(n), Y(n)),$$

where the right hand side denotes the Hom in the category of étale sheaves. A homomorphism $\alpha : X(n)_U \to Y(n)_U$ belongs to $G_n^{\mathrm{p}}(U)$ if there is a profinite étale covering $\tilde{U} \to U$ and a homomorphism of p-divisible groups $\tilde{\alpha} : X_{\tilde{U}} \to Y_{\tilde{U}}$ such that the restriction of $\tilde{\alpha}$ to $X(n)_{\tilde{U}}$ is $\alpha_{\tilde{U}}$.

We denote the sheafification of G_n^{p} by G_n. We define the prosheaf

$$\underline{\mathrm{Hom}}^{\mathrm{et}}(X, Y) = {}''\varprojlim{}'' G_n.$$

The limit is taken with respect to the natural restriction maps $G_n \to G_m$ for $n > m$.

We note that a homomorphism of p-divisible groups $\tilde{\alpha} : X_{\tilde{U}} \to Y_{\tilde{U}}$ defines a homomorphism $\alpha : X(n)_U \to Y(n)_U$ iff

$$\mathrm{pr}_1^\star \tilde{\alpha} - \mathrm{pr}_2^\star \tilde{\alpha} \in p^n \operatorname{Hom}(X_{\tilde{U}\times_U \tilde{U}}, Y_{\tilde{U}\times_U \tilde{U}}). \tag{7.4.3}$$

We consider now a banal local CM-type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p})$. Let $E_\mathfrak{p} = E(r_\mathfrak{p})$ be the corresponding reflex field. Let (X, ι_X) and (Y, ι_Y) be local CM-pairs over $S/\operatorname{Spf} O_{E_\mathfrak{p}}$, which satisfy the Eisenstein condition. As above, we define $\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X, Y)$ by replacing throughout homomorphisms by homomorphisms of $O_{K_\mathfrak{p}}$-modules. The presheaf G_n^{p} is now meant in this sense. The contracting functor (cf. Definition 4.5.3) associates p-adic étale sheaves C_X and C_Y with an $O_{K_\mathfrak{p}}$-module structure. By Theorem 4.5.4,

$$\operatorname{Hom}_{O_{K_\mathfrak{p}}}(X, Y) \cong \operatorname{Hom}_{O_{K_\mathfrak{p}}}(C_X, C_Y).$$

We set $C_{n,X} = C_X/p^n C_X$. One checks easily by the remark after Definition 7.4.1 that

$$G_n^{\mathrm{p}}(U) = \operatorname{Hom}_{O_{K_\mathfrak{p}}}(C_X, C_{n,Y}) = \operatorname{Hom}_{O_{K_\mathfrak{p}}}(C_{n,X}, C_{n,Y}). \tag{7.4.4}$$

In particular $G_n^p = G_n$. We conclude that, for a scheme $S/\operatorname{Spf} O_{E_\mathfrak{p}}$, the pro-sheaf $\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X, Y)$ is a p-adic étale sheaf. Let $\iota : \omega \to S$ be a geometric point. Then we find for the fiber

$$\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X, Y)_\omega = \operatorname{Hom}_{O_{K_\mathfrak{p}}}(X_\omega, Y_\omega),$$

where the right hand side is the Hom in the cateory of p-divisible $O_{K_\mathfrak{p}}$-modules.

Let us assume that S is a scheme over $\operatorname{Spf} O_{\breve{E}_\mathfrak{p}}$. The contracting functor of Theorem 4.5.11 associates to a CM-triple (X, ι_X, λ_X) which satisfies the Eisenstein condition a p-adic étale sheaf C_X with an alternating form

$$\phi_X : C_X \times C_X \to O_{F_\mathfrak{p}}. \tag{7.4.5}$$

We set $\xi_X = \operatorname{Tr}_{F_\mathfrak{p}/\mathbb{Q}_p} \vartheta^{-1}\phi_X$. In particular there is a CM-triple $(\mathbb{X}_\mathfrak{p}, \iota_{\mathbb{X}_\mathfrak{p}}, \lambda_{\mathbb{X}_\mathfrak{p}})$ over $\bar{\kappa}_{\breve{E}_\mathfrak{p}}$ such that

$$(C_{\mathbb{X}_\mathfrak{p}}, \xi_{\lambda_{\mathbb{X}_\mathfrak{p}}}) \cong (\Lambda_\mathfrak{p}, \varsigma_\mathfrak{p}), \tag{7.4.6}$$

cf. (7.2.7).

The group $\mathbf{K}_{\mathfrak{p}}$ acts on the right hand side by similitudes. Therefore we obtain a homomorphism $\mathbf{K}_{\mathfrak{p}} \to \mathrm{Aut}_{O_{K_{\mathfrak{p}}}} \mathbb{X}_{\mathfrak{p}}$ such that the automorphisms in the image respect the polarization $\lambda_{\mathbb{X}_{\mathfrak{p}}}$ up to a factor in $\mathbb{Z}_p^{\times}$.

Definition 7.4.2. Let $\mathfrak{p}$ be banal and let (X, ι_X, λ_X) be a CM-triple on S which satisfies the Eisenstein condition as above. A CL-level structure on (X, ι_X, λ_X) as a class of isomorphisms of p-adic étale sheaves

$$(\Lambda_{\mathfrak{p}}, \varsigma_{\mathfrak{p}}) \xrightarrow{\sim} (C_X, \xi_{\lambda_X}) \quad \mathrm{mod}\ \mathbf{K}_{\mathfrak{p}}^{\star}, \tag{7.4.7}$$

which respect the bilinear forms on both sides up to a factor in $\mathbb{Z}_p^{\times}$. We will write

$$\mathbb{X}_{\mathfrak{p}} \xrightarrow{\sim} X \quad \mathrm{mod}\ \mathbf{K}_{\mathfrak{p}}^{\star}. \tag{7.4.8}$$

for a CL-structure.

More precisely this means the following. Let $M \geq 1$ such that (7.4.1) holds. Then a CL-level structure is a right $\mathbf{K}_{\mathfrak{p}}^{\star}/(\mathbf{K}_{\mathfrak{p}}(p^M))$-torsor

$$T \subset \underline{\mathrm{Isom}}_{O_{K_{\mathfrak{p}}}}(\Lambda_{\mathfrak{p}} \otimes \mathbb{Z}/p^M\mathbb{Z}, C_X \otimes \mathbb{Z}/p^M\mathbb{Z})$$

such that the inclusion is equivariant with respect to the right actions of $\mathbf{K}_{\mathfrak{p}}^{\star}/(\mathbf{K}_{\mathfrak{p}}(p^M))$ on both sides and such that the local sections of T respect the bilinear forms on $\Lambda_{\mathfrak{p}}$ and C_X up to a factor in $(\mathbb{Z}/p^M\mathbb{Z})^{\times}$. If S is connected and $\omega \to S$ is a geometric point a CL-structure is given by a $\mathbf{K}_{\mathfrak{p}}^{\star}$-orbit of an isomorphism $\Lambda_{\mathfrak{p}} \to (C_X)_{\omega}$ which respects the bilinear forms on both sides by a factor in $\mathbb{Z}_p^{\times}$ and such that the orbit is preserved by the action of $\pi_1(S, \omega)$. This explains the notation (7.4.8).

Let (X, ι, λ) be a semi-local CM-triple relative to $(K \otimes \mathbb{Q}_p/F \otimes \mathbb{Q}_p, r)$ over a scheme $S \in (\mathrm{Sch}/\operatorname{Spf} O_{\breve{E}_{\nu}})$, cf. the beginning of section 7.2. We set $X^{\mathrm{ba}} = \prod_{\mathfrak{p},\,\mathrm{banal}} X_{\mathfrak{p}}$. We choose $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as in section 7.2. Then (7.4.6) holds. From this we obtain an action of $\mathbf{K}_p^{\star,\mathrm{ba}}$ on $\mathbb{X}^{\mathrm{ba}}$ which respects the polarization $\prod_{\mathfrak{p},\,\mathrm{banal}} \lambda_{\mathbb{X}_{\mathfrak{p}}}$ up to a factor in $\mathbb{Z}_p^{\times}$.

We define a CL-level structure on $\mathbb{X}^{\mathrm{ba}}$ modulo $\mathbf{K}_p^{\star,\mathrm{ba}}$ as a CL-level stuctures $\bar{\eta}_{\mathfrak{p}} : \mathbb{X}_{\mathfrak{p}} \xrightarrow{\sim} X_{\mathfrak{p}}$ mod $\mathbf{K}_{\mathfrak{p}}^{\star}$, for each banal $\mathfrak{p}$ which respect the bilinear forms up to a factor in $\mathbb{Z}_p^{\times}$ that is independent of $\mathfrak{p}$.

Definition 7.4.3. With the notations of Definition 7.2.2, let $i \in \mathbb{Z}$. Let $\mathcal{M}_{\mathbf{K}_p^{\star}}(i)$ be the following functor on the category of schemes S over $\operatorname{Spf} O_{\breve{E}_{\nu}}$. We will write $\bar{S} = S \times_{\operatorname{Spf} O_{\breve{E}_{\nu}}} \operatorname{Spec} \bar{\kappa}_{E_{\nu}}$. A point of $\mathcal{M}_{\mathbf{K}_p^{\star}}(i)(S)$ is given by the following data:

(1) A CM-triple (X, ι, λ) of type $(K \otimes \mathbb{Q}_p/F \otimes \mathbb{Q}_p, r)$ over S which satisfies the conditions (KC_r) and (EC_r) and is compatible with (V, ς).

(2) A $O_K \otimes \mathbb{Z}_p$-linear quasi-isogeny

$$\rho : \bar{X} := X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \kappa_{\breve{E}_{\nu}}} \bar{S}$$

such that ρ respects the polarization $p^i\lambda$ on X and $\lambda_{\mathbb{X}}$ up to a factor in $\mathbb{Z}_p^{\times}$.

(3) Let $X^{\mathrm{ba}} = \prod_{\mathfrak{p}\ \mathrm{banal}} X_{\mathfrak{p}}$. A CL-level structure

$$\bar{\eta} : \mathbb{X}^{\mathrm{ba}} \xrightarrow{\sim} X^{\mathrm{ba}} \ \mathrm{mod}\ \mathbf{K}_p^{\star,\mathrm{ba}}.$$

We set

$$\tilde{\mathcal{M}}_{\mathbf{K}_p^{\star}} = \coprod_{i \in \mathbb{Z}} \mathcal{M}_{\mathbf{K}_p^{\star}}(i).$$

We formulate a variant of Corollary 7.2.7. Since we assume that banal places exist, we do not need the group $\hat{G}$.

Proposition 7.4.4. *There exists an isomorphism*

$$\tilde{\mathcal{M}}_{\mathbf{K}_p^{\star}} \xrightarrow{\sim} (\hat{\Omega}_{F_v} \times_{\operatorname{Spf} O_{F_v}} \operatorname{Spf} O_{\breve{E}_{\nu}}) \times G^{\mathrm{ba}}(\mathbb{Q}_p)/\mathbf{K}_p^{\star,\mathrm{ba}},$$

which is equivariant with respect to the action of $J(\mathbb{Q}_p)$ on both sides.

Proof. At the banal places we may use lisse p-adic étale sheaves to describe a point of $\tilde{\mathcal{M}}_{\mathbf{K}_p^\star}(S)$. A point consists of a CM-triple $(X_{\mathfrak{p}_v}, \iota_{\mathfrak{p}_v}, \lambda_{\mathfrak{p}_v})$ and a quasi-isogeny $\rho_{\mathfrak{p}_v} : X_{\mathfrak{p}_v,\bar{S}} \to \mathbb{X}_{\mathfrak{p}_v}$ such that $\rho^\star_{\mathfrak{p}_v}(\lambda_{\mathbb{X}_{\mathfrak{p}_v}}) = up^i \lambda_{X_{\mathfrak{p}_v}}$ for some $u \in \mathbb{Z}_p^\times$, $i \in \mathbb{Z}$ and an isomorphism of lisse p-adic étale sheaves on $\bar{S}$,

$$(C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \xrightarrow{\eta} (C, \xi) \subset (C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \otimes \mathbb{Q}, \tag{7.4.9}$$

where η respects the alternating forms up to a factor in $\mathbb{Z}_p^\times$ and such that the restriction of $\xi_{\mathbb{X}^{\mathrm{ba}}}$ with respect to the last inclusion is equal to $up^i\xi$ with the same u and i as above. By (7.2.7) we have

$$(C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \cong (\Lambda^{\mathrm{ba}}, \varsigma^{\mathrm{ba}}), \tag{7.4.10}$$

where the right hand side is the orthogonal direct sum over all $(\Lambda_{\mathfrak{p}}, \varsigma_{\mathfrak{p}})$ for $\mathfrak{p}$ banal.

We denote by $\tilde{\mathcal{M}}^{\mathrm{ba}}_{\mathbf{K}_p^\star}$ the moduli functor described by the data (7.4.9). We claim that there is a natural isomorphism

$$\tilde{\mathcal{M}}^{\mathrm{ba}}_{\mathbf{K}_p^\star} \cong G^{\mathrm{ba}}(\mathbb{Q}_p)/\mathbf{K}_p^{\star,\mathrm{ba}}. \tag{7.4.11}$$

Indeed, the group $G^{\mathrm{ba}}(\mathbb{Q}_p)$ acts naturally on this functor: Let $g \in G^{\mathrm{ba}}(\mathbb{Q}_p)$ such that

$$\phi_{\mathbb{X}}(gx_1, gx_2) = u'p^j \xi_{\mathbb{X}}(x_1, x_2).$$

Then g maps (7.4.9) to

$$(C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \xrightarrow{g\eta} (gC, (1/u'p^j)\xi) \subset C_{\mathbb{X}^{\mathrm{ba}}} \otimes \mathbb{Q}.$$

If we have an arbitrary point (7.4.9), then the composite of the arrow with the inclusion is an element $g \in G^{\mathrm{ba}}(\mathbb{Q}_p)$ and therefore (7.4.9) is isomorphic to

$$C_{\mathbb{X}^{\mathrm{ba}}} \xrightarrow{g} (gC, (1/u'p^j)\xi_{\mathbb{X}^{\mathrm{ba}}}) \subset C_{\mathbb{X}^{\mathrm{ba}}} \otimes \mathbb{Q}.$$

We see that the action is transitive and that the stabilizer of the base point

$$(C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \xrightarrow{\mathrm{id}} (C_{\mathbb{X}^{\mathrm{ba}}}, \xi_{\mathbb{X}^{\mathrm{ba}}}) \subset C_{\mathbb{X}^{\mathrm{ba}}} \otimes \mathbb{Q}$$

is $\mathbf{K}_p^{\star,\mathrm{ba}}$. This shows (7.4.11).

Now we fix $i \in \mathbb{Z}$. We denote by $\mathcal{M}_{r_{\mathfrak{p}_v}}$ the functor of section 6 associated to the special local CM-type $(K_{\mathfrak{p}_v}/F_{\mathfrak{p}_v}, r_{\mathfrak{p}_v})$. There is the natural injection of functors

$$\mathcal{M}_{\mathbf{K}_p^\star}(i) \to \mathcal{M}_{r_{\mathfrak{p}_v}}(i) \times \tilde{\mathcal{M}}^{\mathrm{ba}}_{\mathbf{K}_p^\star}(i).$$

We claim that this map is surjective. Indeed, assume we are given a point $(X_{\mathfrak{p}_v}, \iota_{\mathfrak{p}_v}, \lambda_{\mathfrak{p}_v}, \rho^\star_{\mathfrak{p}_v})$, where $\rho^\star_{\mathfrak{p}_v}(\lambda_{\mathbb{X}_{\mathfrak{p}_v}}) = u_1 p^i \lambda_{\mathfrak{p}_v}$, with $u_1 \in \mathbb{Z}_p^\times$, from the first factor on the right hand side, and a point $(C, \xi) \subset C_{\mathbb{X}^{\mathrm{ba}}}$ (endowed with η), where $u_2 p^i \xi = \xi_{\mathbb{X}^{\mathrm{ba}}}$ with $u_2 \in \mathbb{Z}_p^\times$, from the second factor. These two data form a point of $\mathcal{M}_{\mathbf{K}_p^\star}(i)$ iff $u_1 = u_2$. But in the point from the first factor we can replace $\lambda_{\mathfrak{p}_v}$ by $(u_1/u_2)\lambda_{\mathfrak{p}_v}$ without changing the isomorphism class of this point. Therefore the surjectivity holds and the proposition follows as Proposition 7.2.5. □

Let us fix an open and compact subgroup $\mathbf{K}^p \subset G(\mathbb{A}_f^p)$. We set $\mathbf{K}^\star = \mathbf{K}_p^\star \mathbf{K}^p$ and $\mathbf{K} = \mathbf{K}_p \mathbf{K}^p$ as after (7.1.8). We choose $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as above.

Definition 7.4.5. We define a functor $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}$ on the category of schemes S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$. A point of $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}(S)$ consists of the following data:

(1) a point $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ of $\mathcal{A}_{\mathbf{K}}(S)$,

(2) a CL-level structure

$$\bar{\eta}_p : \mathbb{X}^{\mathrm{ba}} \to A[p^\infty]^{\mathrm{ba}} \bmod \mathbf{K}_p^{\star,\mathrm{ba}}.$$

We denote here by $A[p^\infty]^{\mathrm{ba}}$ the banal part of the p-divisible group of A with its structure of a semi-local CM-triple. The morphism $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star} \to \hat{\mathcal{A}}_{\mathbf{K}}$ is a finite étale covering of formal schemes. Since we assume that $\mathbf{K}^p$ is small enough, $\mathcal{A}_{\mathbf{K}}$ is a proper scheme over $\mathrm{Spec}\, O_{\breve{E}_\nu}$. By the algebraization theorem, there is a unique finite étale morphism of schemes over $\mathrm{Spec}\, O_{\breve{E}_\nu}$

$$\mathcal{A}^\star_{\mathbf{K}^\star} \to \mathcal{A}_{\mathbf{K}} \times_{\mathrm{Spec}\ O_{E,(\mathfrak{p}_\nu)}} \mathrm{Spec}\ O_{\breve{E}_\nu}, \tag{7.4.12}$$

such that the p-adic completion of $\mathcal{A}^\star_{\mathbf{K}^\star}$ is $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}$.

Recall the projective scheme $\mathcal{A}_{\mathbf{K}^\star,E}$ over E from section 1.2 (the canonical model of $\mathrm{Sh}_{\mathbf{K}^\star}$), comp. the proof of Proposition 7.1.5. We will now relate $\mathcal{A}_{\mathbf{K}^\star,E}$ with the general fiber $\mathcal{A}^\star_{\mathbf{K}^\star} \times_{\operatorname{Spec} O_{\breve{E}_\nu}} \operatorname{Spec} \breve{E}_\nu$. We start with a reformulation of the level structure $\bar{\eta}_p$ in Definition 7.4.5.

We assume that S is a scheme over $\operatorname{Spf} O_{\breve{E}_\mathfrak{p}}$, i.e., we pass to the completion of the maximal unramified extension of $E_\mathfrak{p}$. We consider now a polarized local CM-pair (X, ι_X, λ_X) over S of CM-type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p})$, cf. Definition 4.1.2. We will always assume that the Eisenstein conditions are satisfied. By Theorem 4.5.11, λ_X is described by a $O_{F_\mathfrak{p}}$-bilinear form ϕ_{λ_X}, or also

$$\xi_{\lambda_X} : C_X \times C_X \to \mathbb{Z}_p,$$

as defined after (7.4.5). Equivalently, we can consider the $O_{K_\mathfrak{p}}$-anti-hermitian form

$$\varkappa_{\lambda_X} : C_X \times C_X \to K_\mathfrak{p}, \tag{7.4.13}$$

which is defined by

$$\operatorname{Tr}_{K_\mathfrak{p}/F_\mathfrak{p}} a\varkappa_{\lambda_X}(c_1, c_2) = \phi_{\lambda_X}(ac_1, c_2), \quad a \in O_{K_\mathfrak{p}},\ c_1, c_2 \in C_X.$$

Then $\varkappa_{\lambda_X}$ is $O_{K_\mathfrak{p}}$-linear in the first variable and $O_{K_\mathfrak{p}}$-anti-linear in the second variable.

If we define $(\mathbb{X}, \iota_\mathbb{X}, \lambda_\mathbb{X})$ by $(C_\mathbb{X}, \xi_{\lambda_\mathbb{X}}) \cong (\Lambda_\mathfrak{p}, \varsigma_\mathfrak{p})$ (cf. (7.4.6)), we can reformulate (7.4.7): A CL-structure is a class of isomorphisms

$$(\mathbb{X}, \lambda_\mathbb{X}) \to (X, \lambda_X) \mod \mathbf{K}^\star_\mathfrak{p}$$

which respect the polarizations up to a factor in $\mathbb{Z}_p^\times$. This agrees with Definition 7.4.5.

Let $\bar{\kappa}_{E_\mathfrak{p}}$ be the residue class field of $\breve{E}_\mathfrak{p}$. We will consider CM-pairs (Z, ι_Z) of CM-type $r_\mathfrak{p}/2$ over a scheme $S/\operatorname{Spf} O_{\breve{E}_\mathfrak{p}}$. Then Z is a p-divisible group of height $2d_\mathfrak{p}$ and dimension $d_\mathfrak{p}$, where $d_\mathfrak{p} = [K_\mathfrak{p} : F_\mathfrak{p}]$. We will always assume that the Eisenstein condition is fulfilled. Proposition 4.2.2 continues to hold with the same polynomials $\mathbf{E}_{A_\psi}$. The functor C_Z (cf. Definition 4.5.3) exists for local CM-pairs of type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p}/2)$.

We will reformulate CL-level structures as suggested by [24]. There is up to isomorphism a unique CM-pair $(\bar{X}_0, \bar{\iota}_0)$ of CM-type $r_\mathfrak{p}/2$ over $\bar{\kappa}_{E_\mathfrak{p}}$. It lifts uniquely to a CM-pair (X_0, ι_0) over $O_{\breve{E}_\mathfrak{p}}$, and

$$C_{X_0} \cong O_{K_\mathfrak{p}} \tag{7.4.14}$$

is the constant p-adic sheaf. We consider biextensions

$$\beta : X_0 \times X_0 \to \hat{\mathbb{G}}_m$$

or, equivalently, bilinear forms of displays as in Proposition 4.5.9. They are in bijection with bilinear forms

$$\phi : C_{X_0} \times C_{X_0} \to O_{F_\mathfrak{p}}. \tag{7.4.15}$$

Equivalently we use $\xi = \xi_\phi$ or $\varkappa = \varkappa_\phi$ as before (7.4.13).

We define $\varkappa_0 : C_0 \times C_0 \to O_{K_\mathfrak{p}}$ using (7.4.14), by

$$\varkappa_0(x, y) = x\bar{y}. \quad x, y \in O_{K_\mathfrak{p}}. \tag{7.4.16}$$

We denote by $\mathfrak{s}_0 : X_0 \to X_0^\wedge$ the homomorphism associated to ϕ_0. This homomorphism is symmetric.

We note that there is a principal polarization λ on X_0. It is a generator of the free $O_{K_\mathfrak{p}}$-module of rank one $\operatorname{Hom}_{O_{K_\mathfrak{p}}}(X_0, X_0^\wedge)$. In the case where $K_\mathfrak{p}/F_\mathfrak{p}$ is ramified, the corresponding form under the bijection (7.4.15) is

$$\phi_\lambda(x, y) = \operatorname{Tr}_{K_\mathfrak{p}/F_\mathfrak{p}} \Pi^{-1} x\bar{y}.$$

Then $\mathfrak{s}_0 = \lambda\Pi$. In the case where $K_\mathfrak{p}/F_\mathfrak{p}$ is unramified, we choose a unit $\varepsilon \in O_{K_\mathfrak{p}}$ such that $\varepsilon + \bar{\varepsilon} = 0$. Then the corresponding form under the bijection (7.4.15) is

$$\phi_\lambda(x, y) = \operatorname{Tr}_{K_\mathfrak{p}/F_\mathfrak{p}} \varepsilon^{-1} x\bar{y}.$$

Then $\mathfrak{s}_0 = \lambda\varepsilon$.

Let S be a p-adic formal scheme over $\operatorname{Spf} O_{\breve{E}_\mathfrak{p}}$. Let (X, ι_X, λ_X) be a polarized CM-pair of type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p})$ which satisfies the Eisenstein condition as always required. We endow $\underline{\operatorname{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X)$ with an $O_{K_\mathfrak{p}}$-anti-hermitian form with values in $K_\mathfrak{p}$. Let $u_1, u_2 \in C_n(U)$. They are given by homomorphisms $\tilde{u}_1, \tilde{u}_2 : X_0 \to X$ which are defined over a profinite étale covering $\tilde{U} \to U$. We consider the homomorphism

$$\tilde{u}_2^\wedge \lambda_X \tilde{u}_1 : X_0 \to X \to X^\wedge \to X_0^\wedge.$$

This element of $\mathrm{Hom}_{O_{K_\mathfrak{p}}}((X_0)_{\tilde{U}}, (X_0^\wedge)_{\tilde{U}})$ may be written as

$$\tilde{u}_2^\wedge \lambda_X \tilde{u}_1 = \tilde{\delta}(\tilde{u}_1, \tilde{u}_2)\mathfrak{s}_0, \tag{7.4.17}$$

with some constant $\tilde{\delta}(\tilde{u}_1, \tilde{u}_2) \in K_\mathfrak{p}$. In the ramified case,

$$\delta(u_1, u_2) := \tilde{\delta}(\tilde{u}_1, \tilde{u}_2) \mod p^n \Pi^{-1} O_{O_{K_\mathfrak{p}}}$$

is well defined. In the unramified case, the element $\tilde{\delta}(\tilde{u}_1, \tilde{u}_2)$ is well-defined modulo $p^n O_{K_\mathfrak{p}}$. Varying n, we therefore obtain a bilinear form

$$\delta : \underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X) \times \underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X) \to K_\mathfrak{p}.$$

This is a $O_{K_\mathfrak{p}}$-anti-hermitian form. We set

$$\mathfrak{e} = \mathrm{Tr}_{F_\mathfrak{p}/\mathbb{Q}_p} \mathrm{Tr}_{K_\mathfrak{p}/F_\mathfrak{p}} \vartheta^{-1}_{F_\mathfrak{p}/\mathbb{Q}_p} \delta.$$

Then $\mathfrak{e}$ is an alternating form

$$\mathfrak{e} : \underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X) \times \underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X) \to \mathbb{Z}_p.$$

which satisfies $\mathfrak{e}(au_1, u_2) = \mathfrak{e}(u_1, \bar{a}u_2)$, $a \in O_{K_\mathfrak{p}}$.

Proposition 7.4.6. *A CL-level structure on a polarized CM-pair (X, ι_X, λ_X) of type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p})$ over the p-adic formal scheme S can equivalently be given as a class of isomorphisms of p-adic étale sheaves*

$$\eta : (\Lambda_\mathfrak{p}, \varsigma_\mathfrak{p}) \xrightarrow{\sim} (\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X), \mathfrak{e}) \mod \mathbf{K}^\star_\mathfrak{p}, \tag{7.4.18}$$

which respect the bilinear forms on both sides up to a constant in $\mathbb{Z}_p^\times$.

Proof. Indeed, we apply the contracting functor to the right hand side of the isomorphism (7.4.18). We view $\tilde{u}_1, \tilde{u}_2$ from (7.4.17) as homomorphisms

$$\tilde{u}_i : O_{K_\mathfrak{p}} = C_{X_0} \to C_X.$$

Let $\varkappa_{\lambda_X} : C_X \times C_X \to K_\mathfrak{p}$ be the anti-hermitian form induced by λ_X. The definition (7.4.17) of the sesqui-linear form δ which gives rise to $\mathfrak{e}$, reads in terms of the contracting functor as defined by (7.4.17)

$$\varkappa_{\lambda_X}(\tilde{u}_1(x), \tilde{u}_2(y)) = \tilde{\delta}(\tilde{u}_1, \tilde{u}_2)x\bar{y}. \tag{7.4.19}$$

If we identify

$$\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_0, X) = \underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(O_{K_\mathfrak{p}}, C_X) = C_X$$

by sending $\tilde{u}$ to $\tilde{u}(1)$, the form $\tilde{\delta}$ is mapped to the form $\varkappa_{\lambda_X}$. This is immediate by setting $x = y = 1$ in (7.4.19). Therefore we have identified the right hand side of (7.4.18) with (C_X, ξ_{λ_X}). This proves the assertion. □

Proposition 7.4.7. *Let S be a flat proper scheme over $\mathrm{Spec}\, O_{\breve{E}_\mathfrak{p}}$. Let (X, ι_X) and (Y, ι_Y) be CM-pairs of type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p})$ or $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p}/2)$ over S. Let $U \to S$ be a finite étale covering, and $\hat{U} \to \mathrm{Spf}\, O_{\breve{E}_\mathfrak{p}}$ its formal completion along the special fibre. Set $U_\eta = U \times_{\mathrm{Spec}\, O_{\breve{E}_\mathfrak{p}}} \mathrm{Spec}\, \breve{E}_\mathfrak{p}$. Then there is a natural bijective homomorphism*

$$G_n(\hat{U}) \to \mathrm{Hom}_{O_{K_\mathfrak{p}}}(X(n)_{U_\eta}, Y(n)_{U_\eta}). \tag{7.4.20}$$

In particular, the p-adic étale sheaf $\underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(T_p(X_{S_\eta}), T_p(Y_{S_\eta}))$ is unramified along the special fibre of S.

Proof. We consider the natural map $G_n(\hat{U}) \to \mathrm{Hom}_{O_{K_\mathfrak{p}}}(X(n)_{\hat{U}}, Y(n)_{\hat{U}})$. By Grothendieck's existence theorem (EGA III, Thm. 5.1.4), the target of this arrow coincides with $\mathrm{Hom}_{O_{K_\mathfrak{p}}}(X(n)_U, Y(n)_U)$. If we restrict the last set of homomorphisms to the generic fibre we obtain the map (7.4.20).

The injectivity of (7.4.20) follows from the definition of G^n. To prove surjectivity, we can assume that U is connected. By Grothendieck's existence theorem we find a finite connected étale covering $U_1 \to U$ such that the sheaves $C_{n,X_{\hat{S}}}$ and $C_{n,Y_{\hat{S}}}$ become trivial over $\hat{U}_1$.

We write the proof only in the case where X and Y are of CM-type $r_\mathfrak{p}$. The cases where $r_\mathfrak{p}/2$ appears will be obvious. By the choice of U_1, we deduce the isomorphism

$$G_n(\hat{U}_1) \cong \mathrm{Hom}_{O_{K_\mathfrak{p}}}((O_{K_\mathfrak{p}}/p^n O_{K_\mathfrak{p}})^2, (O_{K_\mathfrak{p}}/p^n O_{K_\mathfrak{p}})^2).$$

We choose a geometric point ω of $(U_1)_\eta$. Then we obtain injective homomorphisms

$$\begin{gathered} G_n(\hat{U}_1) \to \mathrm{Hom}_{O_{K_\mathfrak{p}}}(X(n)_{U_{1,\eta}}, Y(n)_{U_{1,\eta}}) = \\ \underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(T_p(X_\eta) \otimes \mathbb{Z}/(p^n), T_p(Y_\eta) \otimes \mathbb{Z}/(p^n))(U_{1,\eta}) \longrightarrow \\ \mathrm{Hom}_{O_{K_\mathfrak{p}}}(T_p(X_\omega), T_p(Y_\omega)) \otimes \mathbb{Z}/(p^n) \cong \mathrm{Hom}_{O_{K_\mathfrak{p}}}((O_{K_\mathfrak{p}})^2, (O_{K_\mathfrak{p}})^2) \otimes \mathbb{Z}/(p^n). \end{gathered}$$

Since we have the same number of elements on both sides, the arrows are bijective. In particular this shows that the étale sheaf $\underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(T_p(X_\eta), T_p(Y_\eta)) \otimes \mathbb{Z}/(p^n)$ becomes trivial over the finite étale covering $U_{1,\eta} \to S_\eta$. Therefore it is unramified along the special fibre of S.

Finally, we obtain the bijectivity of (7.4.20) by exploiting the sheaf property with respect to the covering

$$U_1 \times_U U_1 \rightrightarrows U_1 \to U.$$

□

Corollary 7.4.8. *With the assumptions of the last Proposition, there is a p-adic étale sheaf* $\underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(X, Y)$ *on* S *whose restriction to the special fibre* $S \times_{\mathrm{Spec}\, O_{\breve{E}_\mathfrak{p}}} \mathrm{Spec}\, \bar{\kappa}_{E_\mathfrak{p}}$ *is* $\underline{\mathrm{Hom}}^{\mathrm{et}}_{O_{K_\mathfrak{p}}}(X_{\bar{\kappa}_{E_\mathfrak{p}}}, Y_{\bar{\kappa}_{E_\mathfrak{p}}})$ *and whose restriction to the general fibre* $S \times_{\mathrm{Spec}\, O_{\breve{E}_\mathfrak{p}}} \mathrm{Spec}\, \breve{E}_\mathfrak{p}$ *is* $\underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(T_p(X_{\breve{E}_\mathfrak{p}}), T_p(Y_{\breve{E}_\mathfrak{p}}))$.

Proof. The sheaves G_n over $\hat{S}$ are representable by finite étale morphisms of formal schemes. They come therefore from finite étale morphisms $G_n^{\mathrm{al}} \to S$. We have to compare the general fibre of G_n^{al} with $\underline{\mathrm{Hom}}_{O_{K_\mathfrak{p}}}(T_p(X_\eta), T_p(Y_\eta)) \otimes \mathbb{Z}/(p^n)$.

We have shown that both sheaves are trivialized by a finite étale covering $S_1 \to S$. The homomorphism (7.4.20) gives a canonical isomorphism between these sheaves with constant étale sheaves on $S_1 \times_{\mathrm{Spec}\, O_{\breve{E}_\mathfrak{p}}} \mathrm{Spec}\, \kappa_{\breve{E}_\mathfrak{p}}$. Finally, we consider descent for the general fiber of the covering

$$S_1 \times_S S_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} S_1 \to S$$

We see that the descent data for the two sheaves agree since they are induced from the descent datum on the étale sheaf $\underline{\mathrm{Hom}}(X(n), Y(n))$. □

We now go back to the Definition 7.4.5. We choose for each banal $\mathfrak{p}$ a CM-pair $(X_{\mathfrak{p},0}, \iota_{\mathfrak{p},0})$ of local CM-type $(K_\mathfrak{p}/F_\mathfrak{p}, r_\mathfrak{p}/2)$ over $\mathrm{Spf}\, O_{\breve{E}_\nu}$. We may assume that $C_{X_{\mathfrak{p},0}} = O_{K_\mathfrak{p}}$. We endow $C_{X_{\mathfrak{p},0}}$ with the hermitian form (7.4.16) which corresponds to the symmetric homomorphism $\mathfrak{s}_{\mathfrak{p},0} : X_{\mathfrak{p},0} \to X^\wedge_{\mathfrak{p},0}$. We define $X_0^{\mathrm{ba}} = \prod_{\mathfrak{p},\ \mathrm{banal}} X_{\mathfrak{p},0}$ and we endow it with $\mathfrak{s}_0^{\mathrm{ba}} = \prod \mathfrak{s}_{\mathfrak{p},0}$. Then by Proposition 7.4.6 we may replace (2) in Definition 7.4.5 by

(2′) *A class* $\bar{\eta}_p$ *of isomorphisms of p-adic étale sheaves,*

$$\eta_p : \Lambda^{\mathrm{ba}} \to \underline{\mathrm{Hom}}_{O_K^{\mathrm{ba}}}(X_0^{\mathrm{ba}}, A[p^\infty]^{\mathrm{ba}}) \bmod \mathbf{K}_p^{\star,\mathrm{ba}},$$

which respect the forms on both sides up to a constant in $\mathbb{Z}_p^\times$.

The lisse p-adic sheaf on $\hat{\mathcal{A}}_\mathbf{K}$ given by the right hand side of (2′) is the algebraization of a lisse p-adic sheaf on $\mathcal{A}_\mathbf{K}$ which exists because this scheme is proper over $\mathrm{Spec}\, O_{\breve{E}_\nu}$. We denote this sheaf by the same symbol. Then the scheme $\mathcal{A}^\star_{\mathbf{K}^\star}$ is given by the following functor on the category of schemes S over $\mathrm{Spec}\, O_{\breve{E}_\nu}$: A point of $\mathcal{A}^\star_{\mathbf{K}^\star}(S)$ consists of a point $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ of $\mathcal{A}_\mathbf{K}(S)$ and a class $\bar{\eta}_p$ as in (2′). We deduce the following description of $\mathcal{A}^\star_{\mathbf{K}^\star}$.

Proposition 7.4.9. *The scheme $\mathcal{A}^{\star}_{\mathbf{K}^\star} \times_{\operatorname{Spec} O_{\breve{E}_\nu}} \operatorname{Spec} \breve{E}_\nu$ represents the following functor on the category of $\breve{E}_\nu$-schemes. A T-valued point is a point $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$ of $\mathcal{A}_{\mathbf{K}}(T)$ and a class $\bar{\eta}_p$ of isomorphisms of p-adic étale sheaves*

$$\eta_p : \Lambda^{\mathrm{ba}} \xrightarrow{\sim} \underline{\mathrm{Hom}}_{O_K^{\mathrm{ba}}}(T_p((X_0^{\mathrm{ba}})_{\breve{E}_\nu}), T_p(A)^{\mathrm{ba}}) \bmod \mathbf{K}_p^{\star,\mathrm{ba}}, \tag{7.4.21}$$

which respect the forms on both sides up to a constant in $\mathbb{Z}_p^\times$.

The scheme $\mathcal{A}^{\star}_{\mathbf{K}^\star}$ is the normalization of $\mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} O_{E,(\mathfrak{p}_\nu)}} \operatorname{Spec} O_{\breve{E}_\nu}$ in $\mathcal{A}^{\star}_{\mathbf{K}^\star} \times_{\operatorname{Spec} O_{\breve{E}_\nu}} \operatorname{Spec} \breve{E}_\nu$ and is finite and étale over $\mathcal{A}_{\mathbf{K}} \times_{\operatorname{Spec} O_{E,(\mathfrak{p}_\nu)}} \operatorname{Spec} O_{\breve{E}_\nu}$. □

Theorem 7.4.10. *Let $\breve{E}_\nu^{\mathrm{ab}}$ be the maximal abelian extension of $\breve{E}_\nu$. Then there is an isomorphism*

$$\mathcal{A}_{\mathbf{K}^\star, E} \times_{\operatorname{Spec} E} \operatorname{Spec} \breve{E}_\nu^{\mathrm{ab}} \cong \mathcal{A}^{\star}_{\mathbf{K}^\star} \times_{\operatorname{Spec} O_{\breve{E}_\nu}} \operatorname{Spec} \breve{E}_\nu^{\mathrm{ab}} \tag{7.4.22}$$

which is natural in $\mathbf{K}^\star$.

Proof. We make explicit what a level structure (7.4.21) means after base change to $\breve{E}_\nu^{\mathrm{ab}}$. Over $\breve{E}_\nu^{\mathrm{ab}}$ we may choose an isomorphism $\mathbb{Z}_p(1) \cong \mathbb{Z}_p$ and therefore we do not need to worry about Tate twists. The Tate module $T_p(X_{\mathfrak{p},0})$ of $X_{\mathfrak{p},0}$ over an algebraic closure of $\breve{E}_\nu$ is an $O_{K_\mathfrak{p}}$-module which is free of rank 1. Therefore the Galois group of $\breve{E}_\nu$ acts on the Tate-module via its maximal abelian quotient. We choose an isomorphism

$$T_p(X_{\mathfrak{p},0}) \cong O_{K_\mathfrak{p}} \tag{7.4.23}$$

such that the action of the Galois group of $\breve{E}_\nu^{\mathrm{ab}}$ on both sides is trivial. The symmetric map $\mathfrak{s}_\mathfrak{p} : X_{\mathfrak{p},0} \to X_{\mathfrak{p},0}^\wedge$ induces a hermitian form $\varkappa_{\mathfrak{p},0}$ on the Tate-module (7.4.23). We find

$$\varkappa_{\mathfrak{p},0}(x,y) = c_{\mathfrak{p},0} x\bar{y}, \quad x, y \in O_K$$

for some constant $c_{\mathfrak{p},0} \in O_{F_\mathfrak{p}}^\times$. Note that in the ramified case two isomorphism classes are possible for $\varkappa_{\mathfrak{p},0}$.

We consider a T-valued point $(A, \iota, \bar{\lambda}, \bar{\eta}^p, \bar{\eta}_p)$ from the right hand side of (7.4.22). Let $X = \prod X_\mathfrak{p}$ be the p-divisible group of A. A polarization from $\bar{\lambda}$ induces an anti-hermitian pairing $\varkappa_\mathfrak{p}$ on $T_p(X_\mathfrak{p})$. The anti-hermitian form $\delta_\mathfrak{p}$ on $\underline{\mathrm{Hom}}(T_p(X_{\mathfrak{p},0}), T_p(X_\mathfrak{p}))$ is given by

$$\varkappa_\mathfrak{p}(u_1(x), u_2(y)) = \delta_\mathfrak{p}(u_1, u_2) c_{\mathfrak{p},0} x\bar{y}, \quad x, y \in O_{K_\mathfrak{p}}, \tag{7.4.24}$$

where $u_1, u_2 \in \underline{\mathrm{Hom}}(T_p(X_{\mathfrak{p},0}), T_p(X_\mathfrak{p}))$ are sections.

For an $O_{K_\mathfrak{p}}$-lattice $(\Gamma, \varkappa_\Gamma)$ with an anti-hermitian form $\varkappa_\Gamma : \Gamma \times \Gamma \to K_\mathfrak{p}$, we write $\Gamma[c] = (\Gamma, c\varkappa_\Gamma)$. The equation (7.4.24) gives an isomorphism

$$(\underline{\mathrm{Hom}}(T_p(X_{\mathfrak{p},0}), T_p(X_\mathfrak{p}), c_{\mathfrak{p},0}\delta_\mathfrak{p}) \cong (T_p(X_\mathfrak{p}), \varkappa_\mathfrak{p}).$$

We see that a level structure (7.4.21) at the banal prime $\mathfrak{p}$ is given by an isomorphism

$$\Lambda_\mathfrak{p}[c_{\mathfrak{p},0}] \to (T_p(X_\mathfrak{p}), \varkappa_\mathfrak{p}) \bmod \mathbf{K}_\mathfrak{p}^\star.$$

Choosing a fixed isomorphism $\Lambda_\mathfrak{p}[c_{\mathfrak{p},0}] \cong \Lambda_\mathfrak{p}$, we see that such a level structure at $\mathfrak{p}$ is the same as a class of isomorphisms

$$\bar{\eta}_\mathfrak{p} : \Lambda_\mathfrak{p} \to (T_p(X_\mathfrak{p}), \varkappa_\mathfrak{p}) \bmod \mathbf{K}_\mathfrak{p}^\star.$$

Since we want a level structure for $T_p(A)^{\mathrm{ba}}$, we require that the $\eta_\mathfrak{p}$ must respect the bilinear forms on both sides by the same factor $u \in \mathbb{Z}_p^\times$. For the special prime $\mathfrak{p}_v$ we take an arbitrary isomorphism

$$\eta_{\mathfrak{p}_v} : \Lambda_{\mathfrak{p}_v} \to T_p(X_{\mathfrak{p}_v})$$

which respects the bilinear forms on both sides up to the same factor $u \in \mathbb{Z}_p^\times$. This is possible because by (ii) of Definition 7.1.2, the $O_{K_{\mathfrak{p}_v}}$-lattices of both sides are isomorphic and since, by Lemmas 8.1.2 and 8.1.3, there exist isomorphisms with an arbitrary multiplicator $u \in \mathbb{Z}_p^\times$. We set

$$\tilde{\eta}_p = \eta_{\mathfrak{p}_v} \eta_p : \Lambda \otimes \mathbb{Z}_p \to T_p(A).$$

Finally we set

$$\eta = \tilde{\eta}_p \eta^p : V \otimes \mathbb{A}_f \to \hat{V}(A).$$

Let $\bar{\eta}$ be the class of this isomorphism modulo $\mathbf{K}^\star$. Then $(A, \iota, \bar{\lambda}, \bar{\eta})$ is a T-valued point of $\mathcal{A}_{\mathbf{K}^\star, E}$. Since the last construction can be reversed, we obtain the isomorphism of the theorem. □

We will formulate a more precise version of the last Theorem. Let E_ν^c be the algebraic closure of $\breve{E}_\nu$. The action of the Galois group $\mathrm{Gal}(E_\nu^c/\breve{E}_\nu)$ on $T_p(X_{\mathfrak{p},0})$ is given by a character

$$\chi_{\mathfrak{p},0} : \mathrm{Gal}(E_\nu^c/\breve{E}_\nu) \to O_{K_\mathfrak{p}}^\times, \tag{7.4.25}$$

such that $\sigma(t) = \chi_{\mathfrak{p},0}(\sigma)t$ for $t \in T_p(X_{\mathfrak{p},0})$ and $\sigma \in \mathrm{Gal}(E_\nu^c/\breve{E}_\nu)$. Since the polarization of $X_{\mathfrak{p},0}$ is defined over $\breve{E}_\nu$, we obtain that $\mathrm{Nm}_{K_\mathfrak{p}/F_\mathfrak{p}} \chi_{\mathfrak{p},0}(\sigma) = 1$. We define

$$\chi_0^{\mathrm{ba}}(\sigma) = \prod_{\mathfrak{p},\mathrm{banal}} \chi_{\mathfrak{p},0}(\sigma) \in G^{\mathrm{ba}}(\mathbb{Q}_p).$$

Finally we define $\chi_0 : \mathrm{Gal}(E_\nu^c/\breve{E}_\nu) \to G(\mathbb{Q}_p)$ by setting

$$\chi_0(\sigma) = 1 \times \chi_0^{\mathrm{ba}}(\sigma) \in G_{\mathfrak{p}_v} \times G^{\mathrm{ba}}(\mathbb{Q}_p).$$

We note that this element is in the center of the group $G(\mathbb{Q}_p)$. By definition of the functor $\mathcal{A}_{\mathbf{K}^\star, E}$ before Remark 7.1.6, $\chi_0(\sigma)$ acts on $\mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu} = \mathcal{A}_{\mathbf{K}^\star, E} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E}_\nu$ via the datum (3), i.e. it acts by Hecke operators. We obtain the homomorphism

$$\chi_0^{\mathrm{h}} : \mathrm{Gal}(E_\nu^c/\breve{E}_\nu) \to \mathrm{Aut}^{\mathrm{opp}}\, \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu}. \tag{7.4.26}$$

(We write here the opposite group because the Hecke operators act by definition from the right.)

Corollary 7.4.11. *Let $\sigma \in \mathrm{Gal}(E_\nu^c/\breve{E}_\nu)$. Then the action of $\mathrm{id}_{\mathcal{A}^\star_{\mathbf{K}^\star}} \times \mathrm{Spec}\, \sigma$ on the right hand side of (7.4.22) induces on the left hand side the automorphism $\chi_0^{\mathrm{h}}(\sigma) \times \mathrm{Spec}\, \sigma$.*

Remark 7.4.12. In general, let X a quasi-projective scheme over $\breve{E}_\nu$. Let $\chi : \mathrm{Gal}(E_\nu^c/\breve{E}_\nu) \to \mathrm{Aut}^{\mathrm{opp}}\, X$ be a continuous homomorphism. Then descent says that there is a unique quasi-projective scheme $X(\chi)$ over $\breve{E}_\nu$ and an isomorphism

$$X \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c \to X(\chi) \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c$$

such that, for all $\sigma \in \mathrm{Gal}(E_\nu^c/\breve{E}_\nu)$, the action of $\mathrm{id}_{X(\chi)} \times \mathrm{Spec}\, \sigma$ on the right hand side induces on the left hand side the action $\chi(\sigma) \times \mathrm{Spec}\, \sigma$. We will call $X(\chi)$ the *Galois twist* of X by χ.

Proof. (of Corollary 7.4.11) We take (7.4.22) over the algebraic closure E_ν^c. For $\sigma \in \mathrm{Gal}(E_\nu^c/\breve{E}_\nu)$, we write $\hat{\sigma} := \mathrm{Spec}\, \sigma$. We consider the non-commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu} \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c & \longrightarrow & \mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu} \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c \\ \downarrow {\scriptstyle \mathrm{id}_{\mathcal{A}} \times \hat{\sigma}} & & \downarrow {\scriptstyle \mathrm{id}_{\mathcal{A}^\star} \times \hat{\sigma}} \\ \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu} \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c & \longrightarrow & \mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu} \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c. \end{array} \tag{7.4.27}$$

To understand how this does not commute we consider more generally a scheme S of finite type over $\breve{E}_\nu$ and write $S_{E_\nu^c} = S \times_{\mathrm{Spec}\, \breve{E}_\nu} \mathrm{Spec}\, E_\nu^c$. The morphism $\hat{\sigma}_S := \mathrm{id}_S \times \hat{\sigma} : S_{E_\nu^c} \to S_{E_\nu^c}$ induces maps

$$\sigma_{\mathcal{A}} : \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c}) \to \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c}), \quad \sigma_{\mathcal{A}^\star} : \mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c}) \to \mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c}).$$

Our task is to compare the effect of these maps on an element $\xi \in \mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c}) = \mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c})$. The moduli interpretation describes ξ as a point of $\mathcal{A}_{\mathbf{K}^\star, \breve{E}_\nu}$ by a point $(A, \iota, \lambda, \bar{\eta}^p) \in \mathcal{A}_{\mathbf{K}, \breve{E}_\nu}(S_{E_\nu^c})$ and a rigidification $\bar{\eta}_p : \Lambda^{\mathrm{ba}} \to T_p(A)^{\mathrm{ba}} \bmod \mathbf{K}^{\star,\mathrm{ba}}$. To make this more precise, we choose a geometric point $\omega : \mathrm{Spec}\, E_\nu^c \to S$ which extends naturally to a point $\omega : \mathrm{Spec}\, E_\nu^c \to S_{E^c}$. We define ω' by the commutative diagram

$$\begin{array}{ccc} S_{E_\nu^c} & \xrightarrow{\hat{\sigma}_S} & S_{E_\nu^c} \\ \uparrow {\scriptstyle \omega'} & & \uparrow {\scriptstyle \omega} \\ \mathrm{Spec}\, E_\nu^c & \xrightarrow{\hat{\sigma}} & \mathrm{Spec}\, E_\nu^c. \end{array}$$

The rigidification is given by a homomorphism

$$\eta_p : \Lambda^{\mathrm{ba}} \to T_p(A_\omega)^{\mathrm{ba}}.$$

There is an isomorphism

$$T_p((\hat{\sigma}_S^* A))_{\omega'} = \hat{\sigma}^*(T_p(A_\omega)).$$

By the moduli interpretation, the point $\sigma_{\mathcal{A}}(\xi)$ is given by $(\hat{\sigma}_S^* A, \hat{\sigma}_S^* \iota, \hat{\sigma}_S^* \lambda, \hat{\sigma}_S^* \bar{\eta}^p)$ and the rigidification is given by

$$\Lambda^{\mathrm{ba}} = \hat{\sigma}^*(\Lambda^{\mathrm{ba}}) \xrightarrow{\hat{\sigma}^*(\eta_p)} \hat{\sigma}^*(T_p(A_\omega)). \tag{7.4.28}$$

Now we consider $\sigma_{\mathcal{A}^\star}(\xi)$. We can give the sheaf $T_p((X_0^{\mathrm{ba}})_{\breve{E}_\nu})$ in (7.4.21) equivalently by the $\mathrm{Gal}(E_\nu^c/E_\nu)$-module $\Lambda^{\mathrm{ba}}(\chi_0^{\mathrm{ba}})$, where we indicate that the Galois group acts via the character χ_0^{ba}. Then $\bar{\eta}_p$ of (7.4.21) can be considered as a class of maps

$$\Lambda^{\mathrm{ba}}(\chi_0^{\mathrm{ba}})_{E_\nu^c} \to T_p(A_\omega)^{\mathrm{ba}} \bmod \mathbf{K}^{\star,\mathrm{ba}}. \tag{7.4.29}$$

Since we are over E_ν^c, the action via χ_0^{ba} is trivial and therefore $(A, \iota, \lambda, \bar{\eta}^p, \eta_p)$ describes also a point of $\mathcal{A}^\star_{\mathbf{K}^\star, \breve{E}_\nu}(S_{E_\nu^c})$. But if we want to identify the inverse image of this point by $\hat{\sigma}_S$ we must take into account the twist χ_0^{ba}. This inverse image is again given by $(\hat{\sigma}_S^\star A, \hat{\sigma}_S^\star \iota, \hat{\sigma}_S^\star \lambda, \hat{\sigma}_S^* \bar{\eta}^p)$ as before, but the new rigidification at the banal places is

$$\Lambda^{\mathrm{ba}}(\chi_0^{\mathrm{ba}})_{E_\nu^c} \cong \hat{\sigma}^*(\Lambda^{\mathrm{ba}})(\chi_0^{\mathrm{ba}})_{E_\nu^c} \xrightarrow{\hat{\sigma}^*(\eta_p)} \hat{\sigma}^*(T_p(A_\omega)). \tag{7.4.30}$$

The first isomorphism comes from the fact that both sides are the inverse image of $\Lambda^{\mathrm{ba}}(\chi_0^{\mathrm{ba}})$ considered as a sheaf on $\mathrm{Spec}\, \breve{E}_\nu$. Therefore this isomorphism is the descent datum on the constant sheaf, which is the multiplication by $\chi_0^{\mathrm{ba}}(\sigma)$. We obtain

$$\Lambda^{\mathrm{ba}}(\chi_0^{\mathrm{ba}}) \xrightarrow{\chi_0^{\mathrm{ba}}(\sigma)} \hat{\sigma}^*(\Lambda^{\mathrm{ba}})(\chi_0^{\mathrm{ba}}) \xrightarrow{\hat{\sigma}^*(\eta_p)} \hat{\sigma}^*(T_p(A_\omega)).$$

This proves that $\sigma_{\mathcal{A}^*} = \chi_0^{\mathrm{h}}(\sigma)\sigma_{\mathcal{A}}$. If we apply this to the diagram (7.4.27), we obtain

$$\mathrm{id}_{\mathcal{A}^*} \times \hat{\sigma} = \chi_0^{\mathrm{h}}(\sigma)(\mathrm{id}_{\mathcal{A}} \times \hat{\sigma}).$$

□

We now drop the assumption on $\mathbf{K}_p^\star$ that it be contained in $\mathbf{K}_p$ and come from a product of $\mathbf{K}_{\mathfrak{p}}^\star$. More precisely, let $\mathbf{K}_p^\star \subset G(\mathbb{Q}_p)$ be of the form

$$\mathbf{K}_p^\star = G(\mathbb{Q}_p) \cap \mathbf{K}_{\mathfrak{p}_v} \mathbf{K}_p^{\star,\mathrm{ba}}, \tag{7.4.31}$$

where $\mathbf{K}_p^{\star,\mathrm{ba}}$ is an arbitrary open compact subgroup of $G^{\mathrm{ba}}(\mathbb{Q}_p)$. Since $\mathbf{K}_{\mathfrak{p}_v}$ is a normal subgroup of $G_{\mathfrak{p}_v}$ and $G(\mathbb{Q}_p)$, this class of subgroups is stable under conjugation by elements of $G(\mathbb{Q}_p)$. Therefore, using the naturality of the construction in Proposition 7.4.9, we can extend the definition of $\mathcal{A}^\star_{\mathbf{K}^\star}$ to all such $\mathbf{K}^\star = \mathbf{K}_p^\star \mathbf{K}^p$ by first passing to a small enough normal subgroup of finite index and then dividing out by the factor group.

We make this extension process more explicit by defining the functor $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}$ without using the choice of Λ_p. Thereby the action of the Hecke operators becomes more obvious.

Let $\theta : X \to Y$ be an isogeny of p-divisible $O_F \otimes \mathbb{Z}_p$-modules. Let $\mathfrak{p}_v$ be a prime of O_F over p. We say that θ is an isogeny of order prime to $\mathfrak{p}_v$ if $\theta_{\mathfrak{p}_v}$ is an isomorphism. We use a similar terminology for abelian varieties with action by O_F.

We consider a scheme S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$. We consider abelian schemes $\bar{A}$ over S up to isogeny of order prime to $\mathfrak{p}_v$ which are endowed with an action $\iota : O_K \to \mathrm{End}\, \bar{A}$ and with a $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}$ such that the Rosati involution induces the conjugation on O_K. Moreover, we assume that there is a triple (A, ι, λ) as in Definition 7.1.2 which represents $(\bar{A}, \iota, \bar{\lambda})$ such that (A, ι) satisfies the conditions (KC_r) and (EC_r) and such that (A, ι, λ) satisfies the conditions (i) and (ii) of Definition 7.1.2. Then call $(\bar{A}, \iota, \bar{\lambda})$ an *admissible prime-to-$\mathfrak{p}_v$-isogeny class.* Let $X = \prod_{\mathfrak{p}} X_{\mathfrak{p}}$ be the p-divisible group of A. Then $(C_{X^{\mathrm{ba}}}, \xi_\lambda)$ makes sense and $(C_{X^{\mathrm{ba}}} \otimes \mathbb{Q}, \xi_\lambda)$ depends only on $(\bar{A}, \iota, \bar{\lambda})$. Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as in Definition 7.4.5.

Definition 7.4.13. We define a functor $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}$ on the category of schemes S over $\mathrm{Spf}\, O_{\breve{E}_\nu}$. A point of $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}(S)$ consists of the following data:

(1) an admissible prime-to-$\mathfrak{p}_v$-isogeny class $(\bar{A}, \iota, \bar{\lambda})$ over S.
(2) a class of isomorphisms

$$\bar{\eta}_p : C_{\mathbb{X}^{\mathrm{ba}}} \otimes \mathbb{Q} \to C_{\bar{A}[p^\infty]^{\mathrm{ba}}} \otimes \mathbb{Q} \bmod \mathbf{K}_p^{\star,\mathrm{ba}},$$

which respects the bilinear forms on both sides up to a factor in $\mathbb{Q}_p^\times$.

We explain more detailed what is meant by (2). We assume that S is connected and we choose a geometric point ω of S. Then the meaning of (2) is that we have a class of isomorphisms

$$\eta_p : (C_{\mathbb{X}^{\mathrm{ba}}})_\omega \otimes \mathbb{Q} \to (C_{\bar{A}[p^\infty]^{\mathrm{ba}}})_\omega \otimes \mathbb{Q} \bmod \mathbf{K}_p^{\star,\mathrm{ba}}$$

which respects the bilinear forms on both sides up to a factor in $\mathbb{Q}_p^\times$ and such that the class is preserved by the action of $\pi_1(S, \omega)$.

Let $\mathbf{K}_p^\star$ as in Definition 7.4.5. Then the functors of the Definitions 7.4.5 and 7.4.13 coincide. Indeed, let us start with a point of Definition 7.4.13. Also, fix a triple (A, ι, λ) which represents $(\bar{A}, \iota, \bar{\lambda})$, as before Definition 7.4.13. The sublattice $\Lambda_p^{\mathrm{ba}} \subset C_{\mathbb{X}^{\mathrm{ba}}})_\omega$ is fixed by $\mathbf{K}_p^{\star,\mathrm{ba}}$. Therefore the image C of Λ_p^{ba} by η_p depends only on the class $\bar{\eta}_p$ and is invariant by $\pi_1(S, \omega)$. Therefore C defines a p-adic étale sheaf on S. We endow it with the polarization induced by ς^{ba}, cf. (7.4.6). Therefore, using the contracting functor C defines a p-divisible $O_{K^{\mathrm{ba}}}$-module Y^{ba} with a polarization. Then $Y := X_{\mathfrak{p}_v} \times Y^{\mathrm{ba}}$ is isogenous to the p-divisible group X of A. The polarization on Y differs from the polarization induced from λ on A by a factor in $\mathbb{Z}_p^\times$, as we see by comparing the degrees of the polarizations. Therefore we obtain a point $(A_1, \iota_1, \bar{\lambda}_1)$ of $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}(S)$ which is isogenous to $(\bar{A}, \iota, \bar{\lambda})$. This proves that the point we started with comes from a point of the functor in Definition 7.4.5. It is clear that we have a bijection.

We have an action of $G^{\mathrm{ba}}(\mathbb{Q}_p)$ on the tower $\hat{\mathcal{A}}^\star_{\mathbf{K}^\star}$ for varying $\mathbf{K}_p^\star$. This action extends to the algebraization $\mathcal{A}^\star_{\mathbf{K}^\star}$ and coincides via Theorem 7.4.10 with the Hecke operators on the tower $\mathcal{A}_{\mathbf{K}^\star, E}$.

Corollary 7.4.14. *For every $\mathbf{K}^\star = \mathbf{K}_p^\star \mathbf{K}^p$ with (7.4.31) , there exists a normal scheme $\mathcal{A}^\star_{\mathbf{K}^\star}$ over $\mathrm{Spec}\, O_{\breve{E}_\nu}$ such that for the p-adic completion of this scheme there is an isomorphism*

$$\hat{\mathcal{A}}^\star_{\mathbf{K}^\star} \simeq J(\mathbb{Q}) \backslash [(\hat{\Omega}_{F_v} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{\breve{E}_\nu}) \times G^{\mathrm{ba}}(\mathbb{Q}_p)/\mathbf{K}_p^{\star,\mathrm{ba}} \times G(\mathbb{A}_f^p)/\mathbf{K}^p].$$

For varying $\mathbf{K}^\star$, these schemes form a tower with an action of the group $G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, where the action of $G(\mathbb{Q}_p)$ factors through $G(\mathbb{Q}_p) \to G^{\mathrm{ba}}(\mathbb{Q}_p)$. The isomorphism of formal schemes is compatible with these actions.

The general fiber of $\mathcal{A}^\star_{\mathbf{K}^\star}$ is a Galois twist of $\mathcal{A}_{\mathbf{K}^\star, E} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E}_\nu$ by the character χ_0^{h}, cf. (7.4.26) and Remark 7.4.12. The Galois twist respects the Hecke operators (cf. section 7.6 for an explicit description of χ_0^{h}).

Proof. This is a consequence of Proposition 7.4.4 and the general pattern of p-adic uniformization, cf. (7.3.1). The last assertion follows because $\chi_0 : \mathrm{Gal}(E_\nu^{\mathrm{ab}}/E_\nu) \to G(\mathbb{Q}_p)$ factors through the center. □

7.5. **The rigid-analytic uniformization.** Let $\mathcal{A}_{\mathbf{K}}^{\mathrm{rig}}$ denote the rigid-analytic space over $\mathrm{Sp}\, E_\nu$ associated to $\mathcal{A}_{\mathbf{K}, E}$. Then Theorem 7.3.3 implies the following corollary concerning generic fibers.

Corollary 7.5.1. *Let $\mathbf{K} = \mathbf{K}_p \mathbf{K}^p$ as in (7.1.8). There exists an isomorphism of rigid-analytic spaces over $\mathrm{Sp}\, \breve{E}_\nu$,*

$$\mathcal{A}_{\mathbf{K}}^{\mathrm{rig}} \times_{\mathrm{Sp}\, E_\nu} \mathrm{Sp}\, \breve{E}_\nu \simeq J(\mathbb{Q}) \backslash \left[\left(\Omega_{F_v} \times_{\mathrm{Sp}\, F_v} \mathrm{Sp}\, \breve{E}_\nu \right) \times \hat{G}'(\mathbb{Q}_p)/\hat{G}'(\mathbb{Z}_p) \times G(\mathbb{A}_f^p)/\mathbf{K}^p \right].$$

For varying $\mathbf{K}^p$, this isomorphism is compatible with the action of $G(\mathbb{A}_f^p)$ through Hecke correspondences on both sides. □

Here $\Omega_{F_v} = \mathbb{P}^1_{F_v} \setminus \mathbb{P}^1(F_v)$ is Drinfeld's p-adic halfspace corresponding to the p-adic field F_v.

Similarly, Corollary 7.4.14 implies the following corollary concerning generic fibers for deeper level structures.

Corollary 7.5.2. *Assume that there are banal primes. Let* $\mathbf{K}^\star = \mathbf{K}_p^\star \mathbf{K}^p$ *with* (7.4.31). *Let* $\mathcal{A}_{\mathbf{K}^\star}^{\mathrm{rig}}$ *denote the rigid-analytic space over* $\mathrm{Sp}\, E_\nu$ *associated to* $\mathcal{A}_{\mathbf{K}^\star,E}$. *There exists an isomorphism of rigid-analytic spaces over* $\mathrm{Sp}\, \breve{E}_\nu^{\mathrm{ab}}$,

$$\mathcal{A}_{\mathbf{K}^*}^{\mathrm{rig}} \times_{\mathrm{Sp}\, E_\nu} \mathrm{Sp}\, \breve{E}_\nu^{\mathrm{ab}} \simeq J(\mathbb{Q}) \backslash [(\Omega_{F_v} \times_{\mathrm{Sp}\, F_v} \mathrm{Sp}\, \breve{E}_\nu^{\mathrm{ab}}) \times G^{\mathrm{ba}}(\mathbb{Q}_p)/\mathbf{K}_p^{*,\mathrm{ba}} \times G(\mathbb{A}_f^p)/\mathbf{K}^p].$$

For variable $\mathbf{K}^\star$, *this isomorphism is compatible with the Hecke correspondences by* $G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$. □

7.6. **Determination of the character** χ_0^{h}**.** In this section we give an explicit description of the character χ_0^{h} (7.4.26) which is used in Corollary 7.4.14. In the case where $\mathfrak{p}_v$ is ramified in K/F, we only obtain the restriction of χ_0^{h} to the Galois group of a quadratic extension of $\breve{E}_\nu$. It is enough to describe $\chi_{\mathfrak{p},0}$ (7.4.25) for each banal prime $\mathfrak{p}$. This is done by Proposition 7.6.5 below.

Let K/F be a CM-field. Let $\Xi \subset \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(K, \mathbb{C})$ be a CM-type. We denote the reflex field by H. We define an algebraic torus over $\mathbb{Q}$, with $\mathbb{Q}$-valued points

$$T(\mathbb{Q}) = \{a \in K^\times \mid a\bar{a} \in \mathbb{Q}^\times\}.$$

We use the notation $V = K$ for K regarded as a K-vector space.

We recall the reciprocity law. We define the homomorphism

$$\mu : \mathbb{C}^\times \to (K \otimes_{\mathbb{Q}} \mathbb{C})^\times \cong \prod_{\varphi : K \to \mathbb{C}} \mathbb{C}^\times.$$

The element $\mu(z)$, for $z \in \mathbb{C}$, has component z for $\varphi \in \Xi$ and has component 1 for $\varphi \notin \Xi$ on the right hand side. We find $\mu\bar{\mu} = 1 \otimes z \in (K \otimes_{\mathbb{Q}} \mathbb{C})^\times$. We obtain a homomorphism of algebraic tori

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}.$$

This homomorphism is defined over H,

$$\mu : \mathbb{G}_{m,H} \to T_H.$$

From this we deduce the reciprocity map

$$\mathfrak{r} : \mathrm{Res}_{H/\mathbb{Q}}(\mathbb{G}_{m,H}) \xrightarrow{\mu} \mathrm{Res}_{H/\mathbb{Q}}(T_H) \xrightarrow{\mathrm{Nm}_{H/\mathbb{Q}}} T. \tag{7.6.1}$$

We consider over the algebraic closure $\bar{H} = \bar{\mathbb{Q}}$ the set of tuples $(A, \iota, \bar{\lambda}, \kappa)$, where (A, ι) is an abelian variety over $\bar{H}$ of CM-type Ξ, endowed with a $\mathbb{Q}$-homogeneous polarization $\bar{\lambda}$ which induces on K the conjugation over F and an isomorphism $\kappa : \hat{V}(A) \to V \otimes \mathbb{A}_f$ of $K \otimes \mathbb{A}_f$-modules. We call a second tuple $(A', \iota', \bar{\lambda}', \kappa')$ equivalent to $(A, \iota, \bar{\lambda}, \kappa)$ if there is a quasi-isogeny

$$\alpha : (A, \iota, \bar{\lambda}) \to (A', \iota', \bar{\lambda}') \tag{7.6.2}$$

such that the following diagram

$$\begin{array}{ccc} \hat{V}(A) & \xrightarrow{\quad\alpha\quad} & \hat{V}(A') \\ & {\scriptstyle\kappa}\searrow \quad \swarrow{\scriptstyle\kappa'} & \\ & V \otimes \mathbb{A}_f & \end{array}$$

commutes. We also say that $(A, \iota, \bar{\lambda}, \kappa)$ is quasi-isogenous to $(A', \iota', \bar{\lambda}', \kappa')$.

Let $\mathcal{C}_\Xi$ be the set of tuples $(A, \iota, \bar{\lambda}, \kappa)$ up to equivalence. Let $\sigma \in \mathrm{Gal}(\bar{H}/H)$. Taking the inverse image of $(A, \iota, \bar{\lambda}, \kappa)$ by $\hat{\sigma} := \mathrm{Spec}\, \sigma : \mathrm{Spec}\, \bar{H} \to \mathrm{Spec}\, \bar{H}$ gives a left action of $\mathrm{Gal}(\bar{H}/H)$ on $\mathcal{C}_\Xi$. We denote the inverse image by $\sigma(A, \iota, \bar{\lambda}, \kappa)$.

We formulate the main theorem of complex multiplication of Shimura and Taniyama.

Theorem 7.6.1. *(*[8, Thm. 4.19]*) The Galois group* $\mathrm{Gal}(\bar{H}/H)$ *acts on* $\mathcal{C}_\Xi$ *via its maximal abelian quotient* $\mathrm{Gal}(H^{\mathrm{ab}}/H)$. *Let* $e \in (H \otimes \mathbb{A})^\times$ *and let* $\mathrm{rec}(e) \in \mathrm{Gal}(H^{\mathrm{ab}}/H)$ *be the automorphism given by the reciprocity law of class field theory. The following tuples are equivalent:*

$$\mathrm{rec}(e)(A, \iota, \bar{\lambda}, \kappa) \equiv (A, \iota, \bar{\lambda}, \mathfrak{r}(e_f)\kappa),$$

where e_f *is the finite part of the idèle* e.

Remark 7.6.2. Let $(H^\times)^\wedge \subset (H \otimes \mathbb{A}_f)^\times$ be the closure of $H^\times$. We deduce a homomorphism

$$\mathrm{Gal}(\bar{H}/H) \to (H \otimes \mathbb{A}_f)^\times/(H^\times)^\wedge \xrightarrow{\mathfrak{r}} T(\mathbb{A}_f)/T(\mathbb{Q}), \tag{7.6.3}$$

where the first arrow is deduced from class field reciprocity and the second arrow exists because $T(\mathbb{Q}) = T(\mathbb{Q})^\wedge$. To see this last fact, we note that the group of units in $T(\mathbb{Q})$ is finite. Indeed, the units are elements of $K^\times$ with all absolute values equal to 1 at all places including the infinite ones. Therefore $T(\mathbb{Q}) = T(\mathbb{Q})^\wedge$ by Chevalley's theorem.

Theorem 7.6.1 says that the action of $\mathrm{Gal}(\bar{H}/H)$ on $\mathcal{C}_\Xi$ is via (7.6.3). One can consider the Shimura variety Sh_T. We may choose as usual a bijection

$$\mathrm{Sh}_T(\bar{H}) = \mathrm{Sh}_T(\mathbb{C}) = T(\mathbb{A}_f)/T(\mathbb{Q})^\wedge.$$

Then the theorem may be regarded as a consequence of Langlands' description of the reduction of this Shimura variety at good places [15].

We fix an embedding $\bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$. The p-adic place which is induced on a subfield of $\bar{\mathbb{Q}}$ will be denoted by ν.

Proposition 7.6.3. *Let $L \subset \bar{\mathbb{Q}}$ be a number field such that $H \subset L$. Let $(A_0, \iota_0, \lambda_0)$ be an abelian variety over L with an action $\iota_0 : O_K \to \mathrm{End}\, A_0$ which is of CM-type Ξ. We assume that A_0 has good reduction at ν. The group $\mathrm{Gal}(\bar{L}_\nu/L_\nu)$ acts on the Tate module $T_p(A_0)$ via its maximal abelian quotient $\mathrm{Gal}(L_\nu^{\mathrm{ab}}/L_\nu)$. Let $I_\nu \subset \mathrm{Gal}(L_\nu^{\mathrm{ab}}/L_\nu)$ be the inertia group. The action of I_ν on the Tate module can be described as follows.*

The inverse of the map (7.6.1) induces a homomorphism

$$\rho : L_\nu^\times \xrightarrow{\mathrm{Nm}_{L_\nu/H_\nu}} H_\nu^\times \subset (H \otimes \mathbb{Q}_p)^\times \xrightarrow{\mathfrak{r}^{-1}} (K \otimes \mathbb{Q}_p)^\times.$$

Composing ρ with the reciprocity law of local class field theory yields

$$I_\nu \overset{\mathrm{rec}}{\simeq} O_{L_\nu}^\times \xrightarrow{\rho} (O_K \otimes \mathbb{Z}_p)^\times.$$

The action of an element $\sigma \in I_\nu$ on the Tate-module is the multiplication by the image in the right hand side.

Proof. We set $A = A_0 \otimes_H \bar{H}$ with the O_K-action and the induced polarization. We set $\Lambda = O_K \subset V$ and $\hat{\Lambda} = O_K \otimes \hat{\mathbb{Z}}$. We choose a rigidification $\kappa : \hat{T}(A) \xrightarrow{\sim} \hat{\Lambda}$. We consider the tuple $(A, \iota, \bar{\lambda}, \kappa)$. Let

$$\sigma \in \tilde{I}_\nu \subset \mathrm{Gal}(\bar{H}_\nu/H_\nu) \subset \mathrm{Gal}(\bar{H}/H)$$

be an element of the inertia group at ν. The image in $\mathrm{Gal}(H^{\mathrm{ab}}/H)$ corresponds to an idèle in $(H \otimes \mathbb{A})^\times$ which has components 1 outside ν and a component $e_\nu \in O_{H_\nu}^\times$ at the place ν. We denote the idèle also by e_ν. By Theorem 7.6.1, we have a quasi-isogeny

$$\hat{\sigma}^*(A, \iota, \bar{\lambda}, \kappa) \cong (A, \iota, \bar{\lambda}, \mathfrak{r}(e_\nu)\kappa).$$

Let us moreover assume that σ fixes the elements of L. Since $(A, \iota, \bar{\lambda})$ is defined over L, it is not changed by $\hat{\sigma}^*$. Now we consider the product of the Tate modules for all primes,

$$\hat{T}(A) = \hat{\sigma}^*(\hat{T}(A)) \xrightarrow{\hat{\sigma}^*(\kappa)} \hat{\Lambda}.$$

The first identification is due to the fact that $\hat{T}(A)$ is a projective limit of étale sheaves on $\mathrm{Spec}\, L$.

Lemma 7.6.4. *Denote by $\hat{T}(\sigma)$ the action of σ on $\hat{T}(A)$. Then*

$$\hat{\sigma}^*(\kappa)\hat{T}(\sigma) = \kappa.$$

We postpone the proof of the Lemma. Because we have good reduction, the element $\hat{T}(\sigma)$ acts trivially on the Tate modules $T_\ell(A)$ for $\ell \neq p$. On $T_p(A)$ it acts by multiplication with an element $u(\sigma) \in (O_K \otimes \mathbb{Z}_p)^\times$. Therefore we have a quasi-isogeny

$$(A, \iota, \bar{\lambda}, \mathfrak{r}(e_\nu)\kappa) \cong (A, \iota, \bar{\lambda}, u(\sigma)^{-1}\kappa) \tag{7.6.4}$$

The quasi-isogeny giving this equivalence must be trivial on the Tate modules $V_\ell(A)$ for $\ell \neq p$. It is therefore the identity. The proposition follows therefore from Lemma 7.6.4. ☐

Proof. (of Lemma 7.6.4) We consider an étale sheaf G over $\operatorname{Spec} L$ where L is any field. Let L^s be the separable closure of L. For $\sigma \in \operatorname{Gal}(L^s/L)$ we denote by $G(\sigma) : G(L^s) \to G(L^s)$ the natural action. Let $\underline{\Gamma}$ be a constant sheaf on $\operatorname{Spec} L$ associated to a set Γ. Let

$$\kappa : G \to \underline{\Gamma}$$

be an isomorphism of sheaves on $(\operatorname{Spec} L^s)_{\text{ét}}$. There are canonical isomorphisms $\hat{\sigma}^*(G) \cong G$ and $\hat{\sigma}^*(\Gamma) \cong \Gamma$ because both sheaves are defined over L. We must show that the map

$$\kappa' : G(L^s) \cong \hat{\sigma}^*(G)(L^s) \xrightarrow{\hat{\sigma}^*(\kappa)} \hat{\sigma}^*(\underline{\Gamma})(L^s) = \Gamma$$

coincides with $\kappa G(\sigma^{-1})$.

Let A be a finite étale algebra over L^s. By definition of the inverse image, we have $\hat{\sigma}^*(G)(A) = G(A_{[\sigma]})$. Therefore the L^s-algebra isomorphism $\sigma : L^s \to L^s_{[\sigma]}$ induces a natural map $G[\sigma] \colon G(L^s) \to \hat{\sigma}^*(G)(L^s)$. Our assertion follows from the commutative diagram, in which the composition of the two upper horizontal arrows is $G(\sigma)$,

$$\begin{array}{ccccc} G(L^s) & \xrightarrow{G[\sigma]} & \hat{\sigma}^*(G)(L^s) & \xrightarrow{\sim} & G(L^s) \\ \kappa\downarrow & & \downarrow \hat{\sigma}^*(\kappa) & \swarrow \kappa' & \\ \Gamma & \xrightarrow[\Gamma[\sigma]=\mathrm{id}]{} & \Gamma & & \end{array}$$

□

Let $K/F, r, E \to \bar{\mathbb{Q}}_p, \nu$ be as in section 7.1. Let $\mathfrak{p}$ be a banal prime of K. Let $(X_{\mathfrak{p},0}, \iota_{\mathfrak{p},0})$ be the unique CM-pair of CM-type $r_{\mathfrak{p}}/2$ over $\operatorname{Spec} O_{\breve{E}_\nu}$. We set

$$\Xi_{\mathfrak{p}} = \{\varphi \in \Phi_{\mathfrak{p}} \mid r_\varphi = 2\},$$

where $\Phi_{\mathfrak{p}} = \operatorname{Hom}_{\mathbb{Q}_p\text{-Alg}}(K_{\mathfrak{p}}, \bar{\mathbb{Q}}_p)$, as in (7.1.9). We consider the homomorphism

$$\mu_{\mathfrak{p}} : \bar{\mathbb{Q}}_p^\times \to (K_{\mathfrak{p}} \otimes \bar{\mathbb{Q}}_p)^\times \xrightarrow{\sim} \prod_{\Phi_{\mathfrak{p}}} \bar{\mathbb{Q}}_p^\times$$

such that the component of $\mu_{\mathfrak{p}}(a)$, $a \in \bar{\mathbb{Q}}_p$, is equal to a for $\varphi \in \Xi_{\mathfrak{p}}$ and is 1 for $\varphi \notin \Xi_{\mathfrak{p}}$. This morphism is defined over E_ν. We define the local reciprocity law $\mathfrak{r}_{\mathfrak{p}}$ as

$$\mathfrak{r}_{\mathfrak{p}} : E_\nu^\times \xrightarrow{\mu_{\mathfrak{p}}} (K_{\mathfrak{p}} \otimes_{\mathbb{Q}} E_\nu)^\times \xrightarrow{\operatorname{Nm}_{E_\nu/\mathbb{Q}_p}} K_{\mathfrak{p}}^\times. \tag{7.6.5}$$

Let $I_\nu \subset \operatorname{Gal}(\breve{E}_\nu^{\text{ab}}/\breve{E}_\nu)$ be the inertia group. As before (7.4.25), let E_ν^c be the algebraic closure of $\breve{E}_\nu$ in the completion of $\bar{\mathbb{Q}}_p$. By the reciprocity law of local class field theory, we define

$$\rho_{\mathfrak{p}} : \operatorname{Gal}(E_\nu^c/\breve{E}_\nu) \to \operatorname{Gal}(\breve{E}_\nu^{\text{ab}}/\breve{E}_\nu) \overset{\text{rec}}{\simeq} O_{E_\nu}^\times \xrightarrow{\mathfrak{r}_{\mathfrak{p}}^{-1}} O_{K_{\mathfrak{p}}}^\times.$$

Proposition 7.6.5. *Let $\mathfrak{p}$ be a banal prime of K. Let $\chi_{\mathfrak{p},0} : \operatorname{Gal}(E_\nu^c/\breve{E}_\nu) \to O_{K_{\mathfrak{p}}}^\times$ be the character given by the action on $T_p(X_{\mathfrak{p},0})$, compare (7.4.25). Then the restriction of this character to the subgroup*

$$\operatorname{Gal}(E_\nu^c/\breve{E}_\nu\varphi_0(K_{\mathfrak{p}_v})) \subset \operatorname{Gal}(E_\nu^c/\breve{E}_\nu)$$

coincides with the restriction of $\rho_{\mathfrak{p}}$ to this subgroup.

We remark that $\breve{E}_\nu\varphi_0(K_{\mathfrak{p}_v})$ equals $\breve{E}_\nu$ if $\mathfrak{p}_v$ is unramified in K/F and is a quadratic extension of $\breve{E}_\nu$ if $\mathfrak{p}_v$ is ramified in K/F.

Proof. It follows from the functoriality of rec that the proposition implies the same statement for a finite extension E'_ν of E_ν.

We define a CM-type $\Xi \subset \Phi = \operatorname{Hom}_{\mathbb{Q}\text{-Alg}}(K, \mathbb{C})$ by choosing $\varphi_0 : K \to \mathbb{C}$ with $r_{\varphi_0} = 1$ and setting

$$\Xi = \{\varphi \in \Phi \mid r_\varphi = 2\} \cup \{\varphi_0\}.$$

We denote by H the reflex field of Ξ. We find that $H\varphi_0(K) = E\varphi_0(K)$. We claim that there exists an extension of number fields L/H which is unramified at ν and a tuple $(A, \iota, \bar{\lambda}, \kappa)$ which is defined over L and such that A has good reduction $\tilde{A}$ over O_{L_ν}. Let Y be the p-divisible group of $\tilde{A}$, which we write as $Y = \prod_{\mathfrak{p}} Y_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through the prime ideals of O_K over p. Let $\mathfrak{p}$ be banal. Then $(Y_{\mathfrak{p}}, \iota) \otimes_{O_{L_\nu}} O_{\breve{L}_\nu}$ is a CM-pair of type $r_{\mathfrak{p}}/2$ which satisfies the Kottwitz condition and the Eisenstein condition. Therefore it is isomorphic to $(X_{\mathfrak{p},0}, \iota_{X_{\mathfrak{p},0}})$ which is defined over $O_{\breve{H}_\nu}$. Therefore the proposition follows from Proposition (7.6.3).

It remains to show the existence of L. We fix an open compact subgroup $C \subset T(\mathbb{A}_f)$ which is maximal in p and is small enough. The Shimura variety $\mathrm{Sh}_{\Xi,C}$ which is associated to (T,μ) and C is representable by a moduli problem $\mathcal{A}_{\Xi,C,H}$ which is finite and étale over H. Moreover it has a model $\mathcal{A}_{\Xi,C}$ over O_{H_ν}. It is defined exactly in the same way as $\mathcal{A}_{\mathbf{K}}$. Since for the moduli problem $\mathcal{A}_{\Xi,C}$ each prime $\mathfrak{p}$ of O_K is banal, it is representable by a finite étale scheme over O_{H_ν}. We conclude the $\mathcal{A}_{\Xi,C,H} = \coprod_{i=1}^m \operatorname{Spec} L_i$ for some finite field extensions L_i/H which are unramified over ν. Restricting the universal abelian scheme over $\mathcal{A}_{\Xi,C,H}$ to some $L = L_i$, we obtain a tuple as required. □

8. Appendix: Adjusted invariants

In this appendix we first collect some facts about anti-hermitian forms. Then we give a correction to [18, Prop. 3.2], by introducing the r-adjusted invariant of a CM-triple. Finally, we relate the r-adjusted invariant to the contracting functor of section 4.

8.1. Recollections on binary anti-hermitian forms over p-adic local fields.

We first recall the invariant of an anti-hermitian form in the case relevant to us. A good reference for this material is [13].

Let K/F be a quadratic extension of fields of characteristic 0. We denote by $a \mapsto \bar{a}$ the non-trivial automorphism of K over F. Let V be an 2-dimensional vector space over K. Let

$$\varkappa : V \times V \longrightarrow K,$$

be a sesquilinear form which is linear in the first argument and anti-linear in the second. We assume that $\varkappa$ is anti-hermitian:

$$\varkappa(x,y) = -\overline{\varkappa(y,x)}.$$

We choose a basis $\{v_1, v_2\}$ of V. Then $\det(\varkappa(v_i,v_j))_{i,j\in\{1,2\}} \in F^\times$. We denote by

$$\mathfrak{d}_{K/F}(V,\varkappa) \in F^\times / \operatorname{Nm}_{K/F} K^\times \tag{8.1.1}$$

the residue class of this element. It is independent of the choice of the basis and is called the *discriminant* of $(V,\varkappa)$.

Definition 8.1.1. Let F be a p-adic local field and K/F a quadratic field extension. Let $(V,\varkappa)$ be a K-vector space of dimension 2 with an anti-hermitian form $\varkappa$ which is nondegenerate. We denote by $\operatorname{inv}(V,\varkappa) \in \{\pm 1\}$ the image of $\mathfrak{d}_{K/F}(V,\varkappa)$ under the canonical isomorphism $F^\times / \operatorname{Nm}_{K/F} K^\times \simeq \{\pm 1\}$. The invariant determines $(V,\varkappa)$ up to isomorphism, cf. [13].

We note that an anti-hermitian form $\varkappa$ can equivalently be given by an alternating nondegenerate $\mathbb{Q}_p$-bilinear form

$$\psi : V \times V \longrightarrow \mathbb{Q}_p \tag{8.1.2}$$

such that

$$\psi(ax,y) = \psi(x,\bar{a}y), \quad x,y \in V,\ a \in K.$$

The anti-hermitian form $\varkappa$ is defined by the equation

$$\operatorname{Tr}_{K/\mathbb{Q}_p} a\varkappa(x,y) = \psi(ax,y).$$

In this case we set

$$\operatorname{inv}(V,\psi) = \operatorname{inv}(V,\varkappa).$$

The invariant $\operatorname{inv}(V,\psi)$ determines (V,ψ) up to isomorphism.

Let $\Lambda \subset V$ be an O_K-lattice such that ψ induces a pairing

$$\psi : \Lambda \times \Lambda \longrightarrow \mathbb{Z}_p, \tag{8.1.3}$$

i.e., ψ is integral on Λ. We consider the map

$$\begin{aligned} \Lambda &\longrightarrow \operatorname{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p) \\ y &\longmapsto \ell_y, \quad \text{where } \ell_y(x) = \psi(x,y) \end{aligned} \tag{8.1.4}$$

This is an anti-linear map of O_K-modules. Therefore the image of this map is an O_K-submodule. We denote the length of the cokernel as an O_K-module by $\mathtt{h}(\Lambda, \psi)$.

Lemma 8.1.2 ([13], Thm. 7.1)**.** *Let F be a local p-adic field and K/F an unramified field extension. Let V be a 2-dimensional K-vector space. Let*

$$\psi : V \times V \longrightarrow \mathbb{Q}_p$$

as in (8.1.2). Then $\mathrm{inv}(V, \psi) = 1$ *iff there exists an O_K-lattice $\Lambda \subset V$ such that ψ is integral on Λ and such that* $\mathtt{h}(\Lambda, \psi) = \mathtt{0}$*, i.e., such that $\psi|_{\Lambda\times\Lambda}$ is a perfect pairing. Moreover, Λ is uniquely determined up to* $\mathrm{Aut}(V, \psi)$*.*

Similarly, $\mathrm{inv}(V, \psi) = -1$ *iff there exists an O_K-lattice $\Lambda \subset V$, such that ψ is integral on Λ and such that* $\mathtt{h}(\Lambda, \psi) = \mathtt{1}$*. Moreover, Λ is uniquely determined up to* $\mathrm{Aut}(V, \psi)$*. In this case, $\psi|_{\Lambda\times\Lambda}$ is called* almost perfect.

Proof. This reduces to the analogous statement for the anti-hermitian form $\tilde{\psi} : V \times V \longrightarrow K$ defined by

$$\mathbf{t}(\xi\tilde{\psi}(x_1, x_2)) = \psi(\xi x_1, x_2), \quad x_1, x_2 \in V, \xi \in K.$$

where $\mathbf{t} : K \longrightarrow \mathbb{Q}_p$ is defined by $\mathbf{t}(a) = \mathrm{tr}_{K/\mathbb{Q}_p}(\vartheta^{-1}a)$, where ϑ denotes the different of $K/\mathbb{Q}_p$. Then it follows from loc. cit. □

Lemma 8.1.3 ([13], Prop. 8.1 a))**.** *Let $p \neq 2$, and let F be a local p-adic field and K/F a ramified quadratic field extension. Let V be a 2-dimensional K-vector space. Let*

$$\psi : V \times V \longrightarrow \mathbb{Q}_p.$$

as in (8.1.2). Then there exists an O_K-lattice $\Lambda \subset V$ such that ψ induces a perfect form

$$\psi : \Lambda \times \Lambda \longrightarrow \mathbb{Z}_p.$$

Moreover Λ is unique up to $\mathrm{Aut}(V, \psi)$*.* □

8.2. **The r-adjusted invariant.** Let K be a CM-field, with totally real subfield F. We set $\Phi = \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(K, \bar{\mathbb{Q}})$. Let r be a generalized CM-type of rank n, i.e., $r_\psi + r_{\bar{\psi}} = n$. Let $E = E_r$ be the reflex field, cf. [18, §2]. A CM-*triple* over an O_E-algebra R is a triple (A, ι, λ) where A is an abelian scheme over R with an action $\iota : O_K \to \mathrm{End}\, A$ with satisfies the Kottwitz condition (KC_r) and a polarization λ whose Rosati involution induces the conjugation of K/F. In the case $n = 2$ this is a CM-triple with satisfies the Kottwitz condition. Let v be a place of F. We define an r-*adjusted* invariant $\mathrm{inv}_v^r(A, \iota, \lambda)$ attached to a triple (A, ι, λ) of CM-type r, defined over a field k that is at the same time an O_E-algebra. When v is non-archimedean split in K, then $\mathrm{inv}_v^r(A, \iota, \lambda) = \mathrm{inv}_v(A, \iota, \lambda) = 1$. If v is archimedean, or non-archimedean non-split in K, with residue characteristic of v different from the characteristic of k, then $\mathrm{inv}_v^r(A, \iota, \lambda) = \mathrm{inv}_v(A, \iota, \lambda)$, i.e., the adjusted invariant coincides with the invariant of [18, §3]. Comp. section 2.4 for the definition of the latter invariant for $n = 2$. The case of general n is substantially the same.

Now let v be non-split with residue characteristic equal to the characteristic p of k. We may assume that k is algebraically closed. Let us first assume that the O_E-algebra structure of k is induced by a $O_{\overline{\mathbb{Q}}}$-algebra structure. Let $\tilde{\nu}$ be the induced p-adic place of $\overline{\mathbb{Q}}$. Let

$$\Phi_v = \{\varphi : K \longrightarrow \overline{\mathbb{Q}} \mid \tilde{\nu} \circ \varphi \text{ induces } v \}. \tag{8.2.1}$$

Then

$$\Phi_v = \mathrm{Hom}_{\mathbb{Q}_p}(K_v, \overline{\mathbb{Q}}_{\tilde{\nu}}).$$

Also let

$$r_v = r_{|\Phi_v}.$$

Now define

$$\mathrm{inv}_v^r(A, \iota, \lambda) = \mathrm{inv}_v(A, \iota, \lambda)\, \mathrm{sgn}(r_v), \tag{8.2.2}$$

with

$$\mathrm{sgn}(r_v) = (-1)^{(\frac{n}{2}d_v - \sum_{\varphi\in\Phi_v^+} r_\varphi)} = (-1)^{\frac{1}{2}\sum_{\varphi\in\Phi_v^+}(r_\varphi - r_{\bar{\varphi}})}. \tag{8.2.3}$$

Here Φ_v^+ is a half-system of embeddings in Φ_v, which has cardinality $d_v = [F_v : \mathbb{Q}_p]$. Since $r_\varphi + r_{\overline{\varphi}} = n$ for all $\varphi \in \Phi_v$, and n is supposed to be even, (8.2.3) is independent of Φ_v^+. Note that $\mathrm{sgn}(r_v)$ only depends on the place ν of E induced by $\widetilde{\nu}$.

The correct version of [18, Prop. 3.2] is now as follows.

Proposition 8.2.1. *Let S be an O_E-scheme. Let (A, ι, λ) be a CM-triple over S which satisfies* (KC_r). *Let $c \in \{\pm 1\}$. Then for every place v of F, the set of points $s \in S$ such that*

$$\mathrm{inv}_v^r(A_s, \iota_s, \lambda_s) = c$$

is open and closed in S.

Proof. Clearly we may assume that S is an O_E-scheme of finite type. Further we can assume that S is irreducible. Obviously the invariant is constant on the generic fiber of S. Also, we may assume that v is non-archimedean non-split in K.

First we consider the case when S is an irreducible scheme of finite type over κ_{E_ν}. Since each local ring of S is dominated by a discrete valuation ring R, it is enough to consider the case $S = \operatorname{Spec} R$. We may replace R by a discrete valuation ring that dominates R. Therefore we can assume the R is complete with algebraically closed residue class field, i.e., $R \cong k[[t]]$ for an algebraically closed field k. According to the action of $F \otimes \mathbb{Q}_p$, the p-divisible group X of A is isogenous to a product $\prod_{w|p} X_w$. We consider the factor X_v. Let $\mathcal{P}$ be the display of X_v over R, cf. (3.1.9). We note that P is the value of the crystal of X_v at the pd-thickening $W(R)/R$. By Lemma 8.2.2 below, there is an element $x \in \wedge^n_{O_{K_v} \otimes_{\mathbb{Z}_p} W(R)} P$ such that $Fx = p^{n/2}x$. We define the anti-hermitian form

$$\varkappa : P_{\mathbb{Q}} \times P_{\mathbb{Q}} \to K_v \otimes_{\mathbb{Z}_p} W(R)$$

as in (2.4.3). We consider the hermitian form $h = \wedge^n_{O_{K_v} \otimes_{\mathbb{Z}_p} W(R)} \varkappa$ on $\wedge^n_{O_{K_v} \otimes_{\mathbb{Z}_p} W(R)} P_{\mathbb{Q}}$. From the equation

$$h(Fy_1, Fy_2) = p^n \; {}^F h(y_1, y_2), \quad y_1, y_2 \in \wedge^n_{O_{K_v} \otimes_{\mathbb{Z}_p} W(R)} P_{\mathbb{Q}},$$

we obtain that $h(x, x)$ lies in the invariants $(K_v \otimes_{\mathbb{Z}_p} W(R))^F = K_v$. Because h is hermitian, we obtain $h(x, x) \in F_v$. The element x can be used to determine the invariant of the Dieudonné module $P \otimes_{W(R)} W(L)$ obtained for arbitrary base change $R \to L$ to a perfect field. Therefore $\mathrm{inv}_v(A_s, \iota_s, \lambda_s) = \mathrm{inv}_v(A_\eta, \iota_\eta, \lambda_\eta)$ and $\mathrm{inv}_v^r(A_s, \iota_s, \lambda_s) = \mathrm{inv}_v^r(A_\eta, \iota_\eta, \lambda_\eta)$, where s and η denote the special and the generic point of $\operatorname{Spec} R$. For the comparison with the definition of the invariant of a Dieudonné module we should remark that the equations $Fx = p^{n/2}x$ and $Vx = p^{n/2}x$ are equivalent because $FV = p^n$ on $\wedge^n_{O_{K_v} \otimes_{\mathbb{Z}_p} W(R)} P$.

Now we consider the case when the function field of S has characteristic 0. This case can be reduced to the case when $S = \operatorname{Spec} O_L$, where L is the completion of a subfield of $\overline{\mathbb{Q}}_p$ which contains E and such that its ring of integers $O = O_L$ is a discrete valuation ring with residue field $\overline{\mathbb{F}}_p$. We denote by A_L the generic fiber of A, and by A_k its special fiber.

We decompose the rational p-adic Tate module of A_L, resp. the rational Dieudonné module of A_k, with respect to the actions of $F \otimes \mathbb{Q}_p$,

$$V_p(A_L) = \bigoplus_{w|p} V_w(A_L), \quad M(A_k)_{\mathbb{Q}} = \bigoplus_{w|p} M(A_k)_{\mathbb{Q},w}.$$

Here $V_w(A_L)$ is a free $K \otimes_F F_w$-module of rank n, and $M(A_k)_{\mathbb{Q},w}$ is a free $K \otimes_F F_w \otimes_{\mathbb{Q}_p} W(k)_{\mathbb{Q}}$-module of rank n. Set $\breve{\mathbb{Q}}_p = W(k)_{\mathbb{Q}}$.

Let $S_v = \bigwedge^n_{K_v} V_v(A_L)$ and $N_{\mathbb{Q},v} = \bigwedge^n_{K_v} M(A_k)_{\mathbb{Q},v}$. Both are equipped with hermitian forms (for the first module, cf. [18, section 3, case b)]; for the second module, cf. subsection 2.4). Also, we have $N_{\mathbb{Q},v} = \mathbf{1}_v(\frac{n}{2})$, where $\mathbf{1}_v$ is a multiple of the unit object in the category of Dieudonné modules, comp. (2.4.5), or Lemma 8.2.2. Let U_v be the image under the Fontaine functor of $N_{\mathbb{Q},v}(-\frac{n}{2})$. We need to compare the two hermitian vector spaces $S_v(-\frac{n}{2})$ and U_v.

Let T be the torus over $\mathbb{Q}_p$ which is the kernel of the map defined by the norm of K_v/F_v,

$$1 \longrightarrow T \longrightarrow \operatorname{Res}_{K_v/\mathbb{Q}_p} \mathbb{G}_{m,K_v} \longrightarrow \operatorname{Res}_{F_v/\mathbb{Q}_p} \mathbb{G}_{m,F_v} \longrightarrow 1.$$

Then $H^1(\mathbb{Q}_p, T) = F_v^\times / \operatorname{Nm}(K_v)$. We may regard the isomorphisms of hermitian vector spaces $\operatorname{Isom}(U_v, S_v(-\frac{n}{2}))$ as an etale sheaf on $\operatorname{Spec} F_v$. This is a T-torsor. Its class $\mathrm{cl}(U_v, S_v(-\frac{n}{2}))$ is calculated by [27, Prop. 1. 20].

To evaluate this formula, note that the first summand, $\kappa(b)$, in loc. cit. is trivial. To evaluate the second summand, $\mu^\sharp$, we use the following description of the filtration on $N_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} \overline{\breve{\mathbb{Q}}}_p$. For the filtration of $M(A_k)_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} \overline{\breve{\mathbb{Q}}}_p = \oplus_{\varphi\in\Phi_v} M(A_k)_{\mathbb{Q},v,\varphi}$ we have that the jumps are in degree 0 and 1, with

$$(0) \subset \mathrm{Fil}^1_\varphi \subset^{r_\varphi} M(A_k)_{\mathbb{Q},v,\varphi}. \tag{8.2.4}$$

The upper index means that the cokernel has dimension r_φ. For the filtration of the one-dimensional vector space $N_{\mathbb{Q},v,\varphi}$, this means that the unique jump is in degree $n - r_\varphi$. We use the identification

$$X_*(T) = \mathrm{Ker}\big(\mathrm{Ind}_{F_v}^{K_v}(\mathrm{Ind}_{\mathbb{Q}_p}^{F_v}(\mathbb{Z})) \longrightarrow \mathrm{Ind}_{\mathbb{Q}_p}^{F_v}(\mathbb{Z})\big).$$

Then the corresponding filtration on $N(A)_{\mathbb{Q},v,\varphi}(-\frac{n}{2})$ is given by the cocharacter $\mu \in X_*(T)$ with

$$\mu_\varphi = \frac{n}{2} - r_\varphi, \quad \varphi \in \Phi_v. \tag{8.2.5}$$

We have to determine the image $\mu^\sharp$ of μ in $X_*(T)_\Gamma$. Under the identification $X_*(T)_\Gamma = H^1(\mathbb{Q}_p, T) = \mathbb{Z}/2$, we obtain

$$\mathrm{cl}(U_v, S_v(-\frac{n}{2})) = \mu^\sharp = \sum_{\varphi\in\Phi_v^+} \mu_\varphi = \frac{n}{2} d_v - \sum_{\varphi\in\Phi_v^+} r_\varphi\,, \tag{8.2.6}$$

where we used the notation introduced for (8.2.3). We deduce $\mathrm{inv}_v^r(A_k, \iota_k, \lambda_k) = \mathrm{inv}_v(A_L, \iota_L, \lambda_L)$, as desired. □

In the proof of Proposition 8.2.1, we used the following lemma.

Lemma 8.2.2. *Let $F/\mathbb{Q}_p$ be a finite field extension of degree d and K/F be a quadratic field extension. Let n be an even natural number. Let k be an algebraically closed field of characteristic p. Let (X, ι) be a p-divisible group over $k[[t]]$ of dimension nd and height $2nd$ with an action $O_K \to \mathrm{End}\, X$. Let $(\mathcal{P}, \iota)$ be the display of X, cf. (3.1.9). Then there exists a non-zero element $x \in \wedge^n_{O_K\otimes_{\mathbb{Z}_p} W(k)[[t]]} P$ such that*

$$\wedge^n F(x) = p^{n/2} x.$$

Proof. We consider the $\mathbb{Z}_p$-frame $\mathcal{B}_k = (W(k)[[t]], pW(k)[[t]], k[[t]], \sigma, \dot\sigma)$, where σ is the extension of the Frobenius on $W(k)$ to the power series ring given by $\sigma(t) = t^p$ and where $\dot\sigma = (1/p)\sigma$. The evaluation P_1 of the crystal of X at the pd-thickening $W(k)[[t]]/k[[t]]$ has the structure of a $\mathcal{B}_k$-display. The display $\mathcal{P}$ is obtained by base change with respect to a morphism of frames $\mathcal{B}_k \to \mathcal{W}(k[[t]])$, cf. [34] and [21]. Therefore, it is enough to prove our assertion for the $\mathcal{B}_k$-display of X which we will now denote by $\mathcal{P}$.

We consider first the case when K/F is ramified. When writing $\det_{W(k)[[t]]} F$, we mean this with respect to an arbitrary $W(k)[[t]]$-basis of P. This determinant is well determined up to multiplication with a unit in $W(k)[[t]]$. We know that $\det_{W(k)[[t]]} F = p^{hd} u_1$ for some $u_1 \in (W(k)[[t]])^\times$.

We consider the decomposition $P = \oplus P_\psi$ according to

$$O_K \otimes W(k)[[t]] = \prod_\psi O_K \otimes_{O_{F^t},\tilde\psi} W(k)[[t]].$$

The Frobenius is graded, $F : P_\psi \to P_{\psi\sigma}$. We conclude that $\det_{W(k)[[t]]}(F^f|P_\psi) = p^{nd} u_2$ for some unit $u_2 \in (W(k)[[t]])^\times$. We fix ψ. Up to a unit we have

$$\mathrm{Nm}_{K/F^t} \det_{O_K\otimes_{O_{F^t},\tilde\psi} W(k)[[t]]}(F^f|P_\psi) = \det_{W(k)[[t]]}(F^f|P_\psi). \tag{8.2.7}$$

We fix a normal extension L of $W(k)_{\mathbb{Q}}$ which contains $K \otimes_{O_{F^t},\tilde\psi} W(k)$. The left hand side of (8.2.7) is the product of conjugates $c_1, \ldots, c_{2e} \in O_L[[t]]$ of $\det_{O_K\otimes_{O_{F^t},\tilde\psi} W(k)[[t]]}(F^f|P_\psi)$. These elements have the same order with respect to the prime element ω_L of L which is a prime element in the regular local ring $O_L[[t]]$. We rewrite (8.2.7)

$$c_1 \cdot c_2 \cdot \ldots \cdot c_{2e} = p^{nd} \cdot u_3,$$

for some unit u_3. Since $O_L[[t]]$ is factorial, we find $c_i = p^{fn/2}\mu_i$ for some units μ_i. We conclude that

$$\det_{O_K\otimes_{O_{F^t},\tilde\psi} W(k)[[t]]}(F^f|P_\psi) = p^{fn/2} u_4$$

for some unit $u_4 \in (O_K \otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]])^\times$. Since P_ψ is a free $O_K \otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]]$-module of rank n, we find that, for each element $y_\psi \in \wedge^n_{O_K\otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]]} P_\psi$, there is an equation $\wedge^n F^f y_\psi = p^{fn/2} u(y_\psi) y_\psi$ for some unit $u(y_\psi) \in O_K \otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]]$. On the last ring, σ^f acts via the second factor. There is a unit $\zeta \in O_K \otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]]$ such that

$$\sigma^f(\zeta)\zeta^{-1} = u(y_\psi).$$

Indeed, consider the image $\bar{u}$ of $u(y_\psi)$ in $O_K \otimes_{O_{F^t},\tilde{\psi}} W(k)$ by setting $t = 0$. It is well-known that in this ring $\sigma^f(\bar{\zeta})\bar{\zeta}^{-1} = \bar{u}$ is solvable. One can lift $\bar{\zeta}$ successively modulo t^n to a solution ζ. Then $x_\psi = \zeta y_\psi$ satisfies

$$\wedge^n F^f x_\psi = p^{fn/2} x_\psi.$$

We define $x_{\psi\sigma^i} \in P_{\mathbb{Q}}$ by $\wedge^n F^i x_\psi = p^{in/2} x_{\psi\sigma^i}$ for $i = 1, \ldots, f$. Then $x = (x_\psi) \in P \otimes \mathbb{Q}$ satisfies $\wedge^h F(x) = p^{n/2} x$. Multiplying by a power of p we can arrange that $x \in P$.

The proof in the unramified case is almost the same. We indicate the differences. In this case $\mathrm{Hom}_{\mathbb{Q}_p\text{-Alg}}(F^t, W(k)_{\mathbb{Q}})$ has $2f$ elements. Therefore we have the equation (8.2.7) with f replaced by $2f$,

$$\det_{O_K\otimes_{O_{F^t},\tilde{\psi}} W(k)[[t]]}(F^{2f} | P_\psi) = p^{fn} u_4.$$

We define $x_{\psi\sigma^i} \in P_{\mathbb{Q}}$ by $\wedge^n F^i x_\psi = p^{in/2} x_{\psi\sigma^i}$ for $i = 1, \ldots, 2f$. Then $x = (x_\psi) \in P \otimes \mathbb{Q}$ satisfies $\wedge^h F(x) = p^{n/2} x$. □

Remarks 8.2.3. (i) The remarks and results on a product formula at the end of §3 of [18] become correct when the invariants $\mathrm{inv}_v(A, \iota, \lambda)$ are replaced by the adjusted invariants $\mathrm{inv}_v^r(A, \iota, \lambda)$.

(ii) In the definition of $\mathcal{M}_{r,\mathtt{h},V}$ in [18, (4.3)], the invariants $\mathrm{inv}_v(A, \iota, \lambda)$ have to be replaced by the adjusted invariants $\mathrm{inv}_v^r(A, \iota, \lambda)$.

(iii) One defines in the obvious way the r-adjusted invariant $\mathrm{inv}^r(X, \iota, \lambda)$ of a *local* CM-triple of type r, (X, ι, λ), over a field of characteristic p.

8.3. **r-adjusted invariant and the contracting functor.** In this subsection, we return to the situation in section 2.1. We assume that K/F is a field extension. Let k be an algebraically closed field of characteristic p with an O_F-algebra structure, i.e., $k \in \mathrm{Nilp}_{O_F}$.

We consider the case where r is special. Consider an object $(\mathcal{P}_c, \iota_c, \beta_c) \in \mathfrak{d}\mathfrak{R}_k^{\mathrm{pol}}$ cf. Definition 4.4.10. We write $\mathcal{P}_c = (P_c, F_c, V_c)$ for the corresponding $\mathcal{W}_{O_F}(k)$-Dieudonné module. To avoid too many double notations we denote the Frobenius automorphism on $W_{O_F}(k)$ by τ. The Verschiebung on $W_{O_F}(k)$ is then $\pi\tau^{-1}$. For our purposes it is more convenient to allow quasi-polarizations, i.e., β_c is a $W_{O_F}(k)$-bilinear form

$$P_c \otimes \mathbb{Q} \times P_c \otimes \mathbb{Q} \longrightarrow W_{O_F}(k)_{\mathbb{Q}},$$

such that $(\mathcal{P}_c, \iota_c, p^t\beta_c) \in \mathfrak{d}\mathfrak{R}_k^{\mathrm{pol}}$ for large enough $t \in \mathbb{Z}$. Then β_c is alternating and the following equations hold:

$$\beta_c(F_c u_1, F_c u_2) = \pi\tau(\beta_c(u_1, u_2)), \quad u_1, u_2 \in P_c \otimes \mathbb{Q},$$
$$\beta_c(\iota_c(a)u_1, u_2) = \beta_c(u_1, \iota_c(\bar{a})u_2), \quad a \in K.$$

The polarization β_c defines an anti-hermitian form

$$\varkappa_c : P_c \otimes \mathbb{Q} \times P_c \otimes \mathbb{Q} \longrightarrow K \otimes_{O_F} W_{O_F}(k), \tag{8.3.1}$$

by the formula

$$\mathrm{Tr}_{K/F}\, a\varkappa_c(u_1, u_2) = \beta_c(au_1, u_2),\ a \in K \otimes_{O_F} W_{O_F}(k),\ u_1, u_2 \in P_c \otimes \mathbb{Q}.$$

We note that, by Lemma 3.1.15, $P_c \otimes \mathbb{Q}$ is a free $K \otimes_{O_F} W_{O_F}(k)$-module of rank two.

Since $\mathrm{Lie}\, P_c$ has dimension 2, we have $\mathrm{ord}_\pi \det W_{O_F}(k)(V_c | P_c) = 2$. We recall that, for an arbitrary $K \otimes_{O_F} W_{O_F}(k)$-linear map $V_c^\sharp : P_c \otimes \mathbb{Q} \longrightarrow P_c \otimes \mathbb{Q}$,

$$\mathrm{Nm}_{K/F} \det{}_{K\otimes_{O_F} W_{O_F}(k)}(V_c^\sharp | P_c \otimes \mathbb{Q}) = \det_{W_{O_F}(k)}(V_c^\sharp | P_c \otimes \mathbb{Q}).$$

We conclude that

$$\begin{aligned} &\mathrm{ord}_\Pi \det_{K\otimes_{O_F} W_{O_F}(k)}(V_c | P_c) = 2, \quad K/F \text{ ramified},\\ &\mathrm{ord}_\pi \det_{K\otimes_{O_F} W_{O_F}(k)}(V_c | P_c) = 2, \quad K/F \text{ unramified}. \end{aligned} \tag{8.3.2}$$

With our convention $\pi = \Pi$ in the unramified case, this is the same formula.

Let

$$H_{\mathrm{c}} = \bigwedge^2_{K\otimes_{O_F} W_{O_F}(k)} P_{\mathrm{c}} \otimes \mathbb{Q}.$$

This is a free $K \otimes_{O_F} W_{O_F}(k)$-module of rank 1. There is an element $x_{\mathrm{c}} \in H_{\mathrm{c}}$ such that

$$\wedge^2 V_{\mathrm{c}}\, x_{\mathrm{c}} = \pi x_{\mathrm{c}}. \tag{8.3.3}$$

The existence of x_{c} follows from (8.3.2) and the fact that the τ-conjugacy class of an element $\xi \in K \otimes_{O_F} W_{O_F}(k)$ is determined by its order, compare (2.4.2).

The anti-hermitian form $\varkappa_{\mathrm{c}}$ induces on H_{c} an hermitian form

$$h_{\mathrm{c}} = \wedge^2 \varkappa_{\mathrm{c}} : H_{\mathrm{c}} \times H_{\mathrm{c}} \longrightarrow K \otimes_{O_F} W_{O_F}(k).$$

We find

$$h_{\mathrm{c}}(\wedge^2 V_{\mathrm{c}}\, x_1, \wedge^2 V_{\mathrm{c}}\, x_2) = \pi^2 \tau^{-1}(h_{\mathrm{c}}(x_1, x_2)), \tag{8.3.4}$$

where τ acts on $K \otimes_{O_F} W_{O_F}(k)$ via the second factor. Using (8.3.3) this implies

$$h_{\mathrm{c}}(x_{\mathrm{c}}, x_{\mathrm{c}}) \in F^\times \subset K \otimes_{O_F} W_{O_F}(k).$$

The following definition is analogous to (2.4.7).

Definition 8.3.1. The invariant $\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}) \in \{\pm 1\}$ is defined as the image of $h_{\mathrm{c}}(x_{\mathrm{c}}, x_{\mathrm{c}})$ by the canonical map

$$F^\times \longrightarrow F^\times / \operatorname{Nm}_{K/F} K^\times \xrightarrow{\sim} \{\pm 1\},$$

The following proposition relates this invariant with the invariant (2.4.7) under the contracting functor.

Proposition 8.3.2. *Let K/F be a field extension and let r be special. Recall the reflex field E associated to r. Let $k \in Nilp_{O_{\breve{E}}}$ be an algebraically closed field. Let $(\mathcal{P}, \iota, \beta) \in \mathfrak{dP}^{\mathrm{pol}}_{r,k}$ and let $(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}) \in \mathfrak{dR}^{\mathrm{pol}}_k$ be its image by the contracting functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$, cf. (4.4.14). Then*

$$\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = \mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}}).$$

Here the r-adjusted invariant is given by

$$\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = (-1)^{d-1} \mathrm{inv}(\mathcal{P}, \iota, \beta).$$

Proof. The second assertion follows from the definition of $\mathrm{sgn}(r)$, cf. (8.2.3). Let us prove the first assertion.

We begin with the ramified case. We have the decomposition $P = \oplus_\psi P_\psi$, cf. (4.3.6). By the definition of the contracting functor for Dieudonné modules, we have

$$P_{\mathrm{c}} = P_{\psi_0}, \quad V_{\mathrm{c}} = \Pi^{-ef+1} V,$$

cf. Remark 4.4.12. The bilinear form $\tilde{\beta}_{\mathrm{c}}$ on P_{c} is the restriction of $\tilde{\beta}$ of Proposition 4.4.5. Since we may change $\tilde{\beta}$ by a factor in $F^\times$ without changing the invariant, we may replace $\tilde{\beta}$ by $\vartheta^{-1}\tilde{\beta}$, i.e., we may assume that $\mathrm{Tr}_{F/\mathbb{Q}_p} \tilde{\beta} = \beta$. We define the anti-hermitian form

$$\varkappa : P \otimes \mathbb{Q} \times P \otimes \mathbb{Q} \longrightarrow K \otimes_{\mathbb{Z}_p} W(k)$$

by $\mathrm{Tr}_{K/F}\, \varkappa = \tilde{\beta}$. On

$$H = \bigwedge^2_{K\otimes W(k)} P \otimes \mathbb{Q} \tag{8.3.5}$$

we obtain the hermitian form $h = \wedge^2 \varkappa$. We have the decomposition

$$H = \bigoplus_\psi \bigwedge^2_{K\otimes_{O_{F^t},\tilde{\psi}} W(k)} P_\psi \otimes \mathbb{Q} = \bigoplus_\psi H_\psi.$$

The hermitian form h is the orthogonal sum of the induced forms

$$h_\psi : H_\psi \times H_\psi \longrightarrow K \otimes_{O_{F^t},\tilde{\psi}} W(k).$$

To determine $\mathrm{inv}(\mathcal{P}_c, \iota_c, \beta_c)$, we consider $\varkappa_c$ defined by $\mathrm{Tr}_{K/F} \varkappa_c = \beta_c$ and the hermitian form $h_c = \wedge^2 \varkappa_c$ on $H_c = H_{\psi_0}$. The form h_{ψ_0} coincides with the hermitian form deduced from the form $\tilde{\beta}_c$ above. By definition $\beta_c = \eta_{0.k}^f \tilde{\beta}_c$, cf. (4.4.21). Hence we have

$$h_c = \eta_{0,k}^{2f} h_{\psi_0}.$$

We choose an element $x \in H$ such that

$$\wedge^2 V(x) = px.$$

Let x_{ψ_0} be the ψ_0-component of x. Then $\mathrm{inv}(\mathcal{P}, \iota, \beta)$ is given by the element

$$h_{\psi_0}(x_{\psi_0}, x_{\psi_0}) \in F^\times.$$

We set $z_{\psi_0} = \eta_{0,k}^{-f} x_{\psi_0}$. Then we find

$$h_c(z_{\psi_0}, z_{\psi_0}) = \eta_{0,k}^{2f} h_{\psi_0}(\eta_{0,k}^{-f} x_{\psi_0}, \eta_{0,k}^{-f} x_{\psi_0}) = h_{\psi_0}(x_{\psi_0}, x_{\psi_0}).$$

From $V_c = \Pi^{-d+1} V^f$ and $\wedge^2 V^f x_{\psi_0} = p^f x_{\psi_0}$, we obtain

$$\begin{aligned} \wedge^2 V_c(x_{\psi_0}) &= (-1)^{d-1} \pi (p/\pi^e)^f x_{\psi_0} \\ \wedge^2 V_c(z_{\psi_0}) &= \tau^{-1}(\eta_{0,k}^{-f})(-1)^{d-1} \pi (p/\pi^e)^f \eta_{0,k}^f z_{\psi_0} = (-1)^{d-1} \pi z_{\psi_0}. \end{aligned}$$

By Lemma 8.3.3 below, $h_c(z_{\psi_0}, z_{\psi_0}) \in F^\times$ defines $(-1)^{d-1}\mathrm{inv}(\mathcal{P}_c, \iota_c, \beta_c)$.

We consider now the unramified case. As before, we have H with its hermitian form h, cf. (8.3.5). We consider the decomposition

$$H = \bigoplus_{\psi} \Big(\bigwedge^2_{K \otimes_{O_{K^t}, \tilde{\psi}} W(k)} P_\psi \otimes \mathbb{Q} \Big) = \bigoplus_{\psi} H_\psi, \tag{8.3.6}$$

which has now $2f$ summands. Now H_{ψ_1} and H_{ψ_2} are orthogonal for $\psi_1 \neq \bar{\psi}_2$. We denote by

$$h_\psi : H_\psi \times H_{\bar{\psi}} \longrightarrow K \otimes_{O_K^t, \tilde{\psi}} W(k)$$

the sesquilinear form induced by h. Let $x = (x_\psi) \in H$ such that $\wedge^2 V(x) = px$ or, equivalently, $\wedge^2 V(x_\psi) = p x_{\psi\sigma^{-1}}$ for all ψ. The invariant of $(\mathcal{P}, \iota, \beta)$ is the class in $F^\times / \mathrm{Nm}_{K/F} K^\times$ of $h_\psi(x_\psi, x_{\bar{\psi}}) \in F \subset K \otimes_{O_{K^t}, \tilde{\psi}} W(k)$. This is independent of ψ. Equivalently, we can consider $\mathrm{ord}_\pi h_\psi(x_\psi, x_{\bar{\psi}}) \in \mathbb{Z}/2\mathbb{Z}$. Note that ord_π makes sense for each element of $K \otimes_{O_{K^t}, \tilde{\psi}} W(k)$.

The invariant of $(\mathcal{P}_c, \iota_c, \beta_c)$ is defined by $H_c = H_{\psi_0} \oplus H_{\bar{\psi}_0}$ and h_c, via

$$\mathrm{ord}_\pi h_c(x_{c,\psi_0}, x_{c,\bar{\psi}_0}), \tag{8.3.7}$$

where $x_c = (x_{c,\psi_0}, x_{c,\bar{\psi}_0}) \in H_{\psi_0} \oplus H_{\bar{\psi}_0}$ is the element of (8.3.3). We note that we can change h_c and the elements x_{c,ψ_0}, resp. $x_{c,\bar{\psi}_0}$, by a unit in $K \otimes_{O_{K^t}, \tilde{\psi}} W(k)$ without changing (8.3.7). In particular, (8.3.7) is equal to $\mathrm{ord}_\pi h_{\psi_0}(x_{c,\psi_0}, x_{c,\bar{\psi}_0})$.

For an element $y = (y_{\psi_0}, y_{\bar{\psi}_0}) \in H_c$ we obtain from (4.4.3)

$$\wedge^2 V_c(y_{\psi_0}) = \pi^{-2g_{\bar{\psi}_0}} \wedge^2 V(y_{\psi_0}), \quad \wedge^2 V_c(y_{\bar{\psi}_0}) = \pi^{-2g_{\psi_0}} \wedge^2 V(y_{\bar{\psi}_0}).$$

We set

$$z_{\psi_0} = \pi^{-g_{\psi_0}} x_{\psi_0}, \quad z_{\bar{\psi}_0} = \pi^{-g_{\bar{\psi}_0}} x_{\bar{\psi}_0}.$$

We find

$$\begin{aligned} \wedge^2 V_c(z_{\psi_0}) &= \pi^{-g_{\psi_0}} \pi^{-2g_{\bar{\psi}_0}} p^f x_{\bar{\psi}_0} = \pi^{-g_{\psi_0} - g_{\bar{\psi}_0}} p^f z_{\bar{\psi}_0}, \\ \wedge^2 V_c(z_{\bar{\psi}_0}) &= \pi^{-g_{\bar{\psi}_0} - g_{\psi_0}} p^f z_{\psi_0}. \end{aligned}$$

We have $\mathrm{ord}_\pi(\pi^{-g_{\bar{\psi}_0} - g_{\psi_0}} p^f) = 1$. Therefore we obtain an element x_c as in (8.3.3) if we change z_{ψ_0} and $z_{\bar{\psi}_0}$ by a unit, cf. Lemma 8.3.5 below. Therefore the invariant of $(\mathcal{P}_c, \iota_c, \beta_c)$ is

$$\mathrm{ord}_\pi h_{\psi_0}(z_{\psi_0}, z_{\bar{\psi}_0}) = (-g_{\psi_0} - g_{\bar{\psi}_0}) + \mathrm{ord}_\pi h_{\psi_0}(x_{\psi_0}, x_{\bar{\psi}_0}) = (1-d) + \mathrm{ord}_\pi h_{\psi_0}(x_{\psi_0}, x_{\bar{\psi}_0}).$$

This proves the unramified case. $\square$

In the previous proof, we used two lemmas which we state as Lemmas 8.3.3 and 8.3.5.

Lemma 8.3.3. *Let K/F be ramified and let r be special. Let $y_{\mathrm{c}} \in H_{\mathrm{c}}$ be an element such that*

$$\wedge^2 V_{\mathrm{c}}(y_{\mathrm{c}}) = -\pi y_{\mathrm{c}}.$$

Then $h_{\mathrm{c}}(y_{\mathrm{c}}, y_{\mathrm{c}}) \in F^{\times}$ and the image of this element in $\{\pm 1\}$ is $-\mathrm{inv}(\mathcal{P}_{\mathrm{c}}, \iota_{\mathrm{c}}, \beta_{\mathrm{c}})$.

Proof. We choose an element $\zeta \in W_{O_F}(k)^{\times}$ such that

$$\tau^{-1}(\zeta)\zeta^{-1} = -1.$$

Then $\tau^2(\zeta) = \zeta$ and therefore $\zeta \in O_{F'} \subset W_{O_F}(k)$ where F'/F is the unramified extension of degree 2. More explicitly, we take an element $c \in \kappa_{F'}$ such that $\tau(c) = -c$ and define $\zeta = [c]$ to be the Teichmüller representative.

We set $x_{\mathrm{c}} = \zeta y_{\mathrm{c}}$. Then the equation (8.3.3) is satisfied. We find

$$h_{\mathrm{c}}(x_{\mathrm{c}}, x_{\mathrm{c}}) = \zeta^2 h_{\mathrm{c}}(y_{\mathrm{c}}, y_{\mathrm{c}}). \tag{8.3.8}$$

Since $\zeta^2 \mod \pi = c^2 \in \kappa_F$ is not a square in this field, we conclude that ζ^2 is not in the image of $\mathrm{Nm}_{K/F} : O_K^{\times} \longrightarrow O_F^{\times}$ since the norm is the square on the residue fields. Therefore the image of the right hand side of (8.3.8) in $\{\pm 1\}$ is different from the image of $h_{\mathrm{c}}(y_{\mathrm{c}}, y_{\mathrm{c}})$. □

The last lemma has the following variant which we need in the banal case.

Lemma 8.3.4. *Let K/F be ramified and let r be arbitrary. Let $(\mathcal{P}, \iota, \beta)$ be a CM-triple of type r over an algebraically closed field k. Let $y \in \wedge^2_{K\otimes W(k)} P_{\mathbb{Q}}$ be an element such that*

$$\wedge^2 V(y) = -py.$$

Set $h = \wedge^2 \varkappa$. Then $h(y, y) \in F \subset F \otimes W(k)$ and

$$h(y, y) \equiv (-1)^f \mathrm{inv}(\mathcal{P}, \iota, \beta) \mod \mathrm{Nm}_{K/F} K^{\times}.$$

Proof. We consider the decomposition

$$O_F \otimes W(k) = \prod_{\psi} O_F \otimes_{O_{F^t}, \tilde{\psi}} W(k).$$

We denote by σ the Frobenius acting on $W(k)$. It induces via any of the embeddings $\tilde{\psi}$ the Frobenius $\sigma \in \mathrm{Gal}(F^t/\mathbb{Q}_p)$. The decomposition induces a decomposition $P = \oplus_{\psi} P_{\psi}$ and

$$\wedge^2_{K\otimes W(k)} P_{\mathbb{Q}} = \oplus_{\psi} \wedge^2_{K\otimes_{O_{F^t},\tilde{\psi}} W(k)} P_{\psi,\mathbb{Q}},$$

which is orthogonal with respect to h. By restriction of h, we obtain

$$h_{\psi} : \wedge^2_{K\otimes_{O_{F^t},\tilde{\psi}} W(k)} P_{\psi,\mathbb{Q}} \times \wedge^2_{K\otimes_{O_{F^t},\tilde{\psi}} W(k)} P_{\psi,\mathbb{Q}} \longrightarrow K \otimes_{O_{F^t},\tilde{\psi}} W(k).$$

We find $\zeta \in O_F \otimes W(k)$ such that $\sigma^{-1}(\zeta)\zeta^{-1} = -1$ or equivalently $\sigma(\zeta) = -\zeta$. We set $x = \zeta y$. Then we find

$$\wedge^2 V(x) = \sigma^{-1}(\zeta) \wedge^2 V(y) = -\sigma^{-1}(\zeta) py = -\sigma^{-1}(\zeta)\zeta^{-1} px = px$$

Therefore $\mathrm{inv}(\mathcal{P}, \iota, \beta)$ is the class of

$$h(\zeta y, \zeta y) = \zeta^2 h(y, y) \mod \mathrm{Nm}_{K/F} K^{\times}. \tag{8.3.9}$$

This shows in particular that $h(y, y) \in F^{\times}$ because $\zeta^2 \in F^{\times}$. We can replace in (8.3.9) the left hand side by $\zeta_{\psi} h_{\psi}(y, y)$ which gives for all ψ the same element of F. The equation $\sigma(\zeta) = -\zeta$ may be written as $\sigma(\zeta_{\psi}) = -\zeta_{\psi\sigma}$. If we choose for a given ψ an element $\zeta_{\psi} \in O_F \otimes_{O_{F^t}} W(k)$ such that $\sigma^f(\zeta_{\psi}) = (-1)^f \zeta_{\psi}$, we obtain from this element a unique ζ.

In the case where f is even, we can choose $\zeta_{\psi} = 1$, which proves the Lemma in this case. If f is odd, we obtain that $\zeta \in F' \setminus F$. This implies as in the last Lemma that $\zeta^2 \notin \mathrm{Nm}_{K/F} K^{\times}$. This proves the case where f is odd. □

The following fact is well-known.

Lemma 8.3.5. *Let K/F be unramified. Let $u \in O_K \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)$ be a unit. Then there exists a unit $\zeta \in O_K \otimes_{O_{F^t}, \tilde{\psi}_0} W(k)$ such that*

$$\sigma^{-f}(\zeta) \cdot \zeta^{-1} = u.$$

□

Proposition 8.3.6. *Let r be banal, and let K/F be a field extension. Let $R = k$ be an algebraically closed field. Let $(C_{\mathcal{P}}, \iota, \phi)$ be the image of $(\mathcal{P}, \iota, \beta) \in \mathfrak{d}\mathfrak{P}^{\mathrm{pol}}_{r,k}$ by the polarized contraction functor $\mathfrak{C}^{\mathrm{pol}}_{r,k}$, cf. Theorem 4.5.11. Then*

$$\mathrm{inv}(C_{\mathcal{P}}, \iota, \phi) = \mathrm{inv}^r(\mathcal{P}, \iota, \beta).$$

Here the r-adjusted invariant is given by

$$\mathrm{inv}^r(\mathcal{P}, \iota, \beta) = (-1)^d \mathrm{inv}(\mathcal{P}, \iota, \beta).$$

Proof. We begin with the ramified case. We choose $\bar{\kappa}_E \subset k$. Let ${}^F\eta\eta^{-1} = \pi^e/p = \rho$, $\eta \in O_F \otimes W(\bar{\kappa}_E)$ as in (4.5.19). We define $\tilde{\beta} : P \times P \longrightarrow O_F \otimes W(k)$ by (4.5.8) and the anti-hermitian form $\varkappa : P_{\mathbb{Q}} \times P_{\mathbb{Q}} \longrightarrow K \otimes W(k)$ by $\mathrm{Tr}\, \varkappa = \tilde{\beta}$. This $\varkappa$ differs from the $\varkappa$ of (2.4.3) by a constant in F. We can use it to compute $\mathrm{inv}^r(\mathcal{P}, \iota, \beta)$. We set

$$\tilde{\beta}' = \eta\tilde{\beta}, \quad \varkappa' = \eta\varkappa.$$

We set $V' = \Pi^{-e}V$. Then we have $C_{\mathcal{P}} = \{y \in P \mid V'y = y\}$, cf. Remark 4.5.13. From this, one deduces

$$\rho \cdot {}^F\tilde{\beta}(y_1, y_2) = \tilde{\beta}(y_1, y_2), \quad y_1, y_2 \in C_{\mathcal{P}},$$

cf. (4.5.9). This implies

$${}^F\tilde{\beta}'(y_1, y_2) = \tilde{\beta}'(y_1, y_2)$$

The restriction of $\tilde{\beta}'$ to $C_{\mathcal{P}}$ is the form ϕ, cf. Remark 4.5.13.

We choose an element $x \in \wedge^2 P_{\mathbb{Q}} := \wedge^2_{K \otimes W(k)} P_{\mathbb{Q}}$ such that $\wedge^2 V(x) = (-1)^e px$. By Lemma 8.3.4 the class of $\wedge^2\varkappa(x, x) \in F^\times / \mathrm{Nm}_{K/F} K^\times = \{\pm 1\}$ is $(-1)^{fe}\mathrm{inv}(\mathcal{P}, \iota, \beta) = \mathrm{inv}^r(\mathcal{P}, \iota, \beta)$.

We note that $\wedge^2 V' = (-1)^e \pi^{-e} \wedge^2 V$. We set

$$z = \eta^{-1}x \in \wedge^2 P_{\mathbb{Q}}.$$

Then we find

$$\wedge^2 V'(z) = {}^{F^{-1}}(\eta^{-1})(-1)^e\pi^{-e} \wedge^2 V(x) = {}^{F^{-1}}(\eta^{-1})(-1)^e\pi^{-e}(-1)^e px = {}^{F^{-1}}(\eta^{-1})\eta\pi^{-e}pz = z.$$

Therefore $z \in \wedge^2_K C_{\mathcal{P}} \otimes \mathbb{Q}$. The invariant $\mathrm{inv}(C_{\mathcal{P}}, \iota, \phi)$ is given by $\wedge^2\varkappa'(z, z)$. Therefore the equality of invariants follows from

$$\wedge^2\varkappa'(z, z) = \eta^2 \wedge^2 \varkappa(\eta^{-1}x, \eta^{-1}x) = \wedge^2\varkappa(x, x).$$

Now we consider the case where K/F is unramified. We use the notation H, h, H_ψ, h_ψ from (8.3.6). We have by (4.5.13) that

$$\wedge^2_{O_K} C_{\mathcal{P}} = \{z \in H \mid \wedge^2 V(z) = \pi_r^2 z\}.$$

Using the decomposition (8.3.6), the condition for $z = (z_\psi)$ becomes

$$\wedge^2 V(z_{\psi\sigma}) = \pi^{2a_\psi} z_\psi.$$

We choose $z \neq 0$. Then

$$\mathrm{inv}(C_{\mathcal{P}}, \iota, \phi) = (-1)^{\mathrm{ord}_\pi h_\psi(z_\psi, z_{\bar{\psi}})},$$

for any ψ. We set $g_\psi = a_\psi + a_{\psi\sigma} + \ldots a_{\psi\sigma^{f-1}}$. Then we obtain

$$g_{\psi\sigma} - g_\psi = a_{\psi\sigma^f} - a_\psi = a_{\bar{\psi}} - a_\psi = e - 2a_\psi.$$

We set $u_\psi = \pi^{g_\psi} z_\psi$. Then $u = (u_\psi)$ satisfies

$$\wedge^2 V(u) = \pi^e u.$$

Indeed,

$$\wedge^2 V(u_{\psi\sigma}) = \pi^{g_{\psi\sigma}} \wedge^2 V(z_{\psi\sigma}) = \pi^{g_{\psi\sigma}}\pi^{2a_\psi} z_\psi = \pi^{g_{\psi\sigma}}\pi^{2a_\psi}\pi^{-g_\psi} u_\psi = \pi^e u_\psi.$$

Let again $\eta \in O_F \otimes W(\bar{\kappa}_E)$ be the element defined after (4.5.19). It satisfies $\eta^{-1}\, {}^{F^{-1}}\eta = (p/\pi^e)$. We set $x = \eta u$. Then $\wedge^2 V(x) = px$. Therefore

$$\mathrm{inv}(\mathcal{P}, \iota, \beta) = (-1)^{\mathrm{ord}_\pi h_\psi(x_\psi, x_{\bar{\psi}})},$$

for any ψ. Therefore the unramified case follows from

$$\mathrm{ord}_\pi h_\psi(x_\psi, x_{\bar{\psi}}) = \mathrm{ord}_\pi h_\psi(\eta\pi^{g_\psi} z_\psi, \eta\pi^{g_{\bar{\psi}}} z_{\bar{\psi}}) = (g_\psi + g_{\bar{\psi}}) + \mathrm{ord}_\pi h_\psi(z_\psi, z_{\bar{\psi}}) = ef + \mathrm{ord}_\pi h_\psi(z_\psi, z_{\bar{\psi}}).$$

□

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