A Proposition of Kedlaya

Let k be an algebraically closed field of char.$p > 0$. Let $W = W(k)$ be the Witt ring and $\sigma = \sigma_1^a$, $a > 0$ be a power of the Frobenius. We will denote by $\bar{W}_{\mathbb{Q}}$ an algebraic closure of $W \otimes \mathbb{Q}$. We extend σ in an arbitrary way to an automorphism of $\bar{W}_{\mathbb{Q}}$. Let $f = \sum_{n \in \mathbb{Z}} a_n t^n \in \bar{\mathcal{L}}$, where $a_n \in \bar{W}_{\mathbb{Q}}$ be a Laurent series. If $a_n \in W_{\mathbb{Q}}$ we will write $f \in \mathcal{L}$.

We consider the open unit disc

$$D = \{x \in \bar{W} \mid |x| < 1\}$$

If $y \in D$ and f converges for $|t| = |y|$ then we have $f(y) \in \bar{W}_{\mathbb{Q}}$.

We define

$$f^\sigma = \sum \sigma(a_n) t^{nq}.$$

Consider the map

$$\tau : D \to D, \quad \tau(x) = \sigma^{-1}(x^q).$$

We note that $|\tau(x)| = |x|^q$

Let $x \in D$. Then we have

$$f^\sigma(x) = \sigma(f(\tau(x))), \tag{1}$$

whenever one side of this equation makes sense.

Remark: This suggest the following fact which is easily verified: Let $\delta \in \mathbb{R}$, such that $0 < \delta < 1$.

If f^σ converges for $\delta < |t| < 1$, iff f converges for $\delta^q < |t| < 1$.

We will also use this in the form. If f^σ converges for $u > \operatorname{ord} t > 0$, iff f converges for $qu > \operatorname{ord} t > 0$.

Proposition 1 *Let $a \in \Gamma^c$, $a \neq 0$. We assume that $f = a^\sigma / a$ is a rational function in t.*

Then a is the product of a rational function and a unit in $W[[t]]$.

Proof: We know that $\Gamma^c[1/p]$ is a field. Then $f \in \Gamma^c[1/p]$ and our assumption says that f is in the subring $W_{\mathbb{Q}}((t))$ of $\Gamma^c[1/p]$. We find primitve polynomials $g, h \in W[t]$ (i.e. the greatest common divisor of the coefficients is 1) such that

$$f = g/h,$$

and g and h have no common divisor.

We prove the Proposition by induction on $\deg g + \deg h$. If this is zero we have $a^\sigma = ca$ for some $c \in W_{\mathbb{Q}}, c \neq 0$. By the remark above this implies that a^σ and a converges for $0 < |t| < 1$. But we may apply the same remark to the equation $(a^{-1})^\sigma = c^{-1}a^{-1}$ and conclude that a^{-1} converges in the same range. But then Lemma 5.1 of [Kedl] proves the result.

We assume now that $\deg g + \deg h = d$ and that the Proposition holds for numbers smaller that d.

We may assume that g and h have no zeros outside the open unit disc. Indeed in the opposite case we find a nonconstant primitive polynomial $u \in W[t]$ which has only roots outside the open unit disc and which divides g or h. Let c_0 be the constant coefficient of u. If $\operatorname{ord} c_0 > 0$ then the Newton polygon of u would have a negative slope which would imply a zero in D. Therefore $c_0 \in W$ is a unit. We see that u is a unit in $W[[t]]$. But then by [MZ] Lemma 31 we may write $u = b^\sigma/b$ for some unit $b \in W[[t]]$. Hence in the equation

$$a^\sigma/a = g/h \tag{2}$$

we may put the factor u from the right hand side to the left hand side, and apply the induction assumption.

We will call two elements of $\bar{W}$ equivalent if the differ by a q-th root of unity.

Let $S_1, \ldots, S_n$ the equivalence classes of roots of h. We write

$$S_i = \{r_{i1}, \ldots, r_{iq}\}$$

We denote by m_{ij} the multiplicity of r_{ij} as a root of h. Let m_i the maximum of the m_{ij} for fixed i. We have $m_i > 0$ and we may assume that $m_i = m_{i1}$.

Let e be the polynomial with the roots $\tau(S_i)$, $i = 1, \ldots n$ where every root appears exactly with multiplicity m_i. Then the roots of e^σ are $S_1 \cup S_2 \cup \ldots \cup S_n$, where each root appears with multiplicity m_i. We note that an element $\rho \in \operatorname{Gal}(\bar{W}/W))$ permutes the sets S_i. Moreover if $\rho(S_i) = S_k$ then the multiplicities m_{ij} and m_{kj} for $j = 1, \ldots, q$ are up to permutation the same. In particular we have $m_i = m_j$. This implies that $e^\sigma \in W[t]$ and therefore $e \in W[t]$ is also true.

Therefore e^σ is a multiple of the polynomial h. We obtain the equation

$$(ae)^\sigma = ag(e^\sigma/h). \tag{3}$$

Let $\delta \geq 0$ be the smallest number such that a and a^{-1} converge for $\delta < |t| < 1$. We note that by (3) the Laurent series $(ae)^\sigma$ converges in the same range. Hence the remark before the Proposition shows that ae converges for $\delta^q < |t| < 1$.

The Proposition will follow if we prove:

Lemma 2 *The polynomial e has no roots s with $\delta^q < |s| < 1$.*

We begin to show how the lemma implies the Proposition. By the Lemma e is a unit in the ring of Laurent series converging for $\delta^q < |s| < 1$ and therefore a converges in this domain because ae does. On the other hand we have the equation

$$\frac{(a^{-1})^\sigma}{a^{-1}} = \frac{h}{g}$$

If we apply the same considerations as to the equation (2) we see that also a^{-1} converges in the range $\delta^q < |t| < 1$. By the choice of δ after (3) this is only possible if $\delta = 0$. But then Proposition 5.1 [K] shows that a is of the form $ct^n u$ with $c \in W_{\mathbb{Q}}$ and $u \in W[[t]]$ a unit. This proves the Proposition.

We prove now the Lemma. Let us assume the existence of a zero s of e such that

$$\delta^q < |s| < 1.$$

We may assume that $s = \tau(S_1)$ and in particular $s = \tau(r_{11})$. Since r_{11} is a zero of h it is not a zero of g.

Since $|r_{11}|^q = |s|$ we have $\delta < |r_{11}| < 1$. Therefore $(ae)^\sigma$ converges in r_{11} so that the evaluation $(ae)^\sigma(r_{11})$ makes sense. Note that by our choice of δ the Laurent series a has no zero in r_{11}. It follows immediately from (3) that r_{11} is not a zero of $(ae)^\sigma$. By (1) we find

$$(ae)^\sigma(r_{11}) = \sigma(ae(\tau(r_{11}))).$$

Therefore ae doesn't vanish in $s = \tau(r_{11})$. Since $\tau(r_{1j}) = s$ for $j = 1, \ldots q$ there is no zero of $(ae)^\sigma$ among

$$r_{11}, r_{12}, \ldots, r_{1q}. \tag{4}$$

These elements are neither zeros of a and g as we already remarked. It follows from (3) that these are also not zeros of (e^σ/h). We see that the order of zero of e^σ and h at the elements (4) is the same, namely m_1.

Let $e_1 \in W[t]$ be the polynomial of minimal degree divisible by $(t-s)^{m_1}$. Then e_1 divides e. Moreover e_1^σ divides h. (Note that also all conjugates of the elements r_{1j} appear with multiplicity m_1 as zeros of h) We write:

$$\frac{ge_1^\sigma}{he_1} = \frac{ae_1^\sigma}{ae_1}.$$

If we reduce the fraction on the left hand side by dividing numerator and denominator by e_1^σ we see that the sum of the degree of the numerator and denominator is less thatn $\deg g + \deg h$. Therefore we are done by induction. *Q.E.D.*