B5 Cohomology of algebraic varieties

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1 The ring of Witt vectors

Let p be a fixed prime number. Let R be a commutative ring. Consider the set of infinite vectors:

$$R^{\mathbb{N}} = \{\xi = (x_0, x_1, x_2, \dots) \mid x_i \in R\}$$

This is a ring with componentwise addition and multiplication.
We will endow the same set $W(R) = R^{\mathbb{N}}$ with a new ring structure, such that for each ring homomorphism $S \to R$ the natural map $W(S) \to W(R)$ is a ring homomorphism (functoriality).
E.Witt introduced the following polynomials for $n \in \mathbb{N}$:

$$\mathbf{w}_n(\xi) = x_0^{p^n} + p x_1^{p^{n-1}} + p^2 x_2^{p^{n-2}} + \dots + p^n x_n.$$

Consider the map:

$$\begin{array}{ccc} W(R) & \longrightarrow & R^{\mathbb{N}}. \\ \xi = (x_0, x_1, x_2, \dots) & \mapsto & (\mathbf{w}_0(\xi), \mathbf{w}_1(\xi), \mathbf{w}_2(\xi), \dots) \end{array}$$

If R has no p-torsion this map is injective. One proves that in this case the image of the map is a subring. Therefore $W(R)$ is defined in this case. In general one writes R as a factor of a torsion free ring by an ideal. Then the ring structure is obtained by functoriality.
Example: $W(\mathbb{F}_p) = \mathbb{Z}_p$.
The Witt ring comes with two operators $F, V : W(R) \to W(R)$ which are called Frobenius and Verschiebung:

$${}^V\xi = (0, x_0, x_1, x_2, \dots), \quad {}^F\xi = (x_0^p, 0, 0, \dots) + p(x_1, x_2, x_3, \dots)$$

2 Displays

Let p be a prime number. Let R be a ring such that p is nilpotent in R.

Definition 2.1. *A display over R is a quadruple: $\mathcal{P} = (P, Q, F, \dot{F})$, where P is a finitely generated projective $W(R)$-module and $Q \subset P$ is a submodule, such that P/Q is a finitely generated projective R-module. There are two Frobenius-linear maps*

$$F : P \longrightarrow P, \quad \dot{F} : Q \longrightarrow P,$$

such that for $\xi \in W(R)$ and $x \in P$

$$\dot{F}({}^V\xi x) = \xi F x.$$

and $\dot{F}(Q)$ generates P as a $W(R)$-module.

Theorem 2.2. *(Zink, Asterisque 2002 and Lau, Invent. 2008). There is an equivalence of categories:*

$$\{\textit{nilpotent displays}\} \xrightarrow{\sim} \{\textit{formal } p\textit{-divisible groups}\}.$$

By definition there are decompositions as $W(R)$-modules

$$P = T \oplus L, \quad Q = VW(R)T \oplus L.$$

The Frobenius-linear homomorphism

$$\Phi : F \oplus \dot{F} : T \oplus L \to P$$

is a Frobenius linear isomorphism.
We assume that the modules T, L are free which holds locally. Then Φ is given by an invertible matrix in $GL(h, W(R))$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where the block structure is given by T and L.
Assume a second display $\mathcal{P}'$ is given by a block matrix. Then a morphism $\mathcal{P} \to \mathcal{P}'$ is the same as a matrix:

$$\begin{pmatrix} X & \mathfrak{J} \\ Z & Y \end{pmatrix} \in M(h' \times h, W(R)) \tag{2}$$

of size $h' \times h$ such that the following relation holds:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} {}^F X & \mathfrak{J} \\ p\,{}^F Z & {}^F Y \end{pmatrix} = \begin{pmatrix} X & {}^V\mathfrak{J} \\ Z & Y \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

3 Role within the CRC

B5 is related to other projects by the following topics:

1. Sheaf cohomology: A1, C3, C7, C3, C10
2. Moduli spaces and period maps: C1, C11, C7
3. Shimura varieties C7

There were joint research seminars with C10 and C7.

4 Displays of $\mathcal{O}$-modules

Let $\mathcal{O}$ be the ring of integers of a non-archimedean local field and let R be an $\mathcal{O}$-algebra such that π is nilpotent in R. Recall that a strict $\mathcal{O}$-action on a formal group is an action of $\mathcal{O}$, acting on the tangent space through the structure morphism $\mathcal{O} \to R$.
A formal group with a strict $\mathcal{O}$-action will be called a formal $\mathcal{O}$-module.

Definition 4.1. *An $\mathcal{O}$-display over R is a quadruple $(P, Q, F, \dot{F})$, where P is a finitely generated projective $W_{\mathcal{O}}(R)$-module, Q a $W_{\mathcal{O}}(R)$-submodule of P and where $F : P \to P$ and $\dot{F} : Q \to P$ are Frobenius-linear maps with conditions as in Definition 2.1.*

We have the following analogue of the main theorem of displays:

Theorem 4.2. *(Ahsendorf) The category of formal p-divisible formal $\mathcal{O}$-module over R is equivalent to the category of $\mathcal{O}$-displays over R.*

Let R be an artinian local ring with perfect residue class field. If we replace $W_{\mathcal{O}}(R)$ by the small ring of Witt vectors $\hat{W}_{\mathcal{O}}(R) \subset W_{\mathcal{O}}(R)$ we obtain the notion of a Dieudonné display.

Theorem 4.3. *(Cheng) The category of p-divisible groups over R with a strict $\mathcal{O}$-action is equivalent to the category of Dieudonné $\mathcal{O}$-displays over R.*

5 Truncated p-divisible groups and displays

Assume for simplicity that $pR = 0$. If we replace $W(R)$ by $W_n(R)$ in the matrix description (1), (2) of displays above we obtain the category

$$\mathcal{D}_n(R).$$

Let $\mathcal{BT}_n(R)$ be the category of truncated p-divisible groups of level n. There is a functor (Lau, Journ. of the AMS 2013)

$$\Phi_n : \mathcal{BT}_n(R) \to \mathcal{D}_n(R).$$

An object $G \in \mathcal{BT}_n(R)$ over a reduced ring R has order of nilpotence e if the iterated Frobenius morphism:

$$Frob_G^{e+1} : G[p] \to G[p]^{e+1}$$

is zero. There are also the truncated displays $\mathcal{D}_n^e$ of order of nilpotence e. Let $m > 0$ such that $n \geq m(e+1)$.

Theorem 5.1. *(Lau, Zink) Let R be reduced. There is a functor*

$$BT_m : \mathcal{D}_n^e(R) \to \mathcal{BT}_m(R)$$

such that for each $G \in \mathcal{BT}_n$ of order of nilpotence e

$$BT_m(\Phi_n(G)) \cong G[p^m].$$

With a different m this holds for non reduced rings.

6 Cohomology groups of Shimura curves

Let F be a totally real field in which $p \geq 5$ is unramified. Let D be a quaternion algebra over F which is ramified at one real place and unramified at places dividing p. Let S_D be the product of the prime ideals ramified in D. Let N be a squarefree ideal of F which is prime to pS_D. Let $M_{K_0(N)}$ be the associated Shimura curve. Let σ be a regular Serre weight and $\mathcal{F}_\sigma$ the a associated sheaf on $M_{K_0(N)}$. Let $\mathbb{T} = \mathbb{T}(K_0(N), \mathcal{F}_\sigma)$ be the corresponding Hecke algebra and $\mathfrak{m}$ a maximal non-Eisenstein ideal. For a rank two Galois representation $\bar{\rho} : G_F \to GL(2, \bar{\mathbb{F}}_p)$, we call $a = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{G_F}(\bar{\rho}, H^1_{et}(M_{K_0(N)} \otimes \bar{F}, \mathcal{F})_{\mathfrak{m}})$ the multiplicity of $\bar{\rho}$ in the cohomology.

Theorem 6.1. *(Cheng, Israel Jounal 2013) Let $\bar{\rho}$ be a modular Galois representation of weight σ with conductor dividing pNS_D satisfying the following conditions:*

1. *the restriction $\bar{\rho}|_{G_{F'}}$ is irreducible, where $F' = F(\sqrt{(-1)^{(p-1)/2}p})$,*
2. *if $v \mid p$, then $End_{\bar{\mathbb{F}}_p[G_{F_v}]}(\bar{\rho}|_{G_{F_v}}) = \bar{\mathbb{F}}_p$.*
3. *if $v \mid N$, then $\bar{\rho}$ is ramified at v.*
4. *if $v \mid S_D$, and $\mathrm{Norm}(v)^2 \equiv 1 \pmod{p}$, then $\bar{\rho}$ is ramified at v.*

Then $a \leq 1$. If $a = 1$, then $H^1_{et}(M_{K_0(N)} \otimes \bar{F}, \mathcal{F})_{\mathfrak{m}}$ is free $\mathbb{T}_{\mathfrak{m}}$-module of rank two.

Our main goal is to understand the space

$$\mathcal{H} = \varinjlim_K H^1(M_K \otimes \bar{F}, \bar{\mathbb{F}}_p),$$

which is closely related to the mod p Langlands correspondence.

7 Overconvergent de Rham-Witt complex

Let p be a prime number. Let k be a perfect field of char. p. For a smooth algebra A over k we defined the overconvergent de Rham-Witt complex

$$W^\dagger\Omega^{\cdot}_{A/k}. \tag{3}$$

If $X = \operatorname{Spec} A$ is affine one can lift A to a weakly complete smooth algebra $A^\dagger$ over $W(k)$. We proved: The de Rham complex of $A^\dagger$ is noncanonically quasi isomorphic to (3).
If X is a smooth scheme over k we obtain presheaves $\operatorname{Spec} A \mapsto W^\dagger\Omega^i_{A/k}$ for each i.

Theorem 7.1. *(Davis,Langer,Zink: Annal.ENS 2011; Crelle 2012)*

(1) $W^\dagger\Omega^i_{X/k}$ is a sheaf for each i.

$$H^j(X, W^\dagger\Omega^i_{X/k}) = 0, \quad \textit{for } X \textit{ affine}, \ j > 0.$$

(2) The hypercohomology groups

$$\mathbb{H}^i(X, W^\dagger\Omega^{\cdot}_{X/k}) \otimes \mathbb{Q}$$

are canonically isomorphic to the rigid cohomology of Berthelot.

(3) If X is a projective scheme we have a comparison isomorphism with crystalline cohomology

$$\mathbb{H}^i(X, W^\dagger\Omega^{\cdot}_{X/k}) \cong H^i_{crys}(X/W(k)).$$

Program: Overconvergent De Rham Witt cohomology for singular varieties. Overconvergent de Rham-Witt cohomology is finitely generated modulo torsion.

8 Display structure on de Rham-Witt cohomology

We set $P_0 := \mathbb{H}^i(X, W\Omega^{\cdot}_{X/R})$ for a smooth and projective scheme X over a ring R. Now i is fixed. We assume that R is a local artin ring whose residue field k is perfect of char. p. The cohomology comes naturally with a chain of $W(R)$-modules

$$P_i \to P_{i-1} \to \dots \to P_0,$$

and Frobenius-linear maps $F_i : P_i \to P_0$.

Theorem 8.1. *(Langer, Zink, Doc. Math. 2008) If the cohomology of X has good base change properties (e.g. for abelian varieties and $K3$-surfaces), the data above are a display structure on the cohomology.*

The display property means that all data may be decribed by an invertible block matrix similiar to (1), (2):

$$\mathcal{A} \in GL(h, W(R)), \quad h = \text{ Betti number.}$$

The whole category of displays is completely determined in terms of those matrices which should be regarded as the period matrices of de Rham-Witt cohomology.
It is decisive for the following to replace $W(R)$ by a smaller ring $\hat{W}(R)$ but we ignore this here. If X is an ordinary $K3$-surface then the display determines the Grothendieck crystal of the $K3$-surface.
Conjecture: Let $S \to R$ be a surjection of artinian rings. Let X be an ordinary $K3$-surface. We have a canonical bijection:

$$\{\text{deformations of } X\}/S \xrightarrow{\sim} \{\text{deformations of the display of } X\}/S.$$

9 Applications of Displays

Theorem 9.1. *(Lau, Nicole, Vasiu, Annals of Math. 2013) Let X be a non ordinary p-divisible group over an algebraically close field of char. p of dimension d and codimension c. Let e be the integral part of $2cd/(c+d)$. Let $n > e$ and let ϕ be an endomorphism of the truncated group $X[n]$.*
Then the truncation $\phi[n-e]$ is the truncation of an endomorphism of X.

Theorem 9.2. *(Vasiu, Zink, Doc. Math. 2010) Let $\mathcal{O}$ be a discrete valuation ring of mixed characteristic $0, p$. Let $Y \to \mathcal{O}$ be a smooth scheme. Let $U \subset Y$ be an open subscheme which contains the general fibre and whose complement has codimension ≥ 2.*
Then each abelian scheme over U extends to an abelian scheme over Y.