De Jong's Theorem on Homomorphisms of p-divisible Groups

W.Messing and Th.Zink

January 8, 2002

1 Frobenius Modules

Definition 1 *Let p be a fixed prime number. A frame (A, σ) is a pair such that A is a ring without p-torsion, and σ is an endomorphism of A such that*

$$\sigma(a) = a^p \mod pA$$

Examples: Let R is a reduced ring of characteristic p. Then we set $A = W(R)$ and we take for σ the Frobenius endomorphism of $W(R)$.

Let (A, σ) be a frame. Then we extend σ to the power series ring $A[[t]]$ by setting $\sigma t = t^p$. We obtain a new frame $(A[[t]], \sigma)$.

Let k be a perfect field. We define σ on $W(k)[[t]]$ as above. Consider the ring $W(k)[[t]][t^{-1}]$. This is a Dedekind ring (in fact a principal ideal domain) and its completion in the prime ideal generated by p is a Cohen ring Γ for $K = k((t))$. Then σ extends canonically to Γ. We obtain a frame (Γ, σ). This is the frame which concerns us.

Let M, N be A-modules. Let $F : M \to N$ be a σ-linear map. We denote its linearization by :

$$\begin{array}{rccc} F^\sharp : & A \otimes_{\sigma,A} M & \to & N. \\ & a \otimes m & \mapsto & aFm \end{array}$$

Definition 2 *A Frobenius module over A is a finitely generated projective A-module M of some rank r, equipped with a σ-linear endomorphism $F : M \to M$, such that the map*

$$\wedge^r F^\sharp : A \otimes_{\sigma,A} \wedge^r M \to \wedge^r M,$$

is of the form $p^s u$, where s is some number and $u : A \otimes_{\sigma,A} \wedge^r M \to \wedge^r M$ is an isomorphism.

If M is free and we compute $\det F$ in some basis of M, then the last requirement says that $\det F = p^s \epsilon$ for some unit $\epsilon \in A$. We will write $s = \mathrm{ord}_p \det F$.

Our aim is to prove the following deep theorem of de Jong:

Theorem 3 *Let M and N be Frobenius modules over $W(k)[[t]]$. Let*

$$\phi : M \otimes_{W(k)[[t]]} \Gamma \to N \otimes_{W(k)[[t]]} \Gamma$$

be a morphism of Frobenius modules. Then we have $\phi(M) \subset N$.

For technical resons a slight generalization of definition 1 is comfortable: A quasi Frobenius module is an A-module M as in the definition equipped with a σ-linear homomorphism:

$$F : M \to M[\frac{1}{p}],$$

such that for a suitable integer $m \geq 0$ the pair $(M, p^m F)$ is a Frobenius module. For brevity we will write $M(m) = (M, p^m F)$.

In the category of (quasi) Frobenius modules we may form tensor products and exterior products:

Let (M_1, F_1) and (M_2, F_2) Frobenius modules. Then $(M_1 \otimes_A M_2, F_1 \otimes F_2)$ is a Frobenius module. Indeed, let r_i be the rank of M_i for $i = 1, 2$. We have to show that $\wedge^{r_1} F_1^\sharp \otimes \wedge^{r_2} F_2^\sharp = p^m$(isomorphism). But this question is local for the Zariski topology on $\mathrm{Spec}\, A$. Thus we may take basis of the modules involved and consider determinants. Let X_i be the matrix of $F_i^\sharp$ for $i = 1, 2$ Then our result follows from the formula:

$$\det X_1 \otimes X_2 = (\det X_1)^{r_2} (\det X_2)^{r_1}$$

This may be verified by reducing it to the case where A is an algebraically closed field, and where X_1 has triangular form.

The Frobenius module $U = (A, \sigma)$ is called the unit Frobenius module. Indeed there is a canonical isomorphism of Frobenius modules:

$$M_1 \otimes U \cong M_1$$

If (M, F) is a Frobenius module of rank r, then we may form the exterior products $(\wedge^n M, \wedge^n F)$, for any integer $n \geq 0$. One verifies that this is a Frobenius module by using the following formula for a square matrix X of size r:

$$\det \wedge^n X = (\det X)^{\begin{pmatrix} r-1 \\ n-1 \end{pmatrix}}$$

In the category of quasi Frobenius modules we have an internal Hom: Let (M_1, F_1) and (M_2, F_2) be quasi Frobenius modules. Indeed we define a σ-linear operator F on $\mathrm{Hom}_A(M, N) \otimes \mathbb{Q}$. If $\phi : M \to N$ is a A-linear map, then $F\phi$ is given by the following commutative diagram:

$$\begin{array}{ccc} A \times_{\sigma,A} (M_1 \otimes \mathbb{Q}) & \xrightarrow{F_1^\sharp} & (M_1 \otimes \mathbb{Q}) \\ {\scriptstyle 1_A \otimes \phi}\downarrow & & \downarrow{\scriptstyle F\phi} \\ A \times_{\sigma,A} (M_2 \otimes \mathbb{Q}) & \xrightarrow{F_2^\sharp} & (M_2 \otimes \mathbb{Q}) \end{array}$$

We note that $F_1^\sharp \otimes \mathbb{Q}$ is an isomorphism. One checks that $(\mathrm{Hom}_A(M_1, M_2), F)$ is a quasi Frobenius module. Clearly $F\phi = \phi$, iff $\phi : M_1 \to M_2$ is a homomorphism of quasi Frobenius modules. If $(M_2, F_2) = (A, \sigma)$ we set

$$(\hat{M}_1, \hat{F}) = (\mathrm{Hom}_A(M_1, A), F),$$

and call this the dual Frobenius module. The natural perfect pairing

$$(\ ,\ ) : \hat{M}_1 \times M_1 \to A \tag{1}$$

satisfies the relation:

$$(F\hat{m}, Fm) = \sigma(\hat{m}, m)$$

The submodule $AFM \subset M$ is a Frobenius submodule since it is isomorphic to $(A \otimes_{\sigma,A} M, \sigma \otimes F)$. Clearly the pairing (1) induces a pefect pairing between $AF\hat{M}_1$ and AFM_1.

The usual isomorphism:

$$\hat{M}_1 \otimes M_2 \to \mathrm{Hom}_A(M_1, M_2)$$

becomes an isomorphism of Frobenius modules.

Definition 4 *A homomorphism $\alpha : M_1 \to M_2$ of quasi Frobenius modules of the same rank r is called an isogeny if $\wedge^r \alpha$ is of the form $p^s u$, where $s \geq 0$ is an integer, and $u : \wedge^r M_1 \to \wedge^r M_2$ is an isomorphism.*

Proposition 5 *Let $\phi : (A, \sigma) \to (B, \sigma)$ be a homomorphism of frames. We assume that pA is a prime ideal of A contained in the radical of A, and that p is not a unit in B.*

Let $\alpha : (M_1, F_1) \to (M_2, F_2)$ be a homomorphism of Frobenius modules over A. If $\alpha \otimes id_B$ is an isogeny (respectively an isomorphism), then α is an isogeny (respectively an isomorphism).

Proof: The A-modules M_1 and M_2 have the same rank r. It suffices to show that $\wedge^r \alpha$ is an isomorphism. Consider the commutative diagram:

$$\begin{array}{ccc} \wedge^r M_1 & \xrightarrow{\wedge^r \alpha} & \wedge^r M_2 \\ {\scriptstyle \wedge^r \det F_1} \downarrow & & \downarrow {\scriptstyle \wedge^r \det F_2} \\ \wedge^r M_1 & \xrightarrow{\wedge^r \alpha} & \wedge^r M_2 \end{array} \tag{2}$$

We write $\det F_i = p^{a_i} u_i$, where a_i are numbers, and u_i are σ-linear isomorphisms. If we tensor the diagram with B we obtain $a_1 = a_2$. Hence the diagram remains commutative if we replace $\det F_i$ by u_i. We divide $\wedge^r \alpha$ by the maximal possible power of p, and call the resulting homomorphism β. This power is bounded because it is bounded over B. If we divide the morphisms in the diagram (2) by the maximal power of p and consider the result modulo p, we obtain a commutative diagram:

$$\begin{array}{ccc} L_1 & \xrightarrow{\bar{\beta}} & L_2 \\ {\scriptstyle \bar{u}_1} \downarrow & & \downarrow {\scriptstyle \bar{u}_2} \\ L_1 & \xrightarrow{\bar{\beta}} & L_2 \end{array} \tag{3}$$

We set $\bar{A} = A/pA$. The maps u_i are Frobenius linear isomorphisms of $\bar{A}$-modules. It suffices to show that $\bar{\beta}$ is an isomorphism. Take an open subset $\operatorname{Spec} \bar{A}_f \subset \operatorname{Spec} \bar{A}$ such that there are isomorphisms $L_i \cong \bar{A}_f$. Then we may write: $\bar{\beta}(x) = \rho x$ for all $x \in \bar{A}_f$ and some $\rho \in \bar{A}_f$. We note that $\rho \neq 0$. Indeed, since $\bar{A}$ is an integral domain $\rho = 0$ would imply that $\bar{\beta} = 0$. But then contrary to our assumption β would be divisible by p.

We write $\bar{u}_i(x) = \epsilon_i x^p$, where $\epsilon_i \in \bar{A}_f$ are units for $i = 1, 2$. Because the last diagram is commutative we have:

$$\epsilon_2 \rho^p = \rho \epsilon_1$$

Since $\bar{A}$ is an integral domain we may divide this equation by ρ. We obtain that ρ is a unit. *Q.E.D.*

Remark: Let us fix an integer $s > 0$. We consider a finitely generated projective A-module M with a σ^s-linear endomorphism $F : M \to M$. The definition of a Frobenius module may be given in this situation. Everything done in this section remains true for these more general Frobenius modules.

Remark: In the theorem of de Jong one can replace the Frobenius modules by slightly more general objects. To see this we prove:

Lemma 6 *Let M be a finitely generated torsion free $W[[t]]$-module. We set*

$$M_1 = \{m \in M \otimes_{W[[t]]} W[[t]]_{(p)} \mid p^s m \in M \textit{ for some number } s\}$$

Then M_1 is a free $W[[T]]$-module.

Proof: Using that $W[[t]]$ is UFD one shows that $M_1 = M$, if M is free.

If M is not free we find a free $W[[t]]$-module N of the same rank as M such that $M \subset N \subset M \otimes_{W[[t]]} W[[t]]_{(p)}$. Here we use that the last $W[[t]]_{(p)}$-module is free. Since $N_1 = N$ we find that $M_1 \subset N$.

By definition N/M_1 has no p-torsion and therefore $depth_{W[[t]]} N/M_1 \geq 1$. But since $W[[t]]$ is regular of dimension 2, this implies that the cohomological dimension of M_1 is zero. This proves the lemma. *Q.E.D.*

Let M be a torsion free $W[[t]]$-module. Let $F : M \to M$ be a σ-linear map such that $F \otimes \mathbb{Q}$ is an isomorphism. Since σ operates on $W[[t]]_{(p)}$ the map F extends to a map $F : M_1 \to M_1$, which induces an isomorphism when tensored with $\mathbb{Q}$. Hence there is an $a \in W[[t]]$ and an integer $m \geq 0$, such that $a \det F = p^m$. Since $W[[t]]$ is factorial this implies that $\det F = p^u(\text{unit})$. Therefore (M_1, F) is a Frobenius module. In this sense the pair (M, F) differs not much from a Frobenius module. One sees easily that de Jong's theorem holds if the pair (M, F) is of this more general type, if it holds for Frobenius modules. We will work exclusively with Frobenius modules.

2 Convergent Wittvectors

Let A be a ring such that $pA = 0$, and let $\nu : A \to \mathbb{R} \cup \{\infty\}$ be a valuation. Then we define for a nonnegative integer n and a Witt vector $\xi = (x_0, x_1, \dots) \in W(A)$:

$$\nu(\xi, n) = -\min\{\nu(x_i)/p^i \mid i = 0, \dots n\} \in \mathbb{R} \cup \{-\infty\}$$

This is an increasing function in n.

Proposition 7 *Let $\xi, \eta \in W(A)$. Then we have:*

$$\begin{array}{lll} \nu(\xi + \eta, n) & \leq & \max\{\nu(\xi, n),\ \nu(\eta, n)\} \\ \nu(\xi\eta, n) & \leq & \max\{\nu(\xi, l) + \nu(\eta, n-l) \mid l = 0, \dots n\} \end{array} \tag{4}$$

If in one of these inequalities there is a strict maximum on the right hand side, we have an equality.

Proof: The addition in $W(A)$ is defined by universal polynomials with integral coefficients:

$$S_n(X_0, \dots X_n, Y_0, \dots Y_n)$$

If we give to the variables X_i and Y_i the weight p^i the polynomial S_n is a homogenous polynomial of degree p^n. Hence S_m is a sum of monomials of the following type:

$$M(X_0, \dots X_m, Y_0, \dots Y_m) \overset{+}{=} X_0^{\alpha_0} \cdot \ldots \cdot X_m^{\alpha_m} Y_0^{\beta_0} \cdot \ldots \cdot Y_m^{\beta_m},$$

such that

$$\sum_{i=0}^{m} p^i \alpha_i + \sum_{i=0}^{m} p^i \beta_i = p^m.$$

We set $\xi = (x_0, x_1, \dots)$, and $\eta = (y_0, y_1, \dots)$. Let us assume that $-\nu(x_i)/p^i \leq c$ and $-\nu(y_i)/p^i \leq c$ for some constant c, and $i = 1, \dots, n$. Then we find for a monomial M appearing in S_m for $m \leq n$:

$$-\nu(M(x_0, \dots x_m, y_0, \dots y_m))/p^m =$$

$$\tfrac{1}{p^m}(\textstyle\sum_{i=0}^{m} p^i \alpha_i(-\nu(x_i)/p^i) + \sum_{i=0}^{m} p^i \beta_i(-\nu(y_i)/p^i) \leq c$$

This shows the first inequality of the proposition.

Let us now assume that $\nu(\xi, n) > \nu(\eta, n)$. Let m be the smallest index such that $-\nu(x_m)/p^m = \nu(\xi, n) = c$. We write in $W_{m+1}(A)$:

$$(x_0, \dots, x_{m-1}, 0) + (y_0, \dots, y_{m-1}, 0) = (z_0, \dots, z_{m-1}, z_m)$$

By the first inequality we find $-\nu(z_i)/p^i < c$. We have the following universal relation in $W_{m+1}(A)$:

$$(a_0, a_1, \dots, a_m)(0, \dots, 0, b) = (a_0, \dots, a_{m-1}, a_m + b)$$

Using this we obtain:

$$(x_0, \dots, x_m) + (y_0, \dots, y_m) = (z_1, \dots, z_{m-1}, z_m + x_m + y_m)$$

We find $\nu(z_m + x_m + y_m)/p^m = -c$. Hence the first inequality is an equality.

Now we turn to the second inequality. The assumption $pA = 0$ implies the following relation in $W(A)$:

$$^{V^i}[a]\ ^{V^j}[b] = \ ^{V^{i+j}}([a^{p^j}][b^{p^i}])$$

Therefore we obtain:

$$\xi\eta = (\sum_i \ ^{V^i}[x_i])(\sum_j \ ^{V^j}[y_j]) = \sum_{i+j \leq n} \ ^{V^{i+j}}[x_i^{p^j} y_j^{p^i}] \, mod \, V^{n+1}$$

Hence $\nu(, n)$ applied to the last sum equals $\nu(\xi\eta, n)$. We set $\omega_{ij} = \ ^{V^{i+j}}[x_i^{p^j} y_j^{p^i}]$. Then we find:

$$\nu(\omega_{ij}, n) = -\frac{1}{p^{i+j}}\nu(x_i^{p^j} y_j^{p^i}) = -\nu(x_i)/p^i - \nu(y_j)/p^j \leq \nu(\xi, i) + \nu(\eta, j)$$

Applying the first inequality to the sum of the ω_{ij} we find the result. Moreover if there is a pair (l, k) of nonnegative integers with $l + k \leq n$, such that $-\nu(x_l)/p^l - \nu(y_k)/p^k$ is strictly maximal among these pairs then:

$$\nu(\xi\eta, n) = -\nu(x_l)/p^l - \nu(y_k)/p^k$$

If for some l the number $\nu(\xi, l) + \nu(\eta, n - l)$ is strictly maximal we find that $-\nu(x_l)/p^l - \nu(y_{n-l})/p^{n-l}$ is strictly maximal. Hence the second inequality is an equality in this case. *Q.E.D.*

Definition 8 *A Witt vector $\xi = (x_0, x_1, \dots) \in W(A)$ is called convergent if there exists constants $C_1, C_2 \in \mathbb{R}$, such that for all n we have:*

$$\nu(x_n)/p^n \geq -C_1 - C_2 n$$

Equivalently we could say that ξ is convergent, if there are constants C_1, C_2, such that

$$\nu(\xi, n) \leq C_1 + C_2 n$$

We obtain immediately:

Corollary 9 *The subset $W^c(A) \subset W(A)$ of convergent Witt vectors is a ring.*

This subring is stable by Frobenius and Verschiebung. Indeed, with the convention $\nu(\xi, n) = 0$ for $n < 0$ we have the following obvious formulas:

$$\begin{array}{ll} \nu({}^{V}\xi, n) & = \frac{1}{p}\nu(\xi, n-1), \quad \xi \in W(A) \\ \nu({}^{F}\xi, n) & = p\nu(\xi, n) \\ \nu(p\xi, n) & = \nu(\xi, n-1) \end{array} \tag{5}$$

Proposition 10 *Let C be a quadratic $h \times h$ matrix with coefficients in $W^c(A)$. Let $y \in W(A)^h$ be a vector such that for some number $r > 0$*

$$Cy = {}^{F^r}y.$$

Then we have $y \in W^c(A)^h$.

Proof: We write the equation in coordinates:

$$\sum_{k=1}^{h} c_{lk} y_k = {}^{F^r}y_l, \quad l = 1, \dots, h.$$

We show a statement which is a little more precise: Let $M \geq 0$ be a constant such that for all l, k:

$$\nu(c_{lk}, 0) \leq M, \quad \nu(c_{lk}, n) \leq Mn \text{ for } n \geq 1.$$

Then for all $k = 1, \dots, n$ we have:

$$\nu(y_k, 0) \leq M, \quad \nu(y_k, n) \leq Mn \text{ for } n \geq 1. \tag{6}$$

We show this by induction beginning with $n = 0$. We choose i such that $\nu(y_i, 0)$ is maximal among $\nu(y_k, 0)$ for $k = 1, \dots, h$. Then we choose j such that $\nu(c_{ij} y_j, 0)$ is maximal among $\nu(c_{ik} y_k, 0)$ for $k = 1, \dots, h$. We compute :

$$\begin{aligned} p^r \nu(y_i, 0) = \ & \nu({}^{F^r} y_i, 0) \leq \nu(c_{ij} y_j, 0) = \\ & \nu(c_{ij}, 0) + \nu(y_j, 0) \leq \nu(c_{ij}, 0) + \nu(y_i, 0) \end{aligned}$$

This implies:

$$(p^r - 1)\nu(y_i, 0) \leq \nu(c_{ij}, 0) \leq M.$$

Next we assume $n \geq 1$ and $\nu(y_k, u) \leq Mu$ for $1 \leq u < n$. Again we choose i such that $\nu(y_i, n)$ is maximal among $\nu(y_k, n)$, and j such that $\nu(c_{ij} y_j, n)$ is maximal among $\nu(c_{ik} y_k, 0)$. Then we obtain:

$$p^r \nu(y_i, n) = \nu({}^{F^r} y_i, n) \leq \nu(c_{ij} y_j, n) \leq \max_{u+v=n} \{\nu(y_j, u) + \nu(c_{ij}, v)\} \tag{7}$$

First we assume that the maximum is taken for a pair u, v with $u \neq 0$ and $v \neq 0$. Then we obtain by induction:

$$p^r \nu(y_i, n) \leq Mu + Mv \leq Mn$$

Hence in this case we are done.

Assume now that $u = 0$. Then we obtain:

$$p^r \nu(y_i, n) \leq \nu(y_j, 0) + \nu(c_{ij}, n) \leq (n+1)M$$

This gives $\nu(y_i, n) \leq nM$.

Finally for $v = 0$ the inequality (2) reads:

$$p^r \nu(y_i, n) \leq \nu(y_j, n) + \nu(c_{ij}, 0) \leq \nu(y_i, n) + M$$

This implies $\nu(y_i, n) \leq M/(p^r - 1) \leq nM$. *Q.E.D.*

We set:

$$\mathring{A} = \{x \in A \mid \nu(a) \geq 0\}$$

Then $W(\mathring{A}) \subset W(A)$ consists of the Witt vectors $\xi \in W(A)$, such that $\nu(n, \xi) \leq 0$ for all numbers n. We have

$$W(\mathring{A}) \subset W^c(A) \subset W(A)$$

Corollary 11 *With the notations of proposition 10 we assume that the coefficients of C are in $W(\mathring{A})$. Then we have $y \in W(\mathring{A})$.*

Proposition 12 *Let $\xi = (x_0, x_1, \dots) \in W^c(A)$ such that $x_0 \in A$ is a unit. Then ξ is a unit in $W^c(A)$.*

Proof: There is an element $\eta \in W(A)$ such that $\xi\eta = 1$. We have to show $\eta \in W^c(A)$. We choose a constant C, such that

$$\begin{array}{ll} -\nu(\xi, 0) \leq C & \nu(\xi, n) \leq nC \text{ for } n \geq 1 \\ \nu(\eta, 0) \leq C & \nu(\eta, 1) \leq C \end{array} \tag{8}$$

This choice is possible since the first component of the Witt vector ξ is not 0. We show by induction that $\nu(\eta, n) \leq 2nC$. Indeed, we have the inequality:

$$\nu(\eta, n) + \nu(\xi, 0) \leq \max\{0;\ \nu(\eta, l) + \nu(\xi, n - l),\ 0 \leq l \leq n - 1\} \tag{9}$$

The opposite inequality would lead by proposition 7 to a contradiction:

$$0 = \nu(1, n) = \nu(\xi\eta, n) = \nu(\eta, n) + \nu(\xi, 0) > 0$$

Using the inequalities (8) and the induction assumption $\nu(\eta, l) \leq 2lC$ for $2 \leq l \leq n - 1$ we find from (8):

$$\nu(\eta, n) \leq C + \max\{0; C + nC, 2lC + (n - l)C\}$$

For $n \geq 2$ the right hand side is $\leq 2nC$. *Q.E.D.*

Corollary 13 *Let L be a perfect field and $\nu : L \to \mathbb{R} \cup \{\infty\}$ a valuation. Then $W^c(L)$ is a dicrete valuation ring with residue field L and prime element p. Moreover the Frobenius F on $W(L)$ induces an automorphism of $W^c(L)$.*

Proof: This follows because $pW^c(L) = VW^c(L)$, from Bourbaki AC Chapt VI §3 $n^0 6$ Proposition 9. *Q.E.D.*

With the notation of the last corollary let $R = \{k \in L \mid \nu(k) \geq 0\}$ be the valuation ring for ν. Recall that the units in $W(R)$ are the Witt vectors whose first components are units in R. We will see that a non-zero element $\xi \in W(K)$ may be written non-uniquely in the form:

$$\xi = \sum_{i=0}^{\rho} p^{r_i} [k_i^{-1}] \omega_i \tag{10}$$

Here $0 \leq r_0 < r_1 < \dots$ is an increasing sequence of nonnegative integers. The k_i are a sequence of elements in L such that

$$\nu(k_0) < \nu(k_1) < \dots$$

The ω_i are units in $W(R)$ and ρ is a nonnegative integer or ∞.

If ξ is given by the expression (10) we have:

$$\nu(\xi, n) = \{ \begin{array}{ll} -\infty & \text{if } n < r_0 \\ \nu(k_i) & \text{if } r_i \leq n < r_{i+1} \end{array}$$

First this is shown in the case of a single summand on the right hand side of (10). The general case follows from the proposition 7.

The existence of a decomposition (10) is easy. We take r_0 maximal such that $\xi \in p^{r_0}W(L)$. Let $\xi = p^{r_0}\eta$. Let k_0^{-1} be the first component of the Witt vector η. Then we may write: $\eta = [k_0^{-1}](1, z_1, z_2, \dots)$. Let z_s be the first component which is not in R. We set $\omega_0 = (1, \dots, z_{s-1}, 0, \dots)$, and $\xi_1 = (0, \dots, 0, z_s, \dots)$. Then we find:

$$\xi = p^{r_0}[k_0^{-1}]\omega_0 + p^{r_0}[k_0^{-1}]\xi_1$$

We proceed with ξ_1 as with ξ.

Let $K = k((t))$ be the field of Laurent polynomials over a perfect field k. Let ν be the discrete valuation, such that $\nu(t) = 1$. Let Γ be the p-adic completion of $W(k)[[t]][t^{-1}]$ as before. Explicitly we have:

$$\Gamma = \{\sum_{m \in \mathbb{Z}} a_m t^m \mid a_m \in W(k), \lim_{m \to -\infty} \mathrm{ord}_p(a_m) = \infty\}$$

As above a Laurent series in $f \in \Gamma$ may be written in the form

$$f = \sum_{i=0}^{\rho} p^{r_i} t^{-m_i} g_i \qquad (11)$$

where g_i are units in $W(k)[[t]]$, and r_i, m_i are integers such that:

$$\begin{array}{rl} 0 \leq & r_0 < r_1 < r_2 < \dots \\ & m_0 < m_1 < m_2 < \dots \end{array}$$

Then we have the natural map:

$$\delta : \Gamma \to W(k((t)))$$

It is the identity on $W(k)$ and maps $t \in \Gamma$ to the Teichmüller representative in $[t] \in W(k((t)))$. If we apply δ to the equation (11) we obtain an expression of the form (10), which we can consider in $W(k((t))^{perf})$ the Witt ring of the perfect hull. This proves the formula:

$$\nu(\delta(f), n) = \{ \begin{array}{ll} -\infty & \text{if } n < r_0 \\ m_i & \text{if } r_i \leq n < r_{i+1} \end{array}$$

We set $\nu(f, n) = \nu(\delta(f), n)$.

From the last formula we obtain easily:

$$\nu(f, n) = \min\{a \in \mathbb{Z} \mid t^a f \in W(k)[[t]] + p^{n+1}\Gamma\}$$

A more elementary way to say this is: Let $f = \sum_{i \in \mathbb{Z}} a_m t^m \in \Gamma$ be a Laurent series. Then $\nu(f, n)$ is the minimal number l such that:

$$\operatorname{ord}_p a_m \geq n + 1 \text{ for } m < -l$$

We set $\Gamma^c = \delta^{-1} W^c(k((t)))$. Using the last characterization of $\nu(n, f)$ one proves easily:

Proposition 14 *Let $f = \sum_{i \in \mathbb{Z}} a_m t^m \in \Gamma$, where $a_m \in W(k)$, be a Laurent series. Then the following conditions are equivalent:*

(*i*) $f \in \Gamma^c$

(*ii*) *There are constants $C_1, C_2 \in \mathbb{R}$ such that $\nu(f, n) \leq C_1 + nC_2$ for all nonnegative intgers n.*

(*iii*) *There are real constants C and $\epsilon > 0$, such that $\operatorname{ord}_p a_i \geq -\epsilon i + C$ for sufficiently small integers i.*

(*iv*) *The Newton polygon of f has a negative slope.*

We introduce on $W(k)$ the absolute value $|a| = p^{-\operatorname{ord}_p a}$. Then the condition (iii) says that there is a constanst $\eta < 1$ such that

$$\lim_{i \to -\infty} |a_i| \eta^i = 0$$

Indeed choose $\eta > p^{-\epsilon}$. This implies that $\sum_{i \in \mathbb{Z}} a_i z^i$ converges in a small annulus $\eta < |z| < 1$, where z is in the algebraic closure of the field of fractions of $W(k)$.

Let us now assume that k is algebraically closed. Let K^{perf} be the perfect closure of $K = k((t))$. Then K^{perf} has the powers t^α for $\alpha \in \mathbb{Z}[\frac{1}{p}],\ 0 < \alpha < 1$ as a K-vector space basis. Since $\Gamma \to W(K^{perf})$ is unramified with residue field extension $K \to K^{perf}$ it follows that an arbitrary element $\xi \in W(K^{perf})$ has a unique expression as a convergent sum:

$$\xi = \sum_{0<\alpha<1} [t^\alpha]\delta(f_\alpha), \tag{12}$$

where $f_\alpha \in \Gamma$. The convergence means that for any given number m we have $f_\alpha \in p^m\Gamma$ for almost all α. This implies that for fixed n we have $\nu(n, f_\alpha) = -\infty$ for almost all n. We note that $f_\alpha \in p^m\Gamma$ for all α implies $\nu(n, \xi) = -\infty$ for $n < m$. Clearly we have:

$$\nu(n, [t^\alpha]\delta(f_\alpha)) = \alpha + \nu(n, f_\alpha)$$

These numbers are different for different α. The remarks above and the proposition 7 then show:

$$\nu(n, \xi) = \max_\alpha\{\alpha + \nu(n, f_\alpha)\}$$

Hence $\xi \in W^c(K^{perf})$ implies that all f_α are in Γ_c. Conversely, if we have $\nu(n, f_\alpha) \leq C_1 + nC_2$ uniformly in α this implies $\xi \in W^c(K^{perf})$.

Proposition 15 *Let Γ be the Cohen ring for $K = k((t))$. Let K^{perf} be the perfect closure of K. Then the natural map:*

$$W^c(K^{perf}) \otimes_{\Gamma^c} \Gamma \to W(K^{perf}) \tag{13}$$

is injective.

Proof: Obviously the p-adic completion of Γ^c is Γ and the p-adic completion of $W^c(K^{perf})$ is $W(K^{perf})$. Therefore the p-adic completion of the left hand side of (13) is $W(K^{perf})$. Hence it suffices to show that $W^c(K^{perf}) \otimes_{\Gamma^c} \Gamma$ is separated in the p-adic topology.

Let us first assume that k is algebraically closed. We have shown that decomposition (12) provides an injection of Γ^c-modules:

$$W^c(K^{perf}) \to \prod_\alpha \Gamma^c$$

It suffices to show that $(\prod_\alpha \Gamma^c)\otimes_{\Gamma^c}\Gamma$ is p-adically separated. This follows since the following natural map is injective:

$$(\prod_\alpha \Gamma^c)\otimes_{\Gamma^c}\Gamma \to \prod_\alpha \Gamma \tag{14}$$

This is standard: Consider an arbitrary finitely generated Γ^c-submodule $M \subset \Gamma$. We obtain a commutative diagram:

$$\begin{array}{ccc} (\prod_\alpha \Gamma^c)\otimes_{\Gamma^c} M & \xrightarrow{\iota} & \prod_\alpha M \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ (\prod_\alpha \Gamma^c)\otimes_{\Gamma^c}\Gamma & \longrightarrow & \prod_\alpha \Gamma \end{array}$$

By reduction to the case $M = \Gamma^c$ one sees that ι is an isomorphism. Clearly π_2 is injective. On the other hand any element of $\prod_\alpha \Gamma^c \otimes_{\Gamma^c}\Gamma$ is in the image of π_1 for some finitely generated submodule M. This proves that (14) is injective.

Next we consider the case where k is not algebraically closed. Let $\bar{k}$ be the algebraic closure. We set $L = \bar{k}((t))$ and denote the corresponding ring of Laurent series by Γ_L. Then we have an injection:

$$\Gamma_L^c \otimes_{\Gamma^c}\Gamma \to \Gamma_L \tag{15}$$

Indeed, let u_i, $i\in I$ be a basis of $\bar{k}$ as a vector space over k. Consider an arbitrary Laurent series in Γ_L:

$$f = \sum_{m\in\mathbb{Z}} c_m t^m, \quad c_m \in W(\bar{k}).$$

We write $c_m = \sum_{i\in I} a_{m,i}[u_i]$ as a p-adically convergent sum. Then $c_m \in p^n W(\bar{k})$, iff $a_{m,i}\in p^n W(k)$ for all $i\in I$. We set $g_i = \sum_{m\in\mathbb{Z}} a_{m,i}t^m$. We see that $f\in\Gamma_L^c$ implies $g_i\in\Gamma^c$ for all $i\in I$. Therefore we obtain an injection:

$$\Gamma_L^c \to \prod_{i\in I}\Gamma^c$$

This shows the injectivity of (15) as above.

Finally we consider the maps:

$$\begin{array}{ccc} W^c(L^{perf})\otimes_{\Gamma^c}\Gamma = W^c(L^{perf})\otimes_{\Gamma_L^c}(\Gamma_L^c\otimes_{\Gamma^c}\Gamma) & \longrightarrow & W^c(L^{perf})\otimes_{\Gamma_L^c}\Gamma_L \\ & & \downarrow \\ & & W(L^{perf}) \end{array}$$

The first arrow is injective by (15) and the second arrow is injective by (13) in the case where k is algebraically closed. *Q.E.D.*

Corollary 16 *Let M and N be Γ^c-modules, such that N is torsion free. Let $\phi : M \to N \otimes_{\Gamma^c} \Gamma$ be a homomorphism of Γ^c-modules. Let $\tilde{\phi} : M \otimes_{\Gamma^c} W^c(K^{perf}) \to N \otimes_{\Gamma^c} W(K^{perf})$ be the $W^c(K^{perf})$-linear morphism induced by ϕ. Then the natural injection:*

$$(\operatorname{Im} \phi \cap N) \otimes_{\Gamma^c} W^c(K^{perf}) \to \operatorname{Im} \tilde{\phi} \cap (N \otimes_{\Gamma^c} W^c(K^{perf})$$

is bijective.

Proof: We may assume that $\phi : M \to N \otimes_{\Gamma^c} \Gamma$ is injective. Then the map:

$$\tilde{\phi} : M \otimes_{\Gamma^c} W^c(K^{perf}) \to N \otimes_{\Gamma^c} \Gamma \otimes_{\Gamma^c} W^c(K^{perf}) \to N \otimes_{\Gamma^c} W(K^{perf})$$

is injective too. Clearly it suffices to show that the inclusion:

$$\phi^{-1}(N) \otimes_{\Gamma^c} W^c(K^{perf}) \subset \tilde{\phi}^{-1}(N \otimes_{\Gamma^c} W^c(K^{perf}))$$

is an equality.

Consider the injection:

$$M/\phi^{-1}(N) \to (N \otimes_{\Gamma^c} \Gamma)/N$$

If we tensor this with $\otimes_{\Gamma^c} W^c(K^{perf})$, we obtain two injections:

$$\begin{array}{l} M \otimes W^c(K^{perf})/\phi^{-1}(N) \otimes W^c(K^{perf}) \\ \qquad\qquad\qquad \downarrow \\ (N \otimes \Gamma) \otimes W^c(K^{perf})/N \otimes W^c(K^{perf}) \to \; N \otimes W(K^{perf})/N \otimes W^c(K^{perf}) \end{array}$$

All tensor products in this diagram are taken over Γ^c. This proves the result. *Q.E.D.*

3 The slope filtration over $W^c(L)$

In this section L will be a perfect field with a valuation ν. Then $W(L)$ respectively $W^c(L)$ is a discrete valuation rings whose maximal ideal is generated by p. We will denote the Frobenius automorphism of these rings by σ.

Let (M, F) be a Frobenius module over $W(L)$. Up to isogeny (M, F) is a direct sum of isoclinic Frobenius modules. This implies in particular that M has a filtration by F-invariant submodules:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m, \tag{16}$$

such that M_i/M_{i-1} is a Frobenius module isoclinic of slope τ_i. Moreover we may arrange $\tau_1 > \ldots > \tau_m$. Then the filtration is unique. We will show that such a filtration exists for Frobenius modules over $W^c(L)$.

Let (M, F) and (N, F) be quasi Frobenius modules over $W(L)$. Let $\lambda_1, \ldots, \lambda_h$ be the slopes of M with multiplicities, where $h = rank_{W(L)} M$, and let $\mu_1, \ldots, \mu_l$, be the slopes of N with multiplicties where $l = rank_{W(L)} N$. One obtains easily:

The slopes of $M \otimes N$ with multiplicities are:

$$\lambda_i + \mu_j, \quad i = 1, \ldots, h, \ j = 1, \ldots, l$$

The slopes of $Hom_{W(L)}(M, N)$ with multiplicites are:

$$\mu_j - \lambda_i, \quad i = 1, \ldots, h, \ j = 1, \ldots, l$$

The slopes of $\wedge^k M$ with multiplicities are:

$$\lambda_{i_1} + \ldots + \lambda_{i_k},$$

where the indices run through all tuples, such that $1 \leq i_1 < \ldots < i_k \leq h$.

Before we turn our attention to Frobenius modules over $W^c(L)$ we prove a general result:

Lemma 17 *Let $R \to R'$ be an injective ring homomorphism. Let M be a finitely generated projective R-module, and let $N' \subset M \otimes_R R'$ be a direct summand of constant rank r.*

Then there is a unique direct summand $N \subset M$ such that $N' = N \otimes_R R'$ if and only if there is a direct summand $L \subset \bigwedge^r M$ such that $L \otimes_R R' = \bigwedge^r N'$.

Proof: Let X be the Grassmannian of rank r submodules of M and Y be $\mathbb{P}(\bigwedge^r \hat{M})$, where $\hat{M} = \mathrm{Hom}_R(M, R)$.

The operation $\bigwedge^r$ induces the Plücker morphism $X \to Y$ which is a closed immersion [Sém. Cartan] 1960/61 Exp. XII prop.2.2. Corresponding to N' and L we obtain a commutative diagram.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array} \tag{17}$$

The lemma says that there is an arrow $\operatorname{Spec} R \to X$ which yields a commutative diagram when inserted into (17). As $X \to Y$ is a monomorphism this arrow is unique if it exists. Therefore the question of existence is local on $\operatorname{Spec} R$. We may assume that $\operatorname{Spec} R \to Y$ factors through an affine open set $\operatorname{Spec} A \subset Y$. Let us denote by $\operatorname{Spec} B \subset X$ the preimage of $\operatorname{Spec} A$. Then we obtain from (17) a commutative diagram of rings:

$$\begin{array}{ccc} B & \xleftarrow{\pi} & A \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ R' & \longleftarrow & R \end{array} \tag{18}$$

We want to show that there is $w : B \to R$ making the diagram commutative. Since $R \to R'$ is injective, we have $u(\operatorname{Ker} \pi) = 0$. But this suffices for the existence of w since π is surjective. This proves the lemma. *Q.E.D.*

We prove a slight generalization of Bourbaki Algèbre Chapt II, §8 N^0 7 Théorème 1:

Corollary 18 *Let S be a local ring, and G be a set of endomorphisms of S. We denote by $R = S^G$ the ring of invariants. Let M be a finitely generated free R-module. Assume we are given a direct summand N' of the S-module $M \otimes_R S$, such that $gN' \subset N'$ for each element $g \in G$.*

Then there is a unique direct summand N of the R-module M, such that $N' = N \otimes_R S$.

Proof: We note that R is a local ring too. The uniqueness follows as in lemma 17. This lemma also shows that we may assume that N' is of rank 1. Let $e_1, \ldots, e_r$ be a basis of M, and let n be a generator of N'. Then we write:

$$n = a_1 e_1 + \ldots + a_r e_r, \tag{19}$$

where $a_i \in S$. Since S is local we may assume without loss of generality that $a_1 = 1$. By assumption there is for each $g \in G$ an element $\lambda(g) \in S$, such that

$$g(n) = \lambda(g)n$$

Inserting for n the right hand side of (19) and comparing the coefficients gives $\lambda(g) = 1$ and $g(a_i) = a_i$. But this implies $a_i \in R$. Then $Rn \subset N$ is the desired direct summand. *Q.E.D.*

Let (M, F) be a quasi Frobenius module over $W^c(L)$.

Proposition 19 *Let*

$$\tau_1 > \tau_2 > \ldots > \tau_m$$

be the slopes of the quasi Frobenius module (M, F). Then M has a unique filtration by quasi Frobenius submodules:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m$$

such that M_i/M_{i-1} is a nonzero isoclinic quasi Frobenius module of slope τ_i for $i = 1, \ldots m$.

For the proof we need another lemma:

Lemma 20 *Let (M, F) be a quasi Frobenius module over $W^c(L)$. Let $\lambda = r/s$ where r and $s > 0$ are integers.*

(i) Assume that all slopes of (M, F) are bigger or equal to λ. Then there is a quasi Frobenius module (N, F) which is isogenous to (M, F), and such that $F^s N \subset p^r N$.

(ii) Assume that all slopes of (M, F) are less or equal to λ. Then there is a quasi Frobenius module (N, F) which is isogenous to (M, F), and such that $p^r N \subset F^s N$.

(iii) Assume that (M, F) is isoclinic of slope λ. Then there is a quasi Frobenius module (N, F) which is isogenous to (M, F), and such that $F^s N = p^r N$.

Proof: The corresponding statement is true for $M' = W(L) \otimes_{W^c(L)} M$.

In the case (i) we find a submodule of finite index $N' \subset M'$ such that $F^s N' \subset p^r N'$. We set $N = N' \cap M \otimes \mathbb{Q}$. Then we find $F^s N \subset p^r N' \cap M \otimes \mathbb{Q} = p^r N$. The other cases are similiar *Q.E.D.*

Remark: We note that $F^s N \subset p^r N$, if and only if $p^r \hat{N} \subset F^s \hat{N}$ holds for the dual module. This follows because $F^s \hat{N}$ is the dual module to $F^s N$ with respect to the pairing:

$$(N \otimes \mathbb{Q}) \times (\hat{N} \otimes \mathbb{Q}) \to W^c(L) \otimes \mathbb{Q}$$

Proof (of proposition 19): To prove the proposition we may change M in its isogeny class. Let $\tau_1 = \tau = r/s$ be the highest slope. Then we find $M \subset N \subset M \otimes \mathbb{Q}$ such that $p^r N \subset F^s N$. Let $N' = W(L) \otimes_{W^c(L)} N$. Let $N'_1 \subset N'$ the isoclinic part of slope τ_1 (compare (16)).

We have to show that there is a direct summand $N_1 \subset N$, such that $N'_1 = W(L) \otimes_{W^c(L)} N_1$. Applying lemma 17 we may assume that N'_1 has rank 1. We set $\Phi = p^r F^{-s} : N \to N$. Let n be a generator of N'_1. Then we find:

$$\Phi n = un,$$

for some unit $u \in W(L)$.

Let us assume that L is algebraically closed. Then we may write $u = a\sigma^s(a^{-1})$. Then we find $\Phi(an) = an$. Then proposition 10 shows that $an \in N$. This proves the proposition in the case where L is algebraically closed. In the general case this shows that there is a direct summand $\bar{N}_1 \subset W^c(\bar{L}) \otimes_{W^c(L)} N$, such that $W(\bar{L}) \otimes_{W(L)} N'_1 = W(\bar{L}) \otimes_{W^c(\bar{L})} \bar{N}_1$. Since the left hand side of the last equation is invariant by the Galois group of $\bar{L}/L$, so is the direct summand $\bar{N}_1 \subset W^c(\bar{L}) \otimes_{W^c(L)} N$. Hence this direct summand descents to a submodule $\bar{N}_1 \subset N$, by corollary 18. *Q.E.D.*

Proposition 21 *Let M, N be Frobenius modules over $W^c(L)$. Assume that M is isoclinic of slope λ, and that all slopes of N are less of equal to λ. Let*

$$\alpha : M \otimes_{W^c(L)} W(L) \to N \otimes_{W^c(L)} W(L)$$

be a homomorphism of Frobenius modules. Then we have $\alpha(M) \subset N$.

Proof: We set $U = (W^c(L), \sigma)$. Tensoring α with the dual quasi Frobenius module $\hat{M}$, we obtain a morphism $U \otimes_{W^c(L)} W(L) \to N \otimes_{W^c(L)} \hat{M} \otimes_{W^c(L)} W(L)$

of quasi Frobenius modules. Twisting this morphism by $U(\ell)$, i.e. replacing F by $p^m F$ we obtain a morphism of Frobenius modules.

Hence we may assume without loss of generality that $M = U(\ell)$. Since the slopes of N are less or equal to ℓ, we find an isogeny $N \to N'$. such that $p^\ell F^{-1} N' \subset N'$.

We set $n = \alpha(1) \in N' \otimes_{W^c(L)} W(L)$. Then we have $p^\ell F^{-1} n = n$. By proposition 10 we find $n \in N'$. This shows $\alpha(M \otimes \mathbb{Q}) \subset N \otimes \mathbb{Q}$. But this suffices since:

$$(N \otimes \mathbb{Q}) \cap (W(L) \otimes_{W^c(L)} N) = N.$$

Indeed, this follows because $x \in W(L)$ is in $W^c(L)$, iff $px \in W^c(L)$. *Q.E.D.*

Proposition 22 *Let M and N be Frobenius modules over Γ^c. Assume that N is isoclinic of slope λ. Suppose we are given a morphism of Frobenius modules*

$$\phi : M \otimes_{\Gamma^c} \Gamma \to N \otimes_{\Gamma^c} \Gamma,$$

such that $\phi \otimes \mathbb{Q}$ is surjective. We set $E = \phi^{-1}(N) \cap M$ and consider the map $\psi : E \to N$ induced by ϕ. Then the map $\psi \otimes \mathbb{Q}$ is surjective too. Assume moreover that the map $M \to N \otimes_{\Gamma^c} \Gamma$ induced by ϕ is injective. Then all slopes of M are less or equal to λ.

Proof Let $\phi_0 : M \to N \otimes_{\Gamma^c} \Gamma$ be the restriction of ϕ. Clearly we may assume that this map is injective. If we tensor ϕ_0 by $\otimes_{\Gamma^c} W^c(K^{perf})$ we obtain by proposition 15 an injection:

$$\phi_1 : M \otimes_{\Gamma^c} W^c(K^{perf}) \to N \otimes_{\Gamma^c} (\Gamma \otimes_{\Gamma^c} W^c(K^{perf})) \to N \otimes_{\Gamma^c} W(K^{perf})$$

Let μ be the highest slope of $M \otimes_{\Gamma^c} W^c(K^{perf})$, and consider the first step in the slope filtration $M(\mu) \subset M \otimes_{\Gamma^c} W^c(K^{perf})$. Then $M(\mu)$ is a Frobenius module which is isoclinic of slope μ. Consider the injection

$$M(\mu) \to N \otimes_{\Gamma^c} W(K^{perf}) \tag{20}$$

induced by ϕ_1. If $\mu \neq \lambda$ the map $M(\mu) \otimes_{W^c(K^{perf})} W(K^{perf}) \to N \otimes_{\Gamma^c} W(K^{perf})$ would be zero. This contradicts the injectivity of (20). Hence all slopes of M are less or equal to λ.

It follows from proposition 21 that ϕ_1 maps $M(\lambda)$ to $N \otimes_{\Gamma^c} W^c(K^{perf})$. Looking for the slope decomposition over $W(K^{perf})$ we find that the map $M(\lambda) \otimes \mathbb{Q} \to N \otimes_{\Gamma^c} W^c(K^{perf}) \otimes \mathbb{Q}$ is surjective. This shows that $\mathrm{Im}(M \otimes_{\Gamma^c} W^c(K^{perf})) \cap (N \otimes_{\Gamma^c} W^c(K^{perf})$ has the same rank as N. It follows by corollary 16 that $\mathrm{Im}(M) \cap N$ has the same rank as N. *Q.E.D.*

De Jong's proof of theorem 3 depends on to further results which are not related to convergent Witt vectors. They will be proved in the next two sections. We state them here, and deduce de Jong's theorem.

Theorem 23 *Let M be a Frobenius module over $W[[t]]$. Let $N^c \subset M \otimes_{W[[t]]} \Gamma^c$ be a F-invariant direct summand. Then there is a unique Frobenius submodule $N \subset M$, such that $N^c = N \otimes_{W[[t]]} \Gamma^c$.*

Corollary 24 *Let (M_1, F) and (M_2, F) be Frobenius modules over $W(k)[[t]]$. Assume we are given a morphism*

$$\phi^c : (M_1 \otimes_{W[[t]]} \Gamma^c, F) \to (M_2 \otimes_{W[[t]]} \Gamma^c, F)$$

Then there is a morphism $\phi : (M_1, F) \to (M_2, F)$ such that $\phi^c = \phi \otimes \Gamma^c$.

Proof: This follows from the last theorem applied to the graph of ϕ^c:

$$N^c \subset (M_1 \otimes_{W(k)[[t]]} \Gamma^c, F) \oplus (M_2 \otimes_{W(k)[[t]]} \Gamma^c, F)$$

We obtain a Frobenius submodule $N \subset M_1 \oplus M_2$. By proposition 5 the projection $N \to M_1$ is an isomorphism, because it becomes an isomorphism over Γ^c. *Q.E.D.*

Finally we will show in the last section:

Proposition 25 *Let $M \to N$ be a homomorphism of quasi Frobenius modules over $W[[t]]$. Assume $M \otimes_{W[[t]]} \Gamma$ is isoclinic of slope λ and that all slopes of $N \otimes_{W[[t]]} \Gamma$ are less or equal to λ. If the map*

$$M \otimes \mathbb{Q} \to N \otimes \mathbb{Q}$$

is injective, then it admits an F-equivariant retraction.

Proof (of de Jong's theorem): We note first that it is enough to prove that $\phi(M \otimes \mathbb{Q}) \subset N \otimes \mathbb{Q}$. Indeed for this it suffices to check that

$$(N \otimes \mathbb{Q}) \cap (N \otimes_{W[[t]]} \Gamma) = N. \tag{21}$$

For this we may assume $N = W[[t]]$. Since $W[[t]]$ is UFD we see that $W[[t]]_{(p)}/W[[t]]$ has no p-torsion. Since $\Gamma/W[[t]]_{(p)}$ has no p-torsion we conclude that (21) holds.

Let $\hat{N}$ be the quasi Frobenius module dual to N. Let us denote by U the unit Frobenius module. Then we have in the category of quasi Frobenius modules over $W[[t]]$:

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(M \otimes \hat{N}, U)$$

This shows that we may a assume that $N = U(\ell)$, which is a rank one Frobenius module of slope ℓ.

Consider the map $\breve{\phi} : M \to N \otimes_{W[[t]]} \Gamma$. Dividing by the kernel we may assume that this map is injective. Indeed, let M' be the factor of M by this kernel. If M' is not free we consider the free module M'_1 given by lemma 6. For a suitable integer a we obtain an injective map $p^a\breve{\phi} : M'_1 \to N \otimes_{W[[t]]} \Gamma$.

We assume now that $\breve{\phi}$ is injective. Let us introduce the notation $M^c = M \otimes_{W[[t]]} \Gamma^c$ and $N^c = N \otimes_{W[[t]]} \Gamma^c$. The map

$$M^c \to N \otimes_{W[[t]]} \Gamma \tag{22}$$

induced by $\breve{\phi}$ is injective too. Indeed, by theorem 23 the kernel would be defined over $W[[t]]$. Therefore it is zero because $\breve{\phi}$ is injective. If we a apply proposition 22 to the morphism (22), we obtain that all slopes of M^c are less or equal to λ. Moreover $E^c = M^c \cap N^c$ is a Frobenius submodule of rank 1 and slope ℓ of M^c.

By theorem 23 there is a Frobenius submodule $E \subset M$, such that $E^c = E \otimes_{W[[t]]} \Gamma^c$. If we apply corollary 24 to the map $E^c \to N^c$ we obtain that $\breve{\phi}(E) \subset N$.

Since the slopes of M^c are less or equal to ℓ the Frobenius submodule $E \subset M$ has by proposition 25 a complement E' up to isogeny, i.e. we find an injection of Frobenius modules $E \oplus E' \to M$ whose cokernel has p-torsion. But then we conclude $\breve{\phi}(E') \subset N$ by induction on the rank of M. *Q.E.D.*

4 Proof of theorem 23

The proof of this theorem seems to require some basic facts about non-archimedian analytic functions for its proof. It would be interesting to give a completely elementary proof.

We fix a perfect field k, and denote by $W = W(k)$ the ring of Witt vectors. We gather a few facts for Laurent series over the non-archimedian field $W_{\mathbb{Q}} = W \otimes \mathbb{Q}$. For the proofs we refer [Güntzer] or [Lazard]. We denote by Ω the algebraic closure. We set $|a| = p^{-\operatorname{ord}_p a}$ for $a \in \Omega$.

Definition 26 *Let I be a nonempty interval of nonnegative real numbers. We denote by $\mathcal{L}(I)$ the set of all formal Laurent series*

$$\sum_{n\in\mathbb{Z}} a_n t^n, \quad a_n \in W_{\mathbb{Q}}(k)$$

such that for all $\rho \in I \setminus 0$

$$\lim_{|n|\to\infty} |a_n|\rho^n = 0,$$

and if $0 \in I$ then $a_n = 0$ for $n < 0$. An element of $\mathcal{L}(I)$ is called a Laurent series convergent in I.

It is not difficult to see that $\mathcal{L}(I)$ has a natural ring structure. In contrast the set $\mathcal{L}$ of all formal Laurent series is not a ring. If $I = \{\rho\}$ is a single point we write $\mathcal{L}(\rho) = \mathcal{L}(I)$. One checks immediately that this is a Banach algebra with the norm:

$$||f||_\rho = \max_{n\in\mathbb{Z}} |a_n|\rho^n,$$

where f denotes the Laurent series $f = \sum_{n\in\mathbb{Z}} a_n t^n$. If $g \in \mathcal{L}(\rho)$ is a second Laurent series, we have

$$||fg||_\rho = ||f||_\rho ||g||_\rho.$$

Let $f \in \mathcal{L}(I)$ for arbitrary I. A root (or I-root) of f will be an element $z \in \Omega$ such that $|z| \in I$, and $\sum_n a_n z^n = 0$.

The following theorem (loc.cit.) is a consequence of the theory of Newton polygons.

Theorem 27 *Assume that a nonzero $f \in \mathcal{L}_I$ has only finitely many I-roots. Then there is a unique monic polynomial $P \in W_{\mathbb{Q}}(k)[t]$ whose roots in Ω have absolute values in I, and a unit u in the ring $\mathcal{L}(I)$ such that*

$$fu = P$$

In particular $f \in \mathcal{L}(I)$ is a unit, iff $f \neq 0$ and has no I-roots.

If the interval I is compact, then an arbitrary $f \in \mathcal{L}(I)$ has only finitely many roots.

This theorem implies that for a compact intervall $\mathcal{L}(I)$ is a principal ideal domain. If ρ is transcendental then there are no elements $z \in \Omega$ with $|z| = \rho$. Therefore $\mathcal{L}(\rho)$ is a field (see [Güntzer] Lemma 2).

We will use the following notation for certain rings of Laurent series:

$$\mathcal{D} = \mathcal{L}([0,1)), \quad \mathcal{A}_\eta = \mathcal{L}([\eta,1)), \text{ for } 0 < \eta < 1. \tag{23}$$

We set $\mathcal{A} = \cup_\eta \mathcal{A}_\eta$. We have $\Gamma^c \subset \mathcal{A}$. One can show that $\Gamma^c \otimes \mathbb{Q}$ consists exactly of the Laurent series $f \in \mathcal{A}$ such that $|a_n|$ is bounded, but we don't need this result.

The Frobenius on $W_\mathbb{Q}(k)$ extends to an endomorphism σ of the abelian group of formal Laurent series:

$$\sigma(\sum_{n\in\mathbb{Z}} a_n t^n) = \sum_{n\in\mathbb{Z}} {}^F a_n t^{pn}$$

We have $\sigma(\mathcal{A}) \subset \mathcal{A}$ and moreover the restriction of σ to $\mathcal{A}$ is a ring homomorphism.

Lemma 28 *Assume that a nonzero $f \in \mathcal{L}$ satisfies an equation $\sigma^u f = cf$ for some number $u \geq 1$, and for some $c \in W_\mathbb{Q}(k)$. Then we have $f \in W_\mathbb{Q}(k)$ and $c \in W(k)^*$.*

Proof: We set $q = p^u$. The assertion follows by comparing the coefficients in the equation:

$$\sum_{n\in\mathbb{Z}} {}^{F^u} a_n t^{nq} = \sum_{n\in\mathbb{Z}} c a_n t^n$$

Q.E.D.

The following proposition goes back to Dwork (see [Katz]).

Proposition 29 *Let k be algebraically closed. Let (M,F) be a Frobenius module over $W[[t]]$.*

The $\mathcal{D}$-module $M \otimes_{W[[t]]} \mathcal{D}$ possesses a basis $\{d_1, \ldots, d_r\}$ which satisfies $F^n d_i = p^{a_i} d_i$ for suitable integers $n > 0, a_1, \ldots, a_r \geq 0$.

Proof:M/tM with the operator F is an F-crystal over k. Hence there are elements $e_1, \ldots, e_r \in M/tM$ which form a basis of $M/tM \otimes_W K$ and such that the following equations

$$F^n e_i = p^{a_i} e_i, \qquad i = 1, \ldots r$$

hold for suitable integers $n > 0$ and $a_1, \dots, a_r \geq 0$. Let us choose arbitrary liftings $e'_1, \dots, e'_r \in M$ of these elements. We write:

$$F^n e'_i = p^{a_i} e'_i + t x_i$$

for suitable $x_i \in M$. Inductively we find for each number N:

$$F^{Nn} e'_i = p^{Na_i} e'_i + \sum_{j=1}^{N} p^{(N-j)a_i} t^{p^{n(j-1)}} F^{n(j-1)}(x_i) \tag{24}$$

Consider for fixed i the sequence:

$$f_{i,N} = p^{-Na_i} F^{Nn} e'_i \tag{25}$$

By the last equation we obtain the congruence:

$$p^{-Na_i} F^{Nn} e'_i = p^{-(N-1)a_i} F^{(N-1)n} e'_i \bmod p^{-Na_i} t^{p^{n(N-1)}} M \tag{26}$$

We choose an isomorphism $M \cong W[[t]]^r$. Clearly sequence $f_{i,N}$ converges in the t-adic topology to an element $\tilde{e}_i$ of $W_{\mathbb{Q}}[[t]]^r = M \otimes_{W[[t]]} W_{\mathbb{Q}}[[t]]$.

Both sides of (26) differ by a vector with components $p^{-Na_i} t^{p^{n(N-1)}} p_i$, where $p_i \in W[[t]]$. Since $||p^{-Na_i} t^{p^{n(N-1)}}||_\rho = p^{Na_i} \rho^{p^{n(N-1)}}$ converges to zero whenever $\rho < 1$ the components of the sequence $f_{i,N}$ converge in the Banach algebra $\mathcal{L}(\rho)$. Since this is true for any $\rho < 1$, we find $\tilde{e}_i \in M \otimes_{W[[t]]} \mathcal{D}$.

From the equation $p^{-a_i} F^n f_{i,N} = f_{i,(N+1)}$ we obtain:

$$F^n \tilde{e}_i = p^{a_i} \tilde{e}_i.$$

It remains to be shown that the elements $\tilde{e}_i$ for $i = 1, \dots, r$ form a basis of $M \otimes \mathcal{D}$, or equivalently that $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r$ is a basis of $\bigwedge^r M \otimes_{W[[t]]} \mathcal{D}$. By the lemma below there exists generator $y \in \bigwedge^r M$ such that $Fy = p^\ell y$, for some number ℓ. We write:

$$\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r = y \otimes f, \quad f \in \mathcal{D} \tag{27}$$

Applying F^u to this equation gives an equality of the form:

$$p^i(\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r) = p^j y \otimes \sigma^u(f),$$

for some numbers i and j. We deduce $p^{j-i}\sigma^u(f) = f$. By lemma 28 we find $i = j$ and $f \in W_{\mathbb{Q}}$.

It remains to verify that $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r \neq 0$. But by construction we have $\tilde{e}_i = e'_i \bmod t$ and hence $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ gives a basis of the k-vector space

$$M \otimes_{W[[t]]} \mathcal{D}/tM \otimes_{W[[t]]} \mathcal{D} = M/tM \otimes_W K$$

Therefore $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_r$ is not zero. *Q.E.D.*

Lemma 30 *Let k be algebraically closed. Let (M, F) be a Frobenius module over $W[[t]]$ of rank 1. Then there is an element $m \in M$, such that $Fm = p^\ell m$ for some number ℓ.*

Proof: We fix an isomorphism $M \cong W[[t]]$. We set $F1 = \lambda$. By assumption $\lambda = p^\ell \eta$ for some unit $\eta \in W[[t]]$. By the next lemma we find a unit $u \in W[[t]]$ such that $\sigma(u)\eta/u = 1$. This implies $Fu = \sigma(u)p^\ell\eta = p^\ell u$. *Q.E.D.*

Lemma 31 *Let $\eta \in W[[t]]$ be a unit. Then there is a unit $x \in W[[t]]$ which solves the equation*

$$\sigma(x)/x = \eta$$

Proof: We write the equation modulo p:

$$x^{p-1} - \eta = 0 \bmod p$$

This has a solution in the algebraically closed field k, which lifts by Hensel's lemma to a solution in $k[[t]]$.

Hence we obtain a solution mod p. Let us assume by induction that we have a solution x mod p^n:

$$\sigma(x) = x\eta \bmod p^n$$

It is enough to show that x lifts to a solution mod p^{n+1}. We set $\sigma(x) - x\eta = p^n\xi$ for $\xi \in W[[t]]$. We have to find $\rho \in W[[t]]$, such that

$$\sigma(x + p^n\rho) - (x + p^n\rho)\eta = 0 \bmod p^{n+1}$$

This amounts to finding a solution of the following congruence:

$$\rho^p - \rho\eta + \xi = 0 \bmod p$$

This congruence is an algebraic equation over $k[[t]]$ which we can solve by Hensel's lemma as above. *Q.E.D.*

We have a cartesian diagram of rings:

$$\begin{array}{ccc} W[[t]] & \rightarrow & \Gamma^c \\ \downarrow & & \downarrow \\ \mathcal{D} & \rightarrow & \mathcal{A} \end{array} \tag{28}$$

Below we formulate de Jong's theorem 23 for the arrow $\mathcal{D} \rightarrow \mathcal{A}$:

Lemma 32 *Let k be algebraically closed. Let (M, F) be a Frobenius module over $W[[t]]$. Let $N' \subset M \otimes_{W[[t]]} \mathcal{A}$ be an F-invariant direct summand, which is free of rank 1. Then there is a free direct summand $N \subset M \otimes_{W[[t]]} \mathcal{D}$ of rank 1, such that $N' = N \otimes_{\mathcal{D}} \mathcal{A}$.*

Proof: By proposition 29 there exists a basis $\{d_1, \ldots, d_r\}$ of $M \otimes_{W[[t]]} \mathcal{D}$ such that for suitable numbers u and a_i:

$$F^u d_i = p^{a_i} d_i$$

Let n be a generator of N'. We can write in $M \otimes_{W[[t]]} \mathcal{A} = M \otimes_{W[[t]]} \mathcal{D} \otimes_{\mathcal{D}} \mathcal{A}$:

$$n = \sum_{i=1}^{r} d_i \otimes h_i, \qquad h_i \in \mathcal{A} \tag{29}$$

Since N' is invariant by F we obtain:

$$F^u n = \sum_{i=1}^{r} d_i \otimes p^{a_i} \sigma^u(h_i) = hn$$

for some $h \in \mathcal{A}$. Comparing coefficients we find that $p^{a_i}\sigma^u(h_i) = hh_i$. Then we obtain:

$$p^{a_i}\sigma^u(h_i)h_j = p^{a_j}\sigma^u(h_j)h_i \tag{30}$$

Assume we have chosen j such that h_j is not identically zero. We choose η such that all Laurent series h_i and $\sigma^u(h_i)$ are in $\mathcal{A}_\eta$. Then we choose a transcendental ρ such that $\eta < \rho < 1$.

We set $q = p^u$. One sees easily that σ^u induces a homomorphism of fields $\sigma^u : \mathcal{L}(\rho) \rightarrow \mathcal{L}(\rho^{1/q})$. Therefore we obtain in $\mathcal{L}(\rho^{1/q})$ the equation:

$$\sigma^u(h_j)\sigma^u(h_i h_j^{-1}) = \sigma^u(h_i)$$

We divide equation (30) in the field $\mathcal{L}(\rho^{1/q})$ by $\sigma^u(h_j)h_j$:

$$p^{a_j}h_ih_j^{-1} = p^{a_i}\sigma^u(h_i)\sigma^u(h_j)^{-1} = p^{a_i}\sigma^u(h_ih_j^{-1}).$$

By lemma 28 we obtain $h_ih_j^{-1} = \lambda_i \in W_{\mathbb{Q}}$, where $\lambda_i = 0$ if $a_i \neq a_j$. We set $f = h_j$ and rewrite (29) as follows:

$$n = \sum_{i=1}^{r} \lambda_i d_i \otimes f \tag{31}$$

We may replace in the basis $d_1, \ldots, d_r$ the element d_j by $\sum_{i=1}^{r} \lambda_i d_i$. Then we may write $n = d_j \otimes f$. Since $\mathcal{A}n$ is a direct summand of $\oplus_{i=1}^{r}\mathcal{A}d_i$, it follows that f is a unit in $\mathcal{A}$. Hence $N = \mathcal{D}d_j$ is the direct summand we wanted. *Q.E.D.*

Proof (of theorem 23): Let (M, F) be a Frobenius module over $W(k)[[t]]$. Let $N^c \subset M \otimes_{W(k)[[t]]} \Gamma^c$ be an F-invariant direct summand. Then we have to show that there is a unique Frobenius submodule $N \subset M$ such that $N^c = N \otimes_{W(k)[[t]]} \Gamma^c$.

Let us begin with the uniqueness. If N exist then $N_1 = N^c \cap M$ is by lemma 6 another Frobenius submodule, which fullfills the theorem and contains N. Since $N \to N_1$ becomes an isomorphism over Γ^c it is by proposition 5 an isomorphism. Therefore N is unique.

Let $B = W[[t]]_{(p)}$ the localization in the prime ideal generated by p. We set $M' = M \otimes_{W(k)[[t]]} B$. It is equivalent to show that there is a direct summand $N' \subset M'$ as B-module such that $N' \otimes_B \Gamma^c = N^c$. Then we have $N' = M' \cap N^c$, and therefore N' is F-invariant. By lemma 17 we may therefore assume that N^c has rank 1.

Moreover we may assume that k is algebraically closed. Indeed, let $\bar{k}$ be the algebraic closure of k. We denote by $\bar{B}$ etc. the objects corresponding to $\bar{k}$. Then the Galois group $G = Gal(\bar{k}/k)$ acts on $\bar{B}$ and the invariants are B. If $\bar{N}' \subset M' \otimes_B \bar{B}$ exists it is stable by G because it is uniquely determined. Therefore we may apply corollary 18. Hence we assume that k is algebraically closed, and N^c is of rank 1.

By lemma 32 we find a free direct summand $N' \subset M \otimes_{W[[t]]} \mathcal{D}$ of rank 1, such that $N^c \otimes_{\Gamma^c} \mathcal{A} = N' \otimes_{\mathcal{D}} \mathcal{A}$. Therfore it suffices to show the following general lemma: *Q.E.D.*

If we localize the cartesian diagram (28) by the multiplicatively closed system $S = W[[t]] \setminus pW[[t]]$ we obtain a cartesian diagram:

$$\begin{array}{ccc} B & \rightarrow & \Gamma^c \\ \downarrow & & \downarrow \\ \mathcal{M} & \rightarrow & \mathcal{A} \end{array} \tag{32}$$

Lemma 33 *Let M be a finitely generated free B-module. Let $N^c \subset M \otimes_B \Gamma^c$ be a direct summand of rank 1, and let $N' \subset M \otimes_B \mathcal{M}$ be a direct summand, which is free of rank 1 as an $\mathcal{M}$-module. We assume that $N^c \otimes_{\Gamma^c} \mathcal{A} = N' \otimes_{\mathcal{M}} \mathcal{A}$. Then there exists a unique direct summand $N \subset M$, which induces N^c and N'.*

Proof: Let $f \in \mathcal{M}$ be an element which becomes a unit in $\mathcal{A}$. We claim that f is a unit in $\mathcal{M}$. To see this we may assume $f \in \mathcal{D}$. By assumption f is a unit in $\mathcal{L}([\rho, 1))$ for some ρ. By theorem 27 we may write in $\mathcal{L}([0, \rho])$:

$$fu = P$$

In $\mathcal{L}(\rho)$ we obtain $u = P/f$. The right hand side of this equation converges in $[\rho, 1)$. We conclude that u converges in $[\rho, 1)$ too, and finally $u \in \mathcal{D}$. Therefore we find in $\mathcal{M}$ the equation $f(u/P) = 1$.

Let n^c respectively n' be generators of N^c respectively N' . Let $e_1, \ldots, e_r$ be a basis of the B-module M. Then we write

$$\begin{array}{rcl} n^c & = & e_1 \otimes g_1 + \ldots + e_r \otimes g_r \\ n' & = & e_1 \otimes f_1 + \ldots + e_r \otimes f_r \end{array}$$

where $g_i \in \Gamma^c$, and $f_i \in \mathcal{M}$. Since Γ^c is a local ring we may assume without generality that $g_1 = 1$.

By assumption there is a unit $h \in \mathcal{A}$, such that $n' = hn$. We conclude that $f_1 = h$, and by the assertion shown above that f_1 is a unit in $\mathcal{M}$. dividing by f_1 we may assume that $f_1 = h = 1$. Then we see from the diagram (32) that $f_i = g_i \in B$ *Q.E.D.*

5 Proof of proposition 25

We will now prove proposition 25 in its dual version:

Proposition 34 *Let k be algebraically closed. Let $M \rightarrow N$ be a morphism of quasi Frobenius modules over $W(k)[[t]]$, such that N is isoclinic of slope λ and all slopes of M are bigger or equal to λ.*

If the map $M \otimes \mathbb{Q} \rightarrow N \otimes \mathbb{Q}$ is surjective, it has an F-equivariant section.

Proof: We set $\lambda = r/s$, where r and $s > 0$ are integers. We set $U = p^{-r}F^s$. We may assume that $UM \subset M$, $UN = N$, and that the cokernel of the map $M \to N$ is annihilated by a power of the maximal ideal of $W[[t]]$. Indeed, we set $B = W[[t]]_{(p)}$, $M' = M \otimes_{W(k)[[t]]} B$, and $N' = N \otimes_{W(k)[[t]]} B$. Let $M'_0 \subset M' \otimes \mathbb{Q}$ be a finitely generated (free) B-submodule which is mapped surjectively to $N'_0 = N'$, and such that $M'_0 \otimes \mathbb{Q} = M' \otimes \mathbb{Q}$. We set $M'_1 = \sum_{i=0}^{\infty} U^i M'_0$ and $N'_1 = \sum_{i=0}^{\infty} U^i N'_0$. Then M'_1, N'_1 are both U-invariant and finitely generated over B. The map $M'_1 \to N'_1$ is surjective. Finally we consider $M_1 = M'_1 \cap (M \otimes \mathbb{Q})$, and $N_1 = N'_1 \cap (N \otimes \mathbb{Q})$. These $W(k)[[t]]$-modules are free by lemma 6. Then we have $UM_1 \subset M_1$, and $UN_1 = N_1$ because N_1 is isoclinic of slope λ. Finally the map $M_1 \to N_1$ becomes surjective if we tensor it with $\mathbb{Q}$ and also if we tensor it with B. Hence the cokernel is annihilated by a power of the maximal ideal of $W(k)[[t]]$.

We define X and Y by the exact sequence:

$$0 \to Y \to M \to N \to X \to 0$$

We deduce an exact sequence:

$$0 \to (Y/tY) \otimes \mathbb{Q} \to (M/tM) \otimes \mathbb{Q} \to (N/tN) \otimes \mathbb{Q} \to 0$$

Then Y/tY is a U-invariant submodule of M/tM, and $\bar{Y} = (Y/tY) \otimes \mathbb{Q} \cap (M/tM)$ is U-invariant too. Let us consider the cokernel $\bar{N}$:

$$0 \to \bar{Y} \to M/tM \to \bar{N} \to 0$$

The operator U acts on $\bar{N}$. On the other hand $\bar{N}$ is isoclinic of slope λ, since $\bar{N} \otimes \mathbb{Q} = N/tN \otimes \mathbb{Q}$ is. This shows that U induces an σ^s-linear isomorphism on $\bar{N}$. By Dieudonné's classifiction of isocrystals we have a U-equivariant section $\bar{N} \to M/tM$. Its image $\bar{E}$ lifts by the lemma below to a U-invariant direct summand E of M. Then the U-equivariant map:

$$Y \oplus E \to M \tag{33}$$

is an isogeny modulo t. Therefore it is an isogeny by proposition 5 and the first remark after it. We find that $E \to N$ is an isogeny, and hence $E \otimes \mathbb{Q} \to N \otimes \mathbb{Q}$ is an isomorphism. This shows that $M \otimes \mathbb{Q} \to N \otimes \mathbb{Q}$ has an U-equivariant section β. But then

$$\frac{1}{s} \sum_{i=0}^{s-1} F^{-i} \beta F^i$$

is the desired F-equivariant section. *Q.E.D.*

Finally we have to prove that $\bar{E}$ lifts. We fix an integer $s > 0$, and we consider the endomorphism $\tau = \sigma^s$ of $W(k)[[t]]$.

Lemma 35 *Let P be a finitely generated projective $W(k)[[t]]$-module and $U : P \to P$ be a τ-linear endomorphism. We set $P_0 = P \otimes_{W(k)[[t]]} W(k) = P/tP$. Then U induces a τ-linear endomorphism $U : P_0 \to P_0$ of the $W(k)$-module P_0.*

Let E_0 be a direct summand of P_0, such that U induces a τ-linear isomorphism.

$$\varphi_0 : E_0 \longrightarrow E_0.$$

Then there exists a direct summand $E \subset P$, which is uniquely determined by the following properties:

(i) *$U(E) \subseteq E$.*

(ii) *E lifts E_0.*

(iii) *$U : E \to E$ is a τ-linear isomorphism.*

We prove this in a more general situation: Let A be a commutative ring and $\mathfrak{a} \subset A$ an ideal, which consists of nilpotent elements. We set $A_0 = A/\mathfrak{a}$ and more generally we denote for an A-module M the A_0-module $M/\mathfrak{a}M$ by M_0. Let $\tau : A \to A$ be a ring homomorphism, such that $\tau(\mathfrak{a}) \subset \mathfrak{a}$, and such that there exists a natural number r with $\tau^r(\mathfrak{a}) = 0$. We denote by $\tau_0 : A_0 \to A_0$ the ring homomorphism induced by τ.

We will apply this to the ring $A = W(k)[[t]]/(t^m)$ for an arbitrary natural number m, the endomorphism τ induced by τ on $W(k)[[t]]$, and the ideal $\mathfrak{a} = tA$. Therefore it is enough to prove the following proposition:

Proposition 36 *Let P be a finitely generated projective A-module and $\varphi : P \to P$ be a τ-linear endomorphism. Then φ induces a τ_0-linear endomorphism $\varphi_0 : P_0 \to P_0$ of the A_0-module P_0.*

Let E_0 be a direct summand of P_0, such that φ_0 induces a τ_0-linear isomorphism.

$$\varphi_0 : E_0 \longrightarrow E_0.$$

Then there exists a direct summand $E \subset P$, which is uniquely determined by the following properties:

(i) $\varphi(E) \subseteq E$.

(ii) E *lifts* E_0.

(iii) $\varphi : E \to E$ *is a* τ*-linear isomorphism.*

(iv) *Let* C *be an* A*-module, which is equipped with a* τ*-linear isomorphism* $\psi : C \to C$. *Let* $\alpha : (C, \psi) \to (P, \varphi)$ *be an* A*-module homomorphism such that* $\alpha \circ \psi = \varphi \circ \alpha$. *Let us assume that* $\alpha_0(C_0) \subset E_0$. *Then we have* $\alpha(C) \subset E$.

Proof: By our assumption on r we have an isomorphism

$$A \otimes_{\tau^r, A} P = A \otimes_{\tau^r, A_0} P_0.$$

We define E to be the image of the A-module homomorphism

$$(\varphi^r)^\# : A \otimes_{\tau^r, A_0} E_0 \longrightarrow P. \tag{34}$$

It follows immediately that $\varphi(E) \subset E$.

Let us prove that E is a direct summand of P. We choose a A_0-submodule $F_0 \subset P_0$, which is complementary to E_0:

$$P_0 = E_0 \oplus F_0.$$

Then we lift F_0 to a direct summand F of P. We consider the map induced by (34)

$$(\varphi^r)^\# : A \otimes_{\tau^r, A_0} E_0 \longrightarrow P/F. \tag{35}$$

By assumption the last map becomes an isomorphism, when tensored with $A_0 \otimes_A$. Hence we conclude by the lemma of Nakayama that (35) is an isomorphism. We see that E is a direct summand:

$$P = E \oplus F$$

Applying Nakayama's lemma to the projective and finitely generated module E, we obtain that:

$$\varphi^\# : A \otimes_{\tau, A} E \longrightarrow E$$

is an isomorphim.

Therefore we have checked the properties $(i) - (iii)$. The last property follows from the commutative diagram:

$$\begin{tikzcd} A \otimes_{\tau^r, A_0} E_0 \arrow[r] & E \arrow[r] & P \\ A \otimes_{\tau^r, A_0} C_0 \arrow[u, "1\otimes\alpha_0"] \arrow[rr, "\sim"] & & C \arrow[u, "\alpha"'] \end{tikzcd} \qquad Q.E.D.$$

References

[J] A.J.de Jong, Homomorphisms of Barsotti-Tate groups and crystals in positive characteristics, Invent. math.

[G] U.Güntzer: Zur Funktionentheorie einer Veränderlichen über einem vollständigen nichtarchimedischen Grundkörper, Arch. f. Math. Vol. XVII (1966).

[L] M.Lazard, Publ.Math. IHES 14.