On the slope filtration

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Abstract

Let X be a p-divisible group over a regular scheme S such that the Newton polygon in each geometric point of S is the same. Then there is a p-divisible group isogenous to X which has a slope filtration.

1 Introduction

Let X be a p-divisible group over a perfect field. The Dieudonné classification implies that X is isogenous to a direct product of isoclinic p-divisible groups. We will study what remains true, if the perfect field is replaced by a ring R such that $pR = 0$.

Let now X be a p-divisible group over R. Let us denote by $Fr_X : X \to X^{(p)}$ the Frobenius homomorphism. We call X isoclinic and slope divisible if there are natural numbers $r \geq 0$ and $s > 0$, such that

$$p^{-r} Fr_X^s : X \to X^{(p^s)}$$

is an isomorphism. Then X is isoclinic of slope r/s, i.e. it is isoclinic of slope r/s over each geometric point of $\operatorname{Spec} R$. We will say that r/s is the slope of X.

If R is a field a p-divisible group is isoclinic iff it is isogenous to a p-divisible group which is isoclinic and slope divisible.

It is stated in a letter of Grothendieck to Barsotti (see [G1]), that over a field $K = R$ any p-divisible group admits a slope filtration:

$$0 = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X \tag{1}$$

This filtration is uniquely determined by the following properties: The inclusions are strict and the factors X_i/X_{i-1} are isoclinic p-divisible groups of slope λ_i, such that $1 \geq \lambda_1 > \ldots > \lambda_m \geq 0$. Moreover the rational numbers λ_i are uniquely determined. A proof of this statement was never published, but can be found here.

The heights of the factors and the numbers λ_i determine the Newton polygon and conversely. If we want a slope filtration over R, we have to assume that the Newton polygon is the same in any point of $\operatorname{Spec} R$. We say in this case that X has a constant Newton polygon.

Theorem: *Let R be a regular ring. Then any p-divisible group over R with constant Newton polygon is isogenous to a p-divisible group X, which admits a strict filtration* (1) *such that the quotients X_i/X_{i-1} are isoclinic and slope divisible of slope λ_i with $1 \geq \lambda_1 > \ldots > \lambda_m \geq 0$.*

In the case where $\dim R = 1$ and R is finitely generated over a perfect field the theorem was proved by Katz [K] using the crystalline theory. Our proof uses only Dieudonné theory over a perfect field. It is based on a purity result (proposition 5) below which was suggested to us when reading the work of Harris and Taylor.

Let S be a regular scheme and U an open subset such that the codimension of the complement is ≥ 2. Then we show that a p-divisible group over U with constant Newton polygon extends up to isogeny to a p-divisible group over S. One might call this Nagata-Zariski purity for p-divisible groups.

We note that there is a difficult purity result of de Jong and Oort, which holds without the regularity assumption for any noetherian scheme S. It says that a p-divisible group X over S, which has constant Newton polygon on U, has constant Newton polygon on S.

Finally I would like to thank Johan de Jong and Michael Harris for pointing out this problem to me, and Frans Oort for helpful remarks.

2 The étale part of a Frobenius module

We will work over a base scheme S over $\mathbb{F}_p$. The Frobenius morphism will be denoted by $Frob_S$

Definition 1 *Fix an integer $a > 0$. A Frobenius module over S is a finitely generated locally free $\mathcal{O}_S$-module $\mathcal{M}$, and a $Frob_S^a$-linear map $\Phi : \mathcal{M} \to \mathcal{M}$.*

There is an important case, where the condition that $\mathcal{M}$ is locally free is automatically satisfied namely if Φ is a $Frob_S^a$-linear isomorphism. This means that the linearization

$$\Phi^\sharp : \mathcal{O}_S \otimes_{Frob_S^a, S} \mathcal{M} \to \mathcal{M}$$

is an isomorphism.

Lemma 2 *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume that R is $\mathfrak{m}$-adically separated. Let M be a finitely generated R-module. Assume that there exists a $Frob_S^a$-linear isomorphism $\Phi : M \to M$. Then M is free.*

Proof: We choose a minimal resolution of M:

$$0 \to U \to P \to M \to 0,$$

where P is a finitely generated free R-module and $U \subset \mathfrak{m}P$. Since $R \otimes_{Frob^a, R} P$ is a free R-module the linearization $\Phi^\sharp$ extends to $R \otimes_{Frob^a, R} P$, i.e. we find a commutative diagram:

$$\begin{array}{ccc} R \otimes_{Frob^a,R} P & \longrightarrow & R \otimes_{Frob^a,R} M \\ \Phi^\sharp \downarrow & & \downarrow \Phi^\sharp \\ P & \longrightarrow & M \end{array} \qquad (2)$$

Since $P/\mathfrak{m}P \cong M/\mathfrak{m}M$ it follows by Nakayama that the left vertical arrow is surjective and hence an isomorphism. The diagram implies $U = \Phi^\sharp(R \otimes_{Frob^a,R} U)$ (with a small abuse of notation). Since P is $\mathfrak{m}$-adically separated it is enough to show that $U \subset \mathfrak{m}^n P$ for each number n. This is true for $n = 1$ by construction. We assume by induction that the inclusion is true for a given n and find:

$$U \subset \Phi^\sharp(R \otimes_{Frob^a,R} \mathfrak{m}^n P) \subset \Phi^\sharp(\mathfrak{m}^{np^a} \otimes_{Frob^a,R} P) \subset \mathfrak{m}^{np^a} P.$$

Q.E.D.

To any Frobenius module we associate the following functor on the category of schemes $T \to S$:

$$C_{\mathcal{M}}(T) = \{x \in \Gamma(T, \mathcal{M}_T) \mid \Phi x = x\}$$

Proposition 3 *The functor $C_{\mathcal{M}}$ is representable by a scheme which is étale and affine over S.*

Proof: Since the functor is a sheaf for the flat (fppf) topology the question is local on S. We may therefore assume that $S = \operatorname{Spec} R$ and that $\mathcal{M}$ is the sheaf associated to a free R-module M. We choose an isomorphism $M \cong R^n$ and write the operator Φ in matrix form:

$$\Phi x = U x^{(p^a)}, \qquad x \in R^n.$$

Here x is a column vector, and $x^{(p^a)}$ is the vector obtained by raising all components to the p^a-th power. U is a square matrix with coefficients in R. Let A be an R-algebra. We set $C_M(A) = C_{\mathcal{M}}(\operatorname{Spec} A)$. Then C_M is just the functor of solutions of the equation:

$$x = U x^{(p^a)}, \qquad x \in A^n.$$

This functor is clearly a closed subscheme of the affine space $\mathbb{A}^n_R$.

To show that C_M is étale one applies the infinitesimal criterion: Let $A \to \bar{A}$ be a surjection of R-algebras with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$. We have to show that the canonical map

$$C_M(A) \to C_M(\bar{A})$$

is bijective. We consider an element $\bar{x} \in C_M(\bar{A})$, and lift it to an element x of $A \otimes_R M \cong A^n$. We set $\rho = \Phi x - x \in \mathfrak{a} \otimes_R M$. Since $\Phi(\mathfrak{a} \otimes_R M) = 0$ we obtain

$$\Phi(x + \rho) = \Phi x = x + \rho.$$

This shows that $x + \rho \in C_M(A)$ is the unique lifting of $\bar{x}$. *Q.E.D.*

To make life easier let us assume that S is an $\mathbb{F}_{p^a}$-scheme. Then $C_{\mathcal{M}}$ may be considered as a sheaf of $\mathbb{F}_{p^a}$-vector spaces. If S is connected and $\eta \in S$ is a point, the natural map

$$C_{\mathcal{M}}(S) \to C_{\mathcal{M}}(\eta)$$

is injective because $C_{\mathcal{M}}$ is unramified and separated over S (e.g. proposition [EGA IV 17.4.9]).

Let us assume that $S = \operatorname{Spec} K$ is the spectrum of an algebraically closed field. Let (M, Φ) be a Frobenius module over K. Then there is a unique decomposition:

$$M = M^{bij} \oplus M^{nil}, \tag{3}$$

into Φ-invariant subspaces, such that Φ is bijective on the first summand and nilpotent on the second summand. Moreover by a theorem of Dieudonné (lemma [Z, 6.25]) we have an isomorphism:

$$K \otimes_{\mathbb{F}_{p^a}} C_M(\operatorname{Spec} K) \to M^{bij} \tag{4}$$

Let us assume that $S = \operatorname{Spec} K$ is the spectrum of separably closed field, and denote the algebraic closure by $\bar{K}$. Since $C_M(K) = C_M(\bar{K})$, the subspace M^{bij} is defined over K by (4). Note that M^{nil} is not defined over K, e.g. $M = K^{p^{-1}}$ ane $\Phi = Frob$.

We note that the submodule M^{bij} is defined over any field K by Galois descent ([G2] B, Exemple 1): If K^s denotes the separable closure and G its Galois group over K, we set:

$$M^{bij} = (K^s \otimes_{\mathbb{F}_{p^a}} C_M(K^s))^G$$

This subspace is characterized as follows: On M^{bij} the operator Φ acts as a $Frob^a$-linear isomorphism, and on the factor M/M^{bij} it acts nilpotently.

We note that the functor $M \mapsto M^{bij}$ is an exact functor in M. To see this it is enough to consider the case of an algebraically closed field K. With this assumption the result follows because the decomposition (3) is functorial in M. The same argument shows that the functor commutes with tensor products.

Assume that $S = \operatorname{Spec} R$ and that (M, Φ) is a Frobenius module over R.

Lemma 4 *Assume that* $\operatorname{Spec} R$ *is connected. Then the natural map*

$$R \otimes_{\mathbb{F}_{p^a}} C_M(R) \to M \tag{5}$$

is an injection onto a direct summand of M.

Proof: Since $\operatorname{Spec} R$ is connected, the natural map $C_M(R) \to C_M(R_\mathfrak{p})$ is for any prime ideal $\mathfrak{p}$ of R injective ([EGA] loc.cit.). Therefore it is enough to show our statement for a local ring R with maximal ideal $\mathfrak{m}$. Indeed the question whether the finitely generated quotient of (5) is projective is local. Since $R_\mathfrak{p} \otimes_{\mathbb{F}_{p^a}} C_M(R)$ is obviously a direct summand of $R_\mathfrak{p} \otimes_{\mathbb{F}_{p^a}} C_M(R_\mathfrak{p})$ we are reduced to the local case.

In this case it is enough to show that the following map is injective:

$$R/\mathfrak{m} \otimes_{\mathbb{F}_{p^a}} C_M(R) \to M/\mathfrak{m}M.$$

Since the map $C_M(R) \to C_M(R/\mathfrak{m})$ is injective we are reduced to the case where R is a field. Then the injectivity follows from the considerations above. *Q.E.D.*

Let $S = \operatorname{Spec} R$, where R is an henselian local ring with maximal ideal $\mathfrak{m}$. Then there is a unique Φ-invariant direct summand $L \subset M$, such that Φ is an $Frob^a$-linear isomorphism on L, and is nilpotent on $M/L + \mathfrak{m}M$. We call L the finite part.

To show this one reduces the problem by Galois descent [G2] to the case where R is strictly henselian. In this case we can set $L = R \otimes_{\mathbb{F}_{p^a}} C_M(R)$. We note also that taking the finite part L is an exact functor in M. This functor also commutes with tensor products.

Let us return to the general situation of definition 1. For each point η of S we define the function:

$$\mu_{(\mathcal{M},\Phi)}(\eta) = \dim_{\mathbb{F}_{p^a}} (C_\mathcal{M})_{\bar{\eta}},$$

where $\bar{\eta}$ is some geometric point over η.

If $\mu_{(\mathcal{M},\Phi)}(\eta) \geq k$ it stays bigger or equal than k in some neighbourhood of η. If this function is constant on S there is a Φ-invariant submodule $\mathcal{L}$ of $\mathcal{M}$, which is locally a direct summand, such that Φ is an $Frob^a$-linear isomorphism on $\mathcal{L}$ and is locally on S nilpotent on $\mathcal{M}/\mathcal{L}$. By this last property $\mathcal{L}$ is uniquely determined. For this result it is not necessary that S is noetherian. Indeed in this case the scheme C associated to $(\mathcal{M}, \Phi)$ is finite étale since all geometric fibres have the same number of points (corollaire [EGA IV 18.2.9]). Then C represents an étale sheaf on S denoted by the same letter. In the sense of étale sheaves we have:

$$\mathcal{L} = \mathcal{O}_S \otimes_{\mathbb{F}_{p^a}} C$$

If the scheme S is perfect the exact sequence:

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{M}/\mathcal{L} \to 0$$

splits canonically. Indeed, it is enough to define this splitting in the case $S = \operatorname{Spec} R$. Then $\Phi : \mathcal{L} \to \mathcal{L}$ is bijective. Assume that Φ^n is zero on $\mathcal{M}/\mathcal{L}$ for some number n. Let $\mathcal{M}^{nil}$ be the kernel of Φ^n on $\mathcal{M}$. Then the projection $\mathcal{M}^{nil} \to \mathcal{M}/\mathcal{L}$ is bijective. Indeed, let $x \in \mathcal{M}$. Then $\Phi^n x \in \mathcal{L}$. Since Φ is bijective on $\mathcal{L}$, we find $y \in \mathcal{L}$ with $\Phi^n y = \Phi^n x$. But then x and $x - y \in \mathcal{M}^{nil}$ have the same image in $\mathcal{M}/\mathcal{L}$. This proves the assertion.

The following purity result is contained in Harris-Taylor [HT] in a special case:

Proposition 5 *Let R be a noetherian local ring of dimension ≥ 2. Let (M, Φ) be a Frobenius module over R. Assume that the function $\mu_{(M,\Phi)}$ is constant outside the closed point of* Spec R*. Then it is constant.*

Proof: We can assume that R is a complete local ring with algebraically closed residue field. Let $S = \operatorname{Spec} R$, and let U the complement of the closed point $s \in S$.

Since C_M is étale over R it admits a unique decomposition:

$$C_M = C_M^f \coprod C_M^0,$$

where C_M^f is finite and étale over Spec R, and where C_M^0 has an empty special fibre. We note that C_M^0 is affine as a closed subscheme of C_M.

We have to show that C_M^0 is empty. Let us assume the opposite. We consider the following function on U:

$$\sharp C^0_{M,\bar{\eta}} = \sharp C_{M,\bar{\eta}} - \sharp C^f_{M,\bar{\eta}}, \qquad \eta \in U. \tag{6}$$

Here $\sharp$ denotes the number of points in the corresponding scheme. The first term on the right hand side of (6) is by assumption constant on U while the second term has this property for obvious reasons.

Hence all geometric fibres of the map

$$C_M^0 \to U,$$

have the same number of points. Together with our assumption that C_M^0 is not empty this shows that the last map is surjective. But this implies that

U is affine (théorème [EGA II 6.7.1)]. Since U is not affine ([G3] Proposition 6.4) we have a contradiction. *Q.E.D.*

If R is a regular local ring of dimension 2, then any Frobenius module $(\mathcal{M}, \Phi)$ over U may be extended to S, because the direct image of $\mathcal{M}$ by $j : U \to S$ is a free R-module M. This implies in particular that any locally constant étale sheaf of $\mathbb{F}_{p^a}$-vector spaces extends to a locally constant étale sheaf on S (purity).

We apply this to finite commutative group schemes as follows: Let G be a finite locally free group scheme over a scheme S. Assume we are given a homomorphism $\Phi : G \to G^{(p^a)}$. Let $G = \operatorname{Spec} \mathcal{M}$ relative to S. Then Φ induces on $\mathcal{M}$ the structure of a Frobenius module.

Let $S = \operatorname{Spec} R$ the spectrum of an henselian local ring. Let $\mathcal{L}$ be the finite part of $\mathcal{M}$ which was defined after lemma 4. Since its formation commutes with tensor products we obtain a finite locally free group scheme $G^{\Phi} = \operatorname{Spec} \mathcal{L}$. Since $\mathcal{L}$ is a direct summand of $\mathcal{M}$ the natural morphism $G \to G^{\Phi}$ is an epimorphism of finite locally free group schemes. Let us denote by $G^{\Phi-nil}$ the kernel. We obtain an exact sequence of finite locally free group schemes:

$$0 \to G^{\Phi-nil} \to G \to G^{\Phi} \to 0 \tag{7}$$

such that Φ induces an isomorphism on G^{Φ} and is nilpotent on the special fibre of $G^{\Phi-nil}$.

Lemma 6 *Let G_i $i = 1, 2, 3$ be finite locally free group schemes over the spectrum S of a henselian local ring. Let $\Phi_i : G_i \to G_i^{(p^a)}$ be homomorphisms. Assume we are given an exact sequence:*

$$0 \to G_1 \to G_2 \to G_3 \to 0,$$

which respects the homomorphisms Φ_i. Then the corresponding sequence

$$0 \to G_1^{\Phi_1} \to G_2^{\Phi_2} \to G_3^{\Phi_3} \to 0,$$

is exact.

Proof: Let S be the spectrum of an algebraically closed field. In view of the decomposition (3) we have a unique Φ_i-equivariant section of the epimorphism $G_i \to G_i^{\Phi}$. Therefore there exists a functorial decomposition:

$$G_i = G_i^{\Phi_i} \oplus G_i^{\Phi_i - nil}$$

This proves the assertion for an algebraically closed field and hence for any field. In the general case we consider the kernel H of the epimorphsim $G_2^{\Phi_2} \to G_3^{\Phi_3}$. Then we obtain a homomorphism of locally free group schemes $G_1^{\Phi_1} \to H$, which is an isomorphism over the closed point of S. Hence it is an isomorphism by the lemma of Nakayama. *Q.E.D.*

We consider a pair (G, Φ) as above over any locally noetherian scheme S. Let k be the maximal value of the corresponding function $\mu = \mu_{(\mathcal{M},\Phi)}$. Then the set $\mu = k$ is an open set U of S. Over U we have an exact sequence of finite locally free group schemes (7) such that Φ induces an isomorphism on G^{Φ} and is locally on S nilpotent on $G^{\Phi - nil}$. If S is irreducible the complement of U is by proposition 5 of pure codimension 1 or empty. The formation of G^{Φ} is by lemma 6 an exact functor in an obvious sense.

Assume that Φ is an isomorphism, i.e. $G = G^{\Phi}$. The étale sheaf C associated to (M, Φ) is a locally constant étale sheaf of $\mathbb{F}_{p^a}$-bigebras. If S is the spectrum of a strictly henselian local ring the canonical isomorphism $\mathcal{M} = \mathcal{O}_S \otimes C$ of bigebras means that G is obtained via base change from a group scheme G_0 over $\mathbb{F}_{p^a}$, and Φ from the identity $G_0 \to G_0^{(p^a)}$.

3 The slope filtration of a p-divisible group

Let X be a p-divisible group over a scheme S and $\lambda \in \mathbb{Q}$. We call X slope divisible with respect to λ if there are locally on S integers $r, s > 0$ such that $\lambda = \frac{r}{s}$ and the following quasiisogeny is an isogeny:

$$p^{-r} Fr_X^s : X \to X^{(p^s)} \tag{8}$$

Recall that a quasiisogeny is an isogeny formally divided by a power of p ([RZ] definition 2.8). We will use the fact that the functor of points of S where a quasiisogeny is an isogeny is representable by a closed subscheme of S ([RZ] proposition 2.9). If X is slope divisible and isoclinic of slope λ (i.e. isoclinic over any geometric point of S) then the isogeny above is an isomorphism.

Theorem 7 *Let S be a regular scheme. Let X be a p-divisible group over S whose Newton polygon is constant. Then there is a p-divisible group Y over S which is isogenous to X, and which has a filtration by closed immersions of p-divisible groups:*

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_k = Y,$$

such that Y_i/Y_{i-1} is isoclinic and slope divisible with respect λ_i, and the group Y_i is slope divisible with respect to λ_i. One has $\lambda_1 > \lambda_2 > \ldots > \lambda_k$.

The existence of the slope filtration over a field which is not necessarily perfect is announced in [G1]. Since a proof was never published we give it here before treating the general case.

Proposition 8 *Let K be a field of characteristic p. Let $G \to H$ be a morphism of p-divisible groups over K. The there is a unique factorization in the category of p-divisible groups*

$$G \to G' \to H' \to H$$

with the following properties:

(i) $G' \to H'$ is an isogeny.

(ii) $H' \to H$ is a monomorphism of p-divisible groups.

(iii) For each number n the morphism $G(n) \to G'(n)$ is an epimorphism of finite group schemes.

This factorization commutes with base change to another field.

Proof: We note that a monomorphism in the category of p-divisible groups is the same thing as a closed immersion. Let A be the kernel of $G \to H$ in the category of flat sheaves of abelian groups. Then A has the following properties:

(i) The kernel $A(n)$ of multiplication by p^n on A is representable by a finite group scheme.

(ii) The group A is the union of the subgroups $A(n)$.

With these assumptions there is a unique p-divisible subgroup $A' \subset A$ such that the quotient is a finite group scheme. Indeed, we consider the following sequence of monomorphisms

$$A(n+1)/A(n) \xrightarrow{p} A(n)/A(n-1) \xrightarrow{p} \ldots \rightarrow A(1), \tag{9}$$

which is induced by the multiplication by p. Since the ranks of the group schemes in (9) cannot decrease infinitely, there is a number n_0 such that $A(n+1)/A(n) \rightarrow A(n)/A(n-1)$ is an isomorphism for $n > n_0$. We set $A' = A/A(n_0)$. Then we obtain $A'(m) = A(n_0+m)/A(n_0)$. Because for A' all homomorphisms in (9) are isomorphisms this group is a p-divisible group. The multiplication by p^{n_0} defines a monomorphism $A' \overset{p^{n_0}}{\hookrightarrow} A$. The cokernel of this monomorphism is a finite locally free group scheme. This is seen in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A'(n_0) & \longrightarrow & A' & \xrightarrow{p^{n_0}} & A' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle p^{n_0}} & & \big\| & & \\
0 & \longrightarrow & A(n_0) & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

Now we may define G' as the quotient G/A', and H' as the quotient of G' by the finite group scheme A/A'. *Q.E.D.*

The group H' is the image of $G \rightarrow H$ in the category of flat sheaves. We call G' the small image of $G \rightarrow H$.

Assume for a moment that K is a perfect field, and let M_G and M_H be the covariant Dieudonné modules. Then $M_{G'}$ is the image of the map $M_G \rightarrow M_H$, while $M_{H'}$ is the smallest direct summand of M_H containing $M_{G'}$.

Let X be a p-divisible group of height h over a perfect field K. We denote by M its covariant Dieudonné module. It is a free $W(K)$-module of rank h. Let $\lambda = r/s$ be the smallest Newton slope of X. By lemma [Z, 6.13] there is a $W(K)$-lattice M' in $M \otimes \mathbb{Q}$ such that $V^s M' \subset p^r M'$. The operator $U = p^{-r} V^s$ acts on $M \otimes \mathbb{Q}$.

Lemma 9 *The submodule $M_0 \subset M \otimes \mathbb{Q}$ defined by:*

$$M_0 = M + UM + U^2M + \ldots + U^{h-1}M,$$

is a Dieudonné module which is invariant by U.

Proof: By [Z] loc.cit. we know that $M \otimes \mathbb{Q}$ contains a U-invariant lattice. Let M' be a lattice which contains M, such that $UM' \subset M'$. We will take M' minimal with respect to inclusion. Then $M \not\subseteq pM'$. We consider the ascending chain of lattices:

$$pM' \subsetneqq pM' + M \subset pM' + M + UM \subset \ldots \subset M'.$$

Since $\dim_k M'/pM' = h$ there is an integer $e \leq h - 1$ such that

$$pM' + M + \ldots + U^e M = pM' + M + \ldots + U^{e+1} M.$$

Hence this is a U-invariant lattice containing M and we conclude by minimality:

$$M' = M + \ldots + U^e M + pM'.$$

But then the lemma of Nakayama shows

$$M' = M + \ldots + U^e M.$$

Since U commutes with F and V, it is easy to see that M' is a Dieudonné module. This proves the lemma. *Q.E.D.*

By this lemma $F^{s(h-1)} M_0$ is the Dieudonné module of a p-divisible group Y over K, which is slope divisible with respect to λ. Clearly Y is the small image of the morphism of p-divisible groups which is defined as the composite of the following quasimorphisms:

$$X^{(p^{(h-1)s})} \times \ldots \times X^{(p^s)} \times X \xrightarrow{\alpha} X^{(p^{(h-1)s})} \to X \tag{10}$$

where the last arrow is the power $Ver^{(h-1)s}$ of the Verschiebung $Ver : X^{(p)} \to X$, and where the restriction of α to the factor $X^{(p^{(h-i)s})}$ is $p^{-(i-1)r} Fr^{(i-1)s}$. We recall here that Ver induces F on the Dieudonné module M, while $Fr : X \to X^p$ induces V (lemma [Z, 5.19]).

If K is not perfect we can still consider the small image Y of (10). Making base change to the perfect hull we see that $Y \to X$ is an isogeny, and that Y is slope divisible, i.e. that

$$\Phi = p^{-r} Fr^s : Y \to Y^{(p^s)}$$

is an isogeny. If we apply (7) to the finite group schemes $Y(n)$ and the operator Φ we obtain an exact sequence of p-divisible groups:

$$0 \to Y^{\Phi-nil} \to Y \to Y^{\Phi} \to 0 \tag{11}$$

The p-divisible group Y^{Φ} is slope divisible and isoclinic, while the smallest slope of $Y^{\Phi-nil}$ is strictly bigger than λ.

Remark: In the notation of lemma 9 the inclusion $F^{s(h-1)}M_0 \subset M$ holds. This follows easily from $r \leq s$. Note that we can take $s \leq h$. It follows that over any field K the degree of the isogeny $Y \to X$ is bounded by $p^{h^2(h-1)}$, i.e. by a constant which depends only on the height h of X.

Definition 10 *Let X be a p-divisible group over a field K. We call X completely slope divisible if it admits a filtration*

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X \tag{12}$$

by p-divisible subgroups, and if there are rational numbers $\lambda_1 > \ldots > \lambda_m$ such that

(i) X_i is slope divisible with respect to λ_i for $i = 1, \ldots, m$.

(ii) X_i/X_{i-1} is isoclinic and slope divisible with respect to λ_i.

Since there are no homomorphims between p-divisible groups with pairwise different Newton slopes ([Z]) it follows easily that the filtration (12) is uniquely determined.

Corollary 11 *If the field K is perfect the sequence (12) splits canonically.*

Proof: By the remark after definition 10 the splitting is unique. We consider the Dieudonné modules M_i of X_i. By induction it is enough to show that the following sequence splits as a sequence of Dieudonné modules:

$$0 \to M_{m-1} \to M_m \to M_m/M_{m-1} \to 0$$

But $\Phi = p^{-r_m}V^s$ acts on this sequence. On M_{m-1} the action is topologically nilpotent and on M_m/M_{m-1} it is bijective. Therefore we conclude by lemma [Z, 6.16]. *Q.E.D.*

Proposition 12 *Let h be a number. Then there is a constant c which depends only on h with the following property.*

Let X be a p-divisible group of height h over a field K. Then there is an isogeny $X' \to X$ whose degree is smaller than c such that X' is completely slope divisible.

Proof: Let λ_i for $i = 1, \dots, m$ be the slopes of X. We may write $\lambda_i = r_i/s$ where s divides $h!$, and $r_1 > \dots > r_m$. By what we have proved we find an isogeny $Y \to X$ of bounded degree such that Y is slope divisible with respect to λ_m. If $m = 1$ then Y is completely slope divisible. For $m > 1$ we argue by induction. We set $\Phi = p^{-r_m}Fr^s$ and obtain the Φ-decomposition (11). By induction there is an isogeny of bounded degree $Y^{\Phi-nil} \to Z$ where Z is completely slope divisible. Then we take the push-out of the sequence (11) by the morphism $Y^{\Phi-nil} \to Z$:

$$0 \to Z \to Z' \to Y^{\Phi-et} \to 0$$

The only thing we have to check is that Z' is slope divisible with respect to r_m/s. But by induction Z is slope divisible with respect to r_{m-1}/s and hence a fortiori with respect to r_m/s. From the exact sequence (11) it follows that $Y^{\Phi-nil}$ is slope divisible with respect to $\lambda_m = r_m/s$. By definition Z' sits in an exact sequence:

$$0 \to Y^{\Phi-nil} \to Y \times Z \to Z' \to 0$$

Since all groups in this sequence except Z' are slope divisible with respect to $\lambda_m = r_m/s$ the same is true for Z'. *Q.E.D.*

Corollary 13 *Let X be a p-divisible groups over a field K. Let $\lambda_1 > \dots > \lambda_m$ be the sequence of slopes of X. Then there is a filtration of X*

$$0 \subset X_1 \subset X_2 \subset \dots \subset X_m = X \qquad (13)$$

by p-divisible subgroups, such that

(i) X_i has the slopes $\lambda_1, \dots, \lambda_i$ for $i = 1, \dots, m$.

(ii) X_i/X_{i-1} is isoclinic of slope λ_i.

Proof: Indeed in the notation of proposition 12 it is enough to consider the image of the filtration on X' by the isogeny $X' \to X$. *Q.E.D.*

Proof of Theorem 7: If S is the spectrum of a field this follows from proposition 12.

If the dimension of S is 1 this was shown under more restrictive conditions by Katz ([K] Corollary 2.6.3). We give an alternative proof which holds in the general situation: Let K be a function field of S. Any isogeny $X_K \to \mathring{Y}$

over K extends to an isogeny $X \to Y$ over S. (See also the discussion in front of the next proposition.) Hence we may assume that X_K is completely slope divisible (definition 10). We have to show that this filtration extends to S. Since the functor of points of S, where (8) is an isogeny is representable by a closed subscheme of S we see that X is slope divisible with respect to λ_k. Therefore we have an isogeny:

$$\Phi = p^{-r} Fr^s : X \to X^{(p^s)}.$$

The function μ associated to $(X(n), \Phi)$ is constant since the Newton polygon is constant. Therefore we may form the finite group schemes $X(n)^{\Phi}$. For varying n this is a p-divisible group Z, since the functor $X(n) \mapsto X(n)^{\Phi}$ is exact. We obtain an exact sequence:

$$0 \to X' \to X \to Z \to 0$$

Then X' has again constant Newton polygon, but the slope λ_k doesn't appear. Since X'_K is completely slope divisible we can finish the proof in the case $\dim S = 1$ by induction.

In arbitrary dimension this consideration shows that a p-divisible group X over S has a slope filtration as in theorem 7, if X_K is completely slope divisible.

By the same method we may find Y with the filtration over an open set $U \subset S$, which contains all points of codimension 1. Indeed, we start again with an isogeny $X_K \to \overset{\circ}{Y}$, where $\overset{\circ}{Y}$ is a p-divisible group over K which is completely slope divisible. The kernel $\overset{\circ}{G}$ of this isogeny is a closed subscheme of some $X_K(n)$. We denote by G the scheme theoretic closure in $X(n)$. Let U be the open subscheme where G is flat. We replace S by U and assume that G is flat over S. Then one checks that G inherits the structure of a group scheme, such that $G \to X(n)$ is a closed immersion of group schemes. We may replace X by $Y = X/G$. This group is slope divisible with respect to λ_k. Therefore we obtain the slope filtration as above. The case where S has arbitrary dimension will now follow from the following proposition:

Proposition 14 *Let S be a regular scheme and $U \subset S$ an open subscheme which contains all points of codimension 1. Suppose that Y is a p-divisible group on U with a filtration as in theorem 7. Then Y extends to S.*

Proof: We set $\operatorname{Spec} \mathcal{A}_i(n) = Y_i(n)$. Let $j : U \to S$ be the immersion. It is enough to prove the following 3 statements.

1) The sheaves $\mathcal{A}'_i(n) = j_* \mathcal{A}_i(n)$ are locally free $\mathcal{O}_S$-modules.

If this is true the bigebra structure on $\mathcal{A}_i(n)$ extends to $\mathcal{A}'_i(n)$. Therefore we can define finite locally free group schemes $Y'_i(n) = \operatorname{Spec} \mathcal{A}'_i(n)$.

2) For varying n the systems $\{Y'_i(n)\}$ define a p-divisible group Y'_i.

3) The induced maps $Y'_i \to Y'_{i+1}$ are closed immersions.

To verify these statements one can make without loss of generality a faithfully flat base change $S' \to S$. Therefore it is enough to consider the case, where S is the spectrum of a complete regular local ring R of dimension ≥ 2 with algebraically closed residue field. By induction on the dimension we may assume that U is the complement of the closed point.

We make an induction on the length of the filtration. For $k = 1$ we extend Y_1 as follows. Let $\lambda_1 = \frac{r}{s}$ such that $\Phi = p^{-r} Fr^s$ is an isogeny and a hence an isomorphism $Y_1 \to Y_1^{(p^s)}$. Since the morphism $Frob^s : S \to S$ is flat, we may apply base change (lemme [EGA, IV 2.3.1)] to the cartesian diagram:

$$\begin{array}{ccc} U & \longrightarrow & S \\ {\scriptstyle Frob^s}\downarrow & & \downarrow{\scriptstyle Frob^s} \\ U & \longrightarrow & S \end{array}$$

This yields an isomorphism:

$$R \otimes_{Frob^s, R} A'_1(n) \cong j_*(\mathcal{O}_U \otimes_{Frob^s, \mathcal{O}_U} \mathcal{A}_1(n)),$$

where $A'_1(n)$ denotes the global sections of $\mathcal{A}'_1(n)$. Therefore the isomorphism Φ induces an isomorphism

$$\Phi^* : R \otimes_{Frob^s, R} A'_1(n) \to A'_1(n)$$

We will denote the associated $Frob^s$-linear map by $\Psi : A'_1(n) \to A'_1(n)$. By the lemma 2 it follows that $A'_1(n)$ is free. This shows the assertion 1). To see the second assertion we have to show that the sequence:

$$0 \to Y'_1(1) \to Y'_1(n) \to Y'_1(n-1) \to 0 \qquad (14)$$

is exact. Since we know that this sequence is exact over U, it suffices to show that the first arrow is a closed immersion. Indeed knowing this we obtain a morphism of finite locally free group schemes over S:

$$Y_1'(n)/Y_1'(1) \to Y_1'(n-1)$$

Since this is an isomorphism over U it must be also an isomorphism over S. Finally consider the locally constant étale sheaves C_n on S associated to $(A_1'(n), \Psi)$. Then $C_n \to C_1$ is an epimorphism of étale sheaves because the restriction to U is. From this we obtain that $A_1'(n) \to A_1'(1)$ is surjective too. Hence the first arrow of (14) is a closed immersion and the sequence is exact.

We assume now by induction that Y_{k-1} with its filtration extends to a p-divisible group Y_{k-1}' on S. We denote by Z' the extension of the p-divisible group $Z = Y_k/Y_{k-1}$ to S.

We show that $\mathcal{A}_k'(n)$ is free. We denote $Frob_U^m$ by $\alpha_m : U_m \to U$, and $Frob_S^m$ by $\beta_m : S_m \to S$. Of course $U_m = U$ but we would like to think of $Frob_U$ as a flat covering. Applying again base change to the cartesian diagram above it is enough to show that $j_*\alpha_m^*\mathcal{A}_k(n)$ is free. But the exact sequence

$$0 \to Y_{k-1}(n) \to Y_k(n) \to Z(n) \to 0, \tag{15}$$

splits over the perfect closure of U, and therefore over some U_m by the discussion in front of proposition 5. Hence over U_m the scheme $Y_k(n) \times_U U_m$ is the product of the schemes $Y_{k-1}(n) \times_U U_m$ and $Z(n) \times_U U_m$. Therefore $Y_k(n) \times_U U_m$ extends to a locally free scheme over S_m. This proves that $j_*\alpha_m^*\mathcal{A}_k(n)$ is free.

Since Y_k is slope divisible with respect to λ_k, we find r and s with $\lambda_k = \frac{r}{s}$ such that $\Phi = p^{-r}Fr^s : Y_k \to Y_k^{(p^s)}$ is an isogeny. The pairs $(Y_k(n), \Phi)$ extend to S. The purity result (propositon 5) for these extensions $(Y_k'(n), \Phi)$, yields an exact sequence of finite locally free group schemes on S:

$$0 \to H_n \to Y_k'(n) \to (Y_k'(n))^{\Phi} \to 0$$

It is clear that H_n must be the extension of $Y_{k-1}(n)$ and $(Y_k'(n))^{\Phi}$ must be the extension Z'. We set $Y' = \varinjlim Y_k'(n)$ as a flat sheaf. Then we obtain an exact sequence of flat sheaves:

$$0 \to Y_{k-1}' \to Y' \to Z' \to 0$$

We know that the outer sheaves are p-divisible groups and therefore Y' is a p-divisible group. This is the desired group in the isogeny class of X.

Q.E.D.

This completes also the proof of theorem 7. Combining this theorem and proposition 14 we obtain:

Corollary 15 *Let S be a regular scheme and $U \subset S$ an open subscheme which contains all points of codimension 1. Suppose that X is a p-divisible group on U such that the Newton polygon is constant on U. Then there is a p-divisible group X' on S , whose restriction to U is isogenous to X. The Newton polygon of X' is constant.*

The reader should compare this with a result of de Jong and Oort ([JO] Theorem 4.13).

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