On the slope filtration

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To appear in Duke Mathematical Journal

Abstract

Let $X$ be a $p$-divisible group over a regular scheme $S$ such that the Newton polygon in each geometric point of $S$ is the same. Then there is a $p$-divisible group isogenous to $X$ which has a slope filtration.

1 Introduction

Let $X$ be a $p$-divisible group over a perfect field. The Dieudonné classification implies that $X$ is isogenous to a direct product of isoclinic $p$-divisible groups. We will study what remains true, if the perfect field is replaced by a ring $R$ such that $pR = 0$.

Let now $X$ be a $p$-divisible group over $R$. Let us denote by $Fr_X : X \to X^{(p)}$ the Frobenius homomorphism. We call $X$ isoclinic and slope divisible if there are natural numbers $r \geq 0$ and $s > 0$, such that

$$p^{-r} Fr_X^s : X \to X^{(p^s)}$$

is an isomorphism. Then $X$ is isoclinic of slope $r/s$, i.e. it is isoclinic of slope $r/s$ over each geometric point of $\text{Spec } R$. We will say that $r/s$ is the slope of $X$.

If $R$ is a field a $p$-divisible group is isoclinic iff it is isogenous to a $p$-divisible group which is isoclinic and slope divisible.

It is stated in a letter of Grothendieck to Barsotti (see [G1]), that over a field $K = R$ any $p$-divisible group admits a slope filtration:

$$0 = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X$$

(1)
This filtration is uniquely determined by the following properties: The inclusions are strict and the factors $X_i/X_{i-1}$ are isoclinic $p$-divisible groups of slope $\lambda_i$, such that $1 \geq \lambda_1 > \ldots > \lambda_m \geq 0$. Moreover the rational numbers $\lambda_i$ are uniquely determined. A proof of this statement was never published, but can be found here.

The heights of the factors and the numbers $\lambda_i$ determine the Newton polygon and conversely. If we want a slope filtration over $R$, we have to assume that the Newton polygon is the same in any point of Spec $R$. We say in this case that $X$ has a constant Newton polygon.

**Theorem:** Let $R$ be a regular ring. Then any $p$-divisible group over $R$ with constant Newton polygon is isogenous to a $p$-divisible group $X$, which admits a strict filtration (1) such that the quotients $X_i/X_{i-1}$ are isoclinic and slope divisible of slope $\lambda_i$ with $1 \geq \lambda_1 > \ldots > \lambda_m \geq 0$.

In the case where $\dim R = 1$ and $R$ is finitely generated over a perfect field the theorem was proved by Katz [K] using the crystalline theory. Our proof uses only Dieudonné theory over a perfect field. It is based on a purity result (proposition 5) below which was suggested to us when reading the work of Harris and Taylor.

Let $S$ be a regular scheme and $U$ an open subset such that the codimension of the complement is $\geq 2$. Then we show that a $p$-divisible group over $U$ with constant Newton polygon extends up to isogeny to a $p$-divisible group over $S$. One might call this Nagata-Zariski purity for $p$-divisible groups.

We note that there is a difficult purity result of de Jong and Oort, which holds without the regularity assumption for any noetherian scheme $S$. It says that a $p$-divisible group $X$ over $S$, which has constant Newton polygon on $U$, has constant Newton polygon on $S$.

Finally I would like to thank Johan de Jong and Michael Harris for pointing out this problem to me, and Frans Oort for helpful remarks.

### 2 The étale part of a Frobenius module

We will work over a base scheme $S$ over $\mathbb{F}_p$. The Frobenius morphism will be denoted by $\text{Frob}_S$.

**Definition 1** Fix an integer $a > 0$. A Frobenius module over $S$ is a finitely generated locally free $\mathcal{O}_S$-module $\mathcal{M}$, and a $\text{Frob}_S^a$-linear map $\Phi : \mathcal{M} \to \mathcal{M}$. 

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There is an important case, where the condition that $\mathcal{M}$ is locally free is automatically satisfied namely if $\Phi$ is a $\text{Frob}_S$-linear isomorphism. This means that the linearization

$$\Phi^\#: \mathcal{O}_S \otimes_{\text{Frob}_S} \mathcal{M} \to \mathcal{M}$$

is an isomorphism.

**Lemma 2** Let $R$ be a local ring with maximal ideal $\mathfrak{m}$. Assume that $R$ is $\mathfrak{m}$-adically separated. Let $M$ be a finitely generated $R$-module. Assume that there exists a $\text{Frob}_S$-linear isomorphism $\Phi : M \to M$. Then $M$ is free.

**Proof:** We choose a minimal resolution of $M$:

$$0 \to U \to P \to M \to 0,$$

where $P$ is a finitely generated free $R$-module and $U \subset \mathfrak{m}P$. Since $R \otimes_{\text{Frob}_S} P$ is a free $R$-module the linearization $\Phi^\#$ extends to $R \otimes_{\text{Frob}_S} P$, i.e. we find a commutative diagram:

$$\begin{array}{ccc}
R \otimes_{\text{Frob}_S} P & \longrightarrow & R \otimes_{\text{Frob}_S} M \\
\Phi^\# & & \Phi^\#
\end{array}$$

$$U \subset \Phi^\#(R \otimes_{\text{Frob}_S} \mathfrak{m}P) \subset \Phi^\#(\mathfrak{m}P \otimes_{\text{Frob}_S} P) \subset \mathfrak{m}P.$$  

Q.E.D.

To any Frobenius module we associate the following functor on the category of schemes $T \to S$:

$$C_M(T) = \{x \in \Gamma(T, \mathcal{M}_T) \mid \Phi x = x\}$$
Proposition 3  The functor $C_{\mathcal{M}}$ is representable by a scheme which is étale and affine over $S$.

Proof: Since the functor is a sheaf for the flat (fpf) topology the question is local on $S$. We may therefore assume that $S = \text{Spec } R$ and that $\mathcal{M}$ is the sheaf associated to a free $R$-module $M$. We choose an isomorphism $M \cong R^n$ and write the operator $\Phi$ in matrix form:

$$\Phi x = U x^{(p^a)}, \quad x \in R^n.$$ 

Here $x$ is a column vector, and $x^{(p^a)}$ is the vector obtained by raising all components to the $p^a$-th power. $U$ is a square matrix with coefficients in $R$. Let $A$ be an $R$-algebra. We set $C_M(A) = C_{\mathcal{M}}(\text{Spec } A)$. Then $C_M$ is just the functor of solutions of the equation:

$$x = U x^{(p^a)}, \quad x \in A^n.$$ 

This functor is clearly a closed subscheme of the affine space $A^n_R$.

To show that $C_M$ is étale one applies the infinitesimal criterion: Let $A \to \bar{A}$ be a surjection of $R$-algebras with kernel $a$, such that $a^2 = 0$. We have to show that the canonical map

$$C_M(A) \to C_M(\bar{A})$$

is bijective. We consider an element $\bar{x} \in C_M(\bar{A})$, and lift it to an element $x$ of $A \otimes_R M \cong A^n$. We set $\rho = \Phi x - x \in a \otimes_R M$. Since $\Phi(a \otimes_R M) = 0$ we obtain

$$\Phi(x + \rho) = \Phi x = x + \rho.$$ 

This shows that $x + \rho \in C_M(A)$ is the unique lifting of $\bar{x}$. Q.E.D.

To make life easier let us assume that $S$ is an $\mathbb{F}_{p^a}$-scheme. Then $C_M$ may be considered as a sheaf of $\mathbb{F}_{p^a}$-vector spaces. If $S$ is connected and $\eta \in S$ is a point, the natural map

$$C_M(S) \to C_M(\eta)$$

is injective because $C_M$ is unramified and separated over $S$ (e.g. proposition [EGA IV 17.4.9]).
Let us assume that \( S = \text{Spec } K \) is the spectrum of an algebraically closed field. Let \((M, \Phi)\) be a Frobenius module over \( K \). Then there is a unique decomposition:

\[
M = M^{bij} \oplus M^{nil},
\]

(3)

into \( \Phi \)-invariant subspaces, such that \( \Phi \) is bijective on the first summand and nilpotent on the second summand. Moreover by a theorem of Dieudonné (lemma \([Z, 6.25]\)) we have an isomorphism:

\[
K \otimes_{F_{pa}} C_M(\text{Spec } K) \rightarrow M^{bij}
\]

(4)

Let us assume that \( S = \text{Spec } K \) is the spectrum of separably closed field, and denote the algebraic closure by \( \bar{K} \). Since \( C_M(K) = C_M(\bar{K}) \), the subspace \( M^{bij} \) is defined over \( K \) by (4). Note that \( M^{nil} \) is not defined over \( K \), e.g. \( M = K^{p^{-1}} \) and \( \Phi = Frob \).

We note that the submodule \( M^{bij} \) is defined over any field \( K \) by Galois descent (\([G2] \text{ B, Exemple 1}\)): If \( K^s \) denotes the separable closure and \( G \) its Galois group over \( K \), we set:

\[
M^{bij} = (K^s \otimes_{F_{pa}} C_M(K^s))^G
\]

This subspace is characterized as follows: On \( M^{bij} \) the operator \( \Phi \) acts as a \( Frob^a \)-linear isomorphism, and on the factor \( M/M^{bij} \) it acts nilpotently.

We note that the functor \( M \mapsto M^{bij} \) is an exact functor in \( M \). To see this it is enough to consider the case of an algebraically closed field \( K \). With this assumption the result follows because the decomposition (3) is functorial in \( M \). The same argument shows that the functor commutes with tensor products.

Assume that \( S = \text{Spec } R \) and that \((M, \Phi)\) is a Frobenius module over \( R \).

**Lemma 4** Assume that \( \text{Spec } R \) is connected. Then the natural map

\[
R \otimes_{F_{pa}} C_M(R) \rightarrow M
\]

(5)

is an injection onto a direct summand of \( M \).
Proof: Since Spec $R$ is connected, the natural map $C_M(R) \rightarrow C_M(R_p)$ is for any prime ideal $p$ of $R$ injective ([EGA] loc.cit.). Therefore it is enough to show our statement for a local ring $R$ with maximal ideal $m$. Indeed the question whether the finitely generated quotient of (5) is projective is local. Since $R_p \otimes_{\mathbb{F}_{p^a}} C_M(R)$ is obviously a direct summand of $R_p \otimes_{\mathbb{F}_{p^a}} C_M(R_p)$ we are reduced to the local case.

In this case it is enough to show that the following map is injective:

$$R/m \otimes_{\mathbb{F}_{p^a}} C_M(R) \rightarrow M/mM.$$  

Since the map $C_M(R) \rightarrow C_M(R/m)$ is injective we are reduced to the case where $R$ is a field. Then the injectivity follows from the considerations above.

Q.E.D.

Let $S = \text{Spec } R$, where $R$ is an henselian local ring with maximal ideal $m$. Then there is a unique $\Phi$-invariant direct summand $L \subset M$, such that $\Phi$ is an $\text{Frob}^{\phi}$-linear isomorphism on $L$, and is nilpotent on $M/L + mM$. We call $L$ the finite part.

To show this one reduces the problem by Galois descent [G2] to the case where $R$ is strictly henselian. In this case we can set $L = R \otimes_{\mathbb{F}_{p^a}} C_M(R)$. We note also that taking the finite part $L$ is an exact functor in $M$. This functor also commutes with tensor products.

Let us return to the general situation of definition 1. For each point $\eta$ of $S$ we define the function:

$$\mu_{(M, \Phi)}(\eta) = \dim_{\mathbb{F}_{p^a}} (C_M)_{\bar{\eta}},$$  

where $\bar{\eta}$ is some geometric point over $\eta$.

If $\mu_{(M, \Phi)}(\eta) \geq k$ it stays bigger or equal than $k$ in some neighbourhood of $\eta$. If this function is constant on $S$ there is a $\Phi$-invariant submodule $L$ of $M$, which is locally a direct summand, such that $\Phi$ is an $\text{Frob}^{\phi}$-linear isomorphism on $L$ and is locally on $S$ nilpotent on $M/L$. By this last property $L$ is uniquely determined. For this result it is not necessary that $S$ is noetherian. Indeed in this case the scheme $C$ associated to $(M, \Phi)$ is finite étale since all geometric fibres have the same number of points (corollaire [EGA IV 18.2.9]). Then $C$ represents an étale sheaf on $S$ denoted by the same letter. In the sense of étale sheaves we have:

$$L = \mathcal{O}_S \otimes_{\mathbb{F}_{p^a}} C$$
If the scheme $S$ is perfect the exact sequence:

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{M}/\mathcal{L} \to 0$$

splits canonically. Indeed, it is enough to define this splitting in the case $S = \text{Spec} \, R$. Then $\Phi : \mathcal{L} \to \mathcal{L}$ is bijective. Assume that $\Phi^n$ is zero on $\mathcal{M}/\mathcal{L}$ for some number $n$. Let $\mathcal{M}^{nil}$ be the kernel of $\Phi^n$ on $\mathcal{M}$. Then the projection $\mathcal{M}^{nil} \to \mathcal{M}/\mathcal{L}$ is bijective. Indeed, let $x \in \mathcal{M}$. Then $\Phi^n x \in \mathcal{L}$. Since $\Phi$ is bijective on $\mathcal{L}$, we find $y \in \mathcal{L}$ with $\Phi^n y = \Phi^n x$. But then $x$ and $x - y \in \mathcal{M}^{nil}$ have the same image in $\mathcal{M}/\mathcal{L}$. This proves the assertion.

The following purity result is contained in Harris-Taylor [HT] in a special case:

**Proposition 5** Let $R$ be a noetherian local ring of dimension $\geq 2$. Let $(M, \Phi)$ be a Frobenius module over $R$. Assume that the function $\mu_{(M, \Phi)}$ is constant outside the closed point of $\text{Spec} \, R$. Then it is constant.

**Proof:** We can assume that $R$ is a complete local ring with algebraically closed residue field. Let $S = \text{Spec} \, R$, and let $U$ the complement of the closed point $s \in S$.

Since $C_M$ is étale over $R$ it admits a unique decomposition:

$$C_M = C_M^{\text{f}} \coprod C_M^{0},$$

where $C_M^{\text{f}}$ is finite and étale over $\text{Spec} \, R$, and where $C_M^0$ has an empty special fibre. We note that $C_M^0$ is affine as a closed subscheme of $C_M$.

We have to show that $C_M^0$ is empty. Let us assume the opposite. We consider the following function on $U$:

$$\sharp C_M^0, \eta = \sharp C_M, \eta - \sharp C_M^{\text{f}}, \eta \in U. \quad (6)$$

Here $\sharp$ denotes the number of points in the corresponding scheme. The first term on the right hand side of (6) is by assumption constant on $U$ while the second term has this property for obvious reasons.

Hence all geometric fibres of the map

$$C_M^0 \to U,$$

have the same number of points. Together with our assumption that $C_M^0$ is not empty this shows that the last map is surjective. But this implies that
is affine (théorème [EGA II 6.7.1]). Since $U$ is not affine ([G3] Proposition 6.4) we have a contradiction. Q.E.D.

If $R$ is a regular local ring of dimension 2, then any Frobenius module $(\mathcal{M}, \Phi)$ over $U$ may be extended to $S$, because the direct image of $\mathcal{M}$ by $j: U \to S$ is a free $R$-module $M$. This implies in particular that any locally constant étale sheaf of $\mathbb{F}_p$-vector spaces extends to a locally constant étale sheaf on $S$ (purity).

We apply this to finite commutative group schemes as follows: Let $G$ be a finite locally free group scheme over a scheme $S$. Assume we are given a homomorphism $\Phi : G \to G^{(p^e)}$. Let $G = \text{Spec} \mathcal{M}$ relative to $S$. Then $\Phi$ induces on $\mathcal{M}$ the structure of a Frobenius module.

Let $S = \text{Spec} R$ the spectrum of an henselian local ring. Let $\mathcal{L}$ be the finite part of $\mathcal{M}$ which was defined after lemma 4. Since its formation commutes with tensor products we obtain a finite locally free group scheme $G^\Phi = \text{Spec} \mathcal{L}$. Since $\mathcal{L}$ is a direct summand of $\mathcal{M}$ the natural morphism $G \to G^\Phi$ is an epimorphism of finite locally free group schemes. Let us denote by $G^\Phi_{-\text{nil}}$ the kernel. We obtain an exact sequence of finite locally free group schemes:

$$0 \to G^\Phi_{-\text{nil}} \to G \to G^\Phi \to 0 \quad (7)$$

such that $\Phi$ induces an isomorphism on $G^\Phi$ and is nilpotent on the special fibre of $G^\Phi_{-\text{nil}}$.

**Lemma 6** Let $G_i$, $i = 1, 2, 3$ be finite locally free group schemes over the spectrum $S$ of a henselian local ring. Let $\Phi_i : G_i \to G_i^{(p^e)}$ be homomorphisms. Assume we are given an exact sequence:

$$0 \to G_1 \to G_2 \to G_3 \to 0,$$

which respects the homomorphisms $\Phi_i$. Then the corresponding sequence

$$0 \to G_1^{\Phi_i} \to G_2^{\Phi_i} \to G_3^{\Phi_i} \to 0,$$

is exact.

**Proof:** Let $S$ be the spectrum of an algebraically closed field. In view of the decomposition (3) we have a unique $\Phi_r$-equivariant section of the epimorphism $G_i \to G_i^\Phi$. Therefore there exists a functorial decomposition:
\[ G_i = G_i^\Phi \oplus G_i^{-nil} \]

This proves the assertion for an algebraically closed field and hence for any field. In the general case we consider the kernel \( H \) of the epimorphism \( G_2^\Phi \to G_3^\Phi \). Then we obtain a homomorphism of locally free group schemes \( G_1^\Phi \to H \), which is an isomorphism over the closed point of \( S \). Hence it is an isomorphism by the lemma of Nakayama. \( Q.E.D. \)

We consider a pair \((G, \Phi)\) as above over any locally noetherian scheme \( S \). Let \( k \) be the maximal value of the corresponding function \( \mu = \mu_{(M, \Phi)} \). Then the set \( \mu = k \) is an open set \( U \) of \( S \). Over \( U \) we have an exact sequence of finite locally free group schemes (7) such that \( \Phi \) induces an isomorphism on \( G^\Phi \) and is locally on \( S \) nilpotent on \( G^\Phi-nil \). If \( S \) is irreducible the complement of \( U \) is by proposition 5 of pure codimension 1 or empty. The formation of \( G^\Phi \) is by lemma 6 an exact functor in an obvious sense.

Assume that \( \Phi \) is an isomorphism, i.e. \( G = G^\Phi \). The étale sheaf \( C \) associated to \((M, \Phi)\) is a locally constant étale sheaf of \( \mathbb{F}_p \)-bigebras. If \( S \) is the spectrum of a strictly henselian local ring the canonical isomorphism \( M = \mathcal{O}_S \otimes C \) of bigebras means that \( G \) is obtained via base change from a group scheme \( G_0 \) over \( \mathbb{F}_{p^s} \), and \( \Phi \) from the identity \( G_0 \to G_0^{(p^s)} \).

### 3 The slope filtration of a \( p \)-divisible group

Let \( X \) be a \( p \)-divisible group over a scheme \( S \) and \( \lambda \in \mathbb{Q} \). We call \( X \) slope divisible with respect to \( \lambda \) if there are locally on \( S \) integers \( r, s > 0 \) such that \( \lambda = \frac{r}{s} \) and the following quasiisogeny is an isogeny:

\[ p^{-r} Fr_X^s : X \to X^{(p^s)} \] (8)

Recall that a quasiisogeny is an isogeny formally divided by a power of \( p \) ([RZ] definition 2.8). We will use the fact that the functor of points of \( S \) where a quasiisogeny is an isogeny is representable by a closed subscheme of \( S \) ([RZ] proposition 2.9). If \( X \) is slope divisible and isoclinic of slope \( \lambda \) (i.e. isoclinic over any geometric point of \( S \)) then the isogeny above is an isomorphism.
Theorem 7 Let $S$ be a regular scheme. Let $X$ be a $p$-divisible group over $S$ whose Newton polygon is constant. Then there is a $p$-divisible group $Y$ over $S$ which is isogenous to $X$, and which has a filtration by closed immersions of $p$-divisible groups:

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_k = Y,$$

such that $Y_i/Y_{i-1}$ is isoclinic and slope divisible with respect to $\lambda_i$, and the group $Y_i$ is slope divisible with respect to $\lambda_i$. One has $\lambda_1 > \lambda_2 > \ldots > \lambda_k$.

The existence of the slope filtration over a field which is not necessarily perfect is announced in [G1]. Since a proof was never published we give it here before treating the general case.

Proposition 8 Let $K$ be a field of characteristic $p$. Let $G \to H$ be a morphism of $p$-divisible groups over $K$. The there is a unique factorization in the category of $p$-divisible groups

$$G \to G' \to H' \to H$$

with the following properties:

(i) $G' \to H'$ is an isogeny.

(ii) $H' \to H$ is a monomorphism of $p$-divisible groups.

(iii) For each number $n$ the morphism $G(n) \to G'(n)$ is an epimorphism of finite group schemes.

This factorization commutes with base change to another field.

Proof: We note that a monomorphism in the category of $p$-divisible groups is the same thing as a closed immersion. Let $A$ be the kernel of $G \to H$ in the category of flat sheaves of abelian groups. Then $A$ has the following properties:

(i) The kernel $A(n)$ of multiplication by $p^n$ on $A$ is representable by a finite group scheme.

(ii) The group $A$ is the union of the subgroups $A(n)$. 

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With these assumptions there is a unique $p$-divisible subgroup $A' \subset A$ such that the quotient is a finite group scheme. Indeed, we consider the following sequence of monomorphisms

$$A(n + 1)/A(n) \xrightarrow{p} A(n)/A(n - 1) \xrightarrow{p} \ldots \to A(1), \quad (9)$$

which is induced by the multiplication by $p$. Since the ranks of the group schemes in (9) cannot decrease infinitely, there is a number $n_0$ such that $A(n + 1)/A(n) \to A(n)/A(n - 1)$ is an isomorphism for $n > n_0$. We set $A' = A/A(n_0)$. Then we obtain $A'(m) = A(n_0 + m)/A(n_0)$. Because for $A'$ all homomorphisms in (9) are isomorphisms this group is a $p$-divisible group.

The multiplication by $p^{n_0}$ defines a monomorphism $A'p^{n_0} \hookrightarrow A$. The cokernel of this monomorphism is a finite locally free group scheme. This is seen in the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & A'(n_0) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A(n_0)
\end{array}
\begin{array}{ccc}
\longrightarrow & A'p^{n_0} & \longrightarrow & A' \\
p^{n_0} & \downarrow & \downarrow & \downarrow \\
A & \longrightarrow & A' & \longrightarrow & 0
\end{array}
$$

Now we may define $G'$ as the quotient $G/A'$, and $H'$ as the quotient of $G'$ by the finite group scheme $A/A'$.

The group $H'$ is the image of $G \to H$ in the category of flat sheaves. We call $G'$ the small image of $G \to H$.

Assume for a moment that $K$ is a perfect field, and let $M_G$ and $M_H$ be the covariant Dieudonné modules. Then $M_G$ is the image of the map $M_G \to M_H$, while $M_H'$ is the smallest direct summand of $M_H$ containing $M_G$.

Let $X$ be a $p$-divisible group of height $h$ over a perfect field $K$. We denote by $M$ its covariant Dieudonné module. It is a free $W(K)$-module of rank $h$. Let $\lambda = r/s$ be the smallest Newton slope of $X$. By lemma [Z, 6.13] there is a $W(K)$-lattice $M'$ in $M \otimes \mathbb{Q}$ such that $V^sM' \subset p^\lambda M'$. The operator $U = p^{-\lambda}V^s$ acts on $M \otimes \mathbb{Q}$.

**Lemma 9** The submodule $M_0 \subset M \otimes \mathbb{Q}$ defined by:

$$M_0 = M + UM + U^2M + \ldots + U^{h-1}M,$$

is a Dieudonné module which is invariant by $U$.  

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**Proof:** By [Z] loc.cit. we know that $M \otimes \mathbb{Q}$ contains a $U$-invariant lattice. Let $M'$ be a lattice which contains $M$, such that $UM' \subset M'$. We will take $M'$ minimal with respect to inclusion. Then $M \nsubseteq pM'$. We consider the ascending chain of lattices:

$$pM' \subsetneq pM' + M \subset pM' + M + UM \subset \ldots \subset M'.$$

Since $\dim_k M'/pM' = h$ there is an integer $e \leq h - 1$ such that

$$pM' + M + \ldots + U^e M = pM' + M + \ldots + U^{e+1} M.$$

Hence this is a $U$-invariant lattice containing $M$ and we conclude by minimality:

$$M' = M + \ldots + U^e M + pM'.$$

But then the lemma of Nakayama shows

$$M' = M + \ldots + U^e M.$$

Since $U$ commutes with $F$ and $V$, it is easy to see that $M'$ is a Dieudonné module. This proves the lemma. 

Q.E.D.

By this lemma $F^{s(h-1)}M_0$ is the Dieudonné module of a $p$-divisible group $Y$ over $K$, which is slope divisible with respect to $\lambda$. Clearly $Y$ is the small image of the morphism of $p$-divisible groups which is defined as the composite of the following quasimorphisms:

$$X^{(p^{(h-1)s})} \times \ldots \times X^{(p^r)} \times X \xrightarrow{\alpha} X^{(p^{(h-1)s})} \to X$$

where the last arrow is the power $Ver^{(h-1)s}$ of the Verschiebung $Ver : X^{(p)} \to X$, and where the restriction of $\alpha$ to the factor $X^{(p^{(h-1)s})}$ is $p^{-(r-1)s} Fr^{(l-1)s}$. We recall here that $Ver$ induces $F$ on the Dieudonné module $M$, while $Fr : X \to X^p$ induces $V$ (lemma [Z, 5.19]).

If $K$ is not perfect we can still consider the small image $Y$ of (10). Making base change to the perfect hull we see that $Y \to X$ is an isogeny, and that $Y$ is slope divisible, i.e. that

$$\Phi = p^{-r} Fr^s : Y \to Y^{(p^r)}$$

is an isogeny. If we apply (7) to the finite group schemes $Y(n)$ and the operator $\Phi$ we obtain an exact sequence of $p$-divisible groups:

$$0 \to Y^{\Phi-nil} \to Y \to Y^\Phi \to 0$$

(11)
The $p$-divisible group $Y^\Phi$ is slope divisible and isoclinic, while the smallest slope of $Y^{\Phi-nil}$ is strictly bigger than $\lambda$.

**Remark:** In the notation of lemma 9 the inclusion $F^{s(h-1)}M_0 \subset M$ holds. This follows easily from $r \leq s$. Note that we can take $s \leq h$. It follows that over any field $K$ the degree of the isogeny $Y \to X$ is bounded by $p^{h^2(h-1)}$, i.e. by a constant which depends only on the height $h$ of $X$.

**Definition 10** Let $X$ be a $p$-divisible group over a field $K$. We call $X$ completely slope divisible if it admits a filtration

$$0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X$$

(12)

by $p$-divisible subgroups, and if there are rational numbers $\lambda_1 > \ldots > \lambda_m$ such that

(i) $X_i$ is slope divisible with respect to $\lambda_i$ for $i = 1, \ldots, m$.

(ii) $X_i/X_{i-1}$ is isoclinic and slope divisible with respect to $\lambda_i$.

Since there are no homomorphisms between $p$-divisible groups with pairwise different Newton slopes ([Z]) it follows easily that the filtration (12) is uniquely determined.

**Corollary 11** If the field $K$ is perfect the sequence (12) splits canonically.

**Proof:** By the remark after definition 10 the splitting is unique. We consider the Dieudonné modules $M_i$ of $X_i$. By induction it is enough to show that the following sequence splits as a sequence of Dieudonné modules:

$$0 \to M_{m-1} \to M_m \to M_m/M_{m-1} \to 0$$

But $\Phi = p^{-r_m}V^s$ acts on this sequence. On $M_{m-1}$ the action is topologically nilpotent and on $M_m/M_{m-1}$ it is bijective. Therefore we conclude by lemma [Z, 6.16].

**Q.E.D.**

**Proposition 12** Let $h$ be a number. Then there is a constant $c$ which depends only on $h$ with the following property.

Let $X$ be a $p$-divisible group of height $h$ over a field $K$. Then there is an isogeny $X' \to X$ whose degree is smaller than $c$ such that $X'$ is completely slope divisible.
Proof: Let \( \lambda_i \) for \( i = 1, \ldots, m \) be the slopes of \( X \). We may write \( \lambda_i = r_i/s \) where \( s \) divides \( h! \), and \( r_1 > \ldots > r_m \). By what we have proved we find an isogeny \( Y \to X \) of bounded degree such that \( Y \) is slope divisible with respect to \( \lambda_m \). If \( m = 1 \) then \( Y \) is completely slope divisible. For \( m > 1 \) we argue by induction. We set \( \Phi = p^{-r_m}Fr^s \) and obtain the \( \Phi \)-decomposition (11). By induction there is an isogeny of bounded degree \( Y^{\Phi-nil} \to Z \) where \( Z \) is completely slope divisible. Then we take the push-out of the sequence (11) by the morphism \( Y^{\Phi-nil} \to Z \):

\[
0 \to Z \to Z' \to Y^{\Phi-ct} \to 0
\]

The only thing we have to check is that \( Z' \) is slope divisible with respect to \( r_m/s \). But by induction \( Z \) is slope divisible with respect to \( r_{m-1}/s \) and hence a fortiori with respect to \( r_m/s \). From the exact sequence (11) it follows that \( Y^{\Phi-nil} \) is slope divisible with respect to \( \lambda_m = r_m/s \). By definition \( Z' \) sits in an exact sequence:

\[
0 \to Y^{\Phi-nil} \to Y \times Z \to Z' \to 0
\]

Since all groups in this sequence except \( Z' \) are slope divisible with respect to \( \lambda_m = r_m/s \) the same is true for \( Z' \).

Q.E.D.

Corollary 13 Let \( X \) be a \( p \)-divisible groups over a field \( K \). Let \( \lambda_1 > \ldots > \lambda_m \) be the sequence of slopes of \( X \). Then there is a filtration of \( X \)

\[
0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X
\]

by \( p \)-divisible subgroups, such that

(i) \( X_i \) has the slopes \( \lambda_1, \ldots, \lambda_i \) for \( i = 1, \ldots, m \).

(ii) \( X_i/X_{i-1} \) is isoclinic of slope \( \lambda_i \).

Proof: Indeed in the notation of proposition 12 it is enough to consider the image of the filtration on \( X' \) by the isogeny \( X' \to X \).

Q.E.D.

Proof of Theorem 7: If \( S \) is the spectrum of a field this follows from proposition 12.

If the dimension of \( S \) is 1 this was shown under more restrictive conditions by Katz ([K] Corollary 2.6.3). We give an alternative proof which holds in the general situation: Let \( K \) be a function field of \( S \). Any isogeny \( X_K \to Y \)
over $K$ extends to an isogeny $X \to Y$ over $S$. (See also the discussion in front of the next proposition.) Hence we may assume that $X_K$ is completely slope divisible (definition 10). We have to show that this filtration extends to $S$. Since the functor of points of $S$, where (8) is an isogeny is representable by a closed subscheme of $S$ we see that $X$ is slope divisible with respect to $\lambda_k$. Therefore we have an isogeny:

$$\Phi = p^{-r} Fr^s : X \to X^{(p^r)}.$$ 

The function $\mu$ associated to $(X(n), \Phi)$ is constant since the Newton polygon is constant. Therefore we may form the finite group schemes $X(n)^\Phi$. For varying $n$ this is a $p$-divisible group $Z$, since the functor $X(n) \mapsto X(n)^\Phi$ is exact. We obtain an exact sequence:

$$0 \to X' \to X \to Z \to 0$$

Then $X'$ has again constant Newton polygon, but the slope $\lambda_k$ doesn’t appear. Since $X'_K$ is completely slope divisible we can finish the proof in the case $\text{dim } S = 1$ by induction.

In arbitrary dimension this consideration shows that a $p$-divisible group $X$ over $S$ has a slope filtration as in theorem 7, if $X_K$ is completely slope divisible.

By the same method we may find $Y$ with the filtration over an open set $U \subset S$, which contains all points of codimension 1. Indeed, we start again with an isogeny $X_K \to \tilde{Y}$, where $\tilde{Y}$ is a $p$-divisible group over $K$ which is completely slope divisible. The kernel $\tilde{G}$ of this isogeny is a closed subscheme of some $X_K(n)$. We denote by $G$ the scheme theoretic closure in $X(n)$. Let $U$ be the open subscheme where $G$ is flat. We replace $S$ by $U$ and assume that $G$ is flat over $S$. Then one checks that $G$ inherits the structure of a group scheme, such that $G \to X(n)$ is a closed immersion of group schemes. We may replace $X$ by $Y = X/G$. This group is slope divisible with respect to $\lambda_k$. Therefore we obtain the slope filtration as above. The case where $S$ has arbitrary dimension will now follow from the following proposition:

**Proposition 14** Let $S$ be a regular scheme and $U \subset S$ an open subscheme which contains all points of codimension 1. Suppose that $Y$ is a $p$-divisible group on $U$ with a filtration as in theorem 7. Then $Y$ extends to $S$. 

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Proof: We set \( \text{Spec} \mathcal{A}_i(n) = Y_i(n) \). Let \( j : U \to S \) be the immersion. It is enough to prove the following 3 statements.

1) The sheaves \( \mathcal{A}'_i(n) = j^* \mathcal{A}_i(n) \) are locally free \( \mathcal{O}_S \)-modules. If this is true the bigebra structure on \( \mathcal{A}_i(n) \) extends to \( \mathcal{A}'_i(n) \). Therefore we can define finite locally free group schemes \( Y'_i(n) = \text{Spec} \mathcal{A}'_i(n) \).

2) For varying \( n \) the systems \( \{Y'_i(n)\} \) define a \( p \)-divisible group \( Y'_i \).

3) The induced maps \( Y'_i \to Y'_{i+1} \) are closed immersions.

To verify these statements one can make without loss of generality a faithfully flat base change \( S' \to S \). Therefore it is enough to consider the case, where \( S \) is the spectrum of a complete regular local ring \( R \) of dimension \( \geq 2 \) with algebraically closed residue field. By induction on the dimension we may assume that \( U \) is the complement of the closed point.

We make an induction on the length of the filtration. For \( k = 1 \) we extend \( Y_1 \) as follows. Let \( \lambda_1 = \frac{r}{s} \) such that \( \Phi = p^{-r} F_r^s \) is an isogeny and a hence an isomorphism \( Y_1 \to Y_1^{(p^s)} \). Since the morphism \( Frob^s : S \to S \) is flat, we may apply base change (lemme [EGA, IV 2.3.1]) to the cartesian diagram:

\[
\begin{array}{ccc}
U & \longrightarrow & S \\
\downarrow \scriptstyle{Frob^s} & & \downarrow \scriptstyle{Frob^s} \\
U & \longrightarrow & S
\end{array}
\]

This yields an isomorphism:

\[
R \otimes_{Frob^s, R} A'_1(n) \cong j_*(\mathcal{O}_U \otimes_{Frob^s, \mathcal{O}_U} \mathcal{A}_1(n)),
\]

where \( A'_1(n) \) denotes the global sections of \( \mathcal{A}'_1(n) \). Therefore the isomorphism \( \Phi \) induces an isomorphism

\[
\Phi^* : R \otimes_{Frob^s, R} A'_1(n) \to A'_1(n)
\]

We will denote the associated \( Frob^s \)-linear map by \( \Psi : A'_1(n) \to A'_1(n) \). By the lemma 2 it follows that \( A'_1(n) \) is free. This shows the assertion 1). To see the second assertion we have to show that the sequence:

\[
0 \to Y'_1(1) \to Y'_1(n) \to Y'_1(n-1) \to 0 \quad (14)
\]
is exact. Since we know that this sequence is exact over \( U \), it suffices to show that the first arrow is a closed immersion. Indeed knowing this we obtain a morphism of finite locally free group schemes over \( S \):
\[
Y'_1(n)/Y'_1(1) \to Y'_1(n-1)
\]
Since this is an isomorphism over \( U \) it must be also an isomorphism over \( S \). Finally consider the locally constant étale sheaves \( C_n \) on \( S \) associated to \((A'_1(n), \Psi)\). Then \( C_n \to C_1 \) is an epimorphism of étale sheaves because the restriction to \( U \) is. From this we obtain that \( A'_1(n) \to A'_1(1) \) is surjective too. Hence the first arrow of (14) is a closed immersion and the sequence is exact.

We assume now by induction that \( Y_{k-1} \) with its filtration extends to a \( p \)-divisible group \( Y'_{k-1} \) on \( S \). We denote by \( Z' \) the extension of the \( p \)-divisible group \( \mathbb{Z} = Y_{k}/Y_{k-1} \) to \( S \).

We show that \( A'_k(n) \) is free. We denote \( \text{Frob}^m_U \) by \( \alpha_m : U_m \to U \), and \( \text{Frob}^m_S \) by \( \beta_m : S_m \to S \). Of course \( U_m = U \) but we would like to think of \( \text{Frob}_S \) as a flat covering. Applying again base change to the cartesian diagram above it is enough to show that \( j_\ast \alpha'_m \ast A_k(n) \) is free. But the exact sequence
\[
0 \to Y_{k-1}(n) \to Y_k(n) \to Z(n) \to 0,
\]
splits over the perfect closure of \( U \), and therefore over some \( U_m \) by the discussion in front of proposition 5. Hence over \( U_m \) the scheme \( Y_k(n) \times_U U_m \) is the product of the schemes \( Y_{k-1}(n) \times_U U_m \) and \( Z(n) \times_U U_m \). Therefore \( Y_k(n) \times_U U_m \) extends to a locally free scheme over \( S_m \). This proves that \( j_\ast \alpha'_m \ast A_k(n) \) is free.

Since \( Y_k \) is slope divisible with respect to \( \lambda_k \), we find \( r \) and \( s \) with \( \lambda_k = \frac{r}{s} \) such that \( \Phi = p^{-r} \text{Fr}^s : Y_k \to Y_k(p^s) \) is an isogeny. The pairs \((Y_k(n), \Phi)\) extend to \( S \). The purity result (proposition 5) for these extensions \((Y_k'(n), \Phi)\) yields an exact sequence of finite locally free group schemes on \( S \):
\[
0 \to H_n \to Y'_k(n) \to (Y'_k(n))^\Phi \to 0
\]
It is clear that \( H_n \) must be the extension of \( Y_{k-1}(n) \) and \((Y'_k(n))^\Phi \) must be the extension \( Z' \). We set \( Y' = \lim_{\to} Y'_k(n) \) as a flat sheaf. Then we obtain an exact sequence of flat sheaves:
\[
0 \to Y'_{k-1} \to Y' \to Z' \to 0
\]
We know that the outer sheaves are $p$-divisible groups and therefore $Y'$ is a $p$-divisible group. This is the desired group in the isogeny class of $X$.

Q.E.D.

This completes also the proof of theorem 7. Combining this theorem and proposition 14 we obtain:

**Corollary 15** Let $S$ be a regular scheme and $U \subset S$ an open subscheme which contains all points of codimension 1. Suppose that $X$ is a $p$-divisible group on $U$ such that the Newton polygon is constant on $U$. Then there is a $p$-divisible group $X'$ on $S$, whose restriction to $U$ is isogenous to $X$. The Newton polygon of $X'$ is constant.

The reader should compare this with a result of de Jong and Oort ([JO] Theorem 4.13).

**References**


